M216: Exercise sheet 7

Warmup questions

1. Let V be a complex inner product space and $\phi \in \mathrm{U}(V)$ a unitary operator. Without recourse to the spectral theorem, show:
   (*a*) All eigenvalues of ϕ have unit length.
   (*b*) Eigenvectors of ϕ with distinct eigenvalues are orthogonal.

2. Let $\phi \in L(V)$ be a diagonalisable linear operator and $v_1, \dots, v_n$ a basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$. Show that if λ is an eigenvalue of ϕ then

$$E_\phi(\lambda) = \operatorname{span}\{v_i \mid \lambda_i = \lambda\}.$$

   Thus λ appears $\dim E_\phi(\lambda)$ times in the list $\lambda_1, \dots, \lambda_n$.

Homework

3. Let $\phi \in L(V)$ be a linear operator on a finite-dimensional complex inner product space V.
   (*a*) Show that ϕ is normal if and only if there is an orthonormal basis $u_1, \dots, u_n$ of V and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that, for all $v \in V$,

$$\phi(v) = \sum_{i=1}^{n} \lambda_i \langle u_i, v\rangle u_i.$$

   Hint: The forward implication is a consequence of the spectral theorem!
   (*b*) If ϕ is normal, find a similar formula for ϕ^*.
   (*c*) Deduce that an operator on V is self-adjoint if and only if it is normal and has all eigenvalues real.

4. Let ϕ be a self-adjoint operator on a finite-dimensional inner product space with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. Show that

$$\lambda_1 = \min_{v \neq 0} \frac{\langle v, \phi(v)\rangle}{\langle v, v\rangle} \qquad \lambda_n = \max_{v \neq 0} \frac{\langle v, \phi(v)\rangle}{\langle v, v\rangle}$$

Extra questions

5. Let $\phi \in L(V)$ be a linear operator on a vector space. A *square root* of ϕ is a linear operator $\psi \in L(V)$ such that $\psi \circ \psi = \phi$.
   Show that any normal operator on a finite-dimensional complex inner product space has a square root.

6. Let $\phi \in L(V)$ be a normal operator on a finite-dimensional complex inner product space.
   (*a*) Show that $\|\phi(v)\| = \|\phi^*(v)\|$, for all $v \in V$.
   (*b*) Deduce that if v is an eigenvector of ϕ with eigenvalue λ then v is also an eigenvector of ϕ^* with eigenvalue $\overline{\lambda}$.
   Hint: $\phi - \lambda\,\mathrm{id}_V$ is also normal.

Please hand in at 4W level 1 by NOON on Friday 23rd November

Home page: `http://go.bath.ac.uk/ma20216`

M216: Exercise sheet 7—Solutions

1. Let v be an eigenvector of ϕ so that $\phi(v) = \lambda v$. Taking ϕ^{-1} of both sides and rearranging gives $\phi^{-1}(v) = \lambda^{-1}v$. Thus v is also an eigenvector of ϕ^{-1} with eigenvalue λ^{-1}.
Now let v, w be eigenvectors of ϕ with eigenvalues λ, μ. Since $\phi^{-1} = \phi^*$ we have

$$\langle \phi^{-1}(v), w\rangle = \langle v, \phi(w)\rangle.$$

   (a) If $v = w$, this reads $\langle \lambda^{-1}v, v\rangle = \langle v, \lambda v\rangle$ so that $(\overline{\lambda})^{-1}\|v\|^2 = \lambda\|v\|^2$ whence $(\overline{\lambda})^{-1} = \lambda$ or $|\lambda| = 1$.

   (b) If $\lambda \neq \mu$, we get $((\overline{\lambda})^{-1} - \mu)\langle v, w\rangle = 0$, or, using the first part, $(\lambda - \mu)\langle v, w\rangle = 0$. Thus $\langle v, w\rangle = 0$ and we are done.

2. If $\lambda_i = \lambda$ then $v_i \in E_\phi(\lambda)$ and since $E_\phi(\lambda)$ is a subspace, we get

$$\operatorname{span}\{v_i \mid \lambda_i = \lambda\} \leq E_\phi(\lambda).$$

For the converse, let $v \in E_\phi(\lambda)$ and write $v = x_1v_1 + \cdots + x_nv_n$. Then $\phi(v) = \lambda v$ and we write each in terms of $v_1, \dots, v_n$:

$$\lambda v = \sum_{i=1}^{n} \lambda x_i v_i$$
$$\phi(v) = \sum_{i=1}^{n} x_i\phi(v_i) = \sum_{i=1}^{n} \lambda_i x_i v_i.$$

Since $v_1, \dots, v_n$ are linearly independent, we conclude that $\lambda x_i = \lambda_i x_i$, for all $1 \leq i \leq n$. Thus $x_i = 0$ unless $\lambda = \lambda_i$ and we conclude that $v \in \operatorname{span}\{v_i \mid \lambda_i = \lambda\}$.

3. (a) If ϕ is normal then, by the spectral theorem, it is orthogonally diagonalisable so we have an orthonormal basis $u_1, \dots, u_n$ of eigenvectors of ϕ with eigenvalues $\lambda_1, \dots, \lambda_n$. Then any $v \in V$ can be written $v = \sum_{i=1}^{n}\langle u_i, v\rangle u_i$ and taking ϕ of this yields

$$\phi(v) = \phi(\sum_{i=1}^{n}\langle u_i, v\rangle u_i) = \sum_{i=1}^{n}\langle u_i, v\rangle\phi(u_i) = \sum_{i=1}^{n}\lambda_i\langle u_i, v\rangle u_i.$$

Conversely, if there is an orthonormal basis $u_1, \dots, u_n$ and scalars $\lambda_1, \dots, \lambda_n$ for which

$$\phi(v) = \sum_{i=1}^{n}\lambda_i\langle u_i, v\rangle u_i,$$

for all $v \in V$, take $v = u_j$ to get

$$\phi(u_j) = \sum_{i=1}^{n}\lambda_i\langle u_i, u_j\rangle u_i = \lambda_j u_j.$$

Thus ϕ is orthogonally diagonalisable and so normal by a result in the notes.

   (b) Let $u_1, \dots, u_n$ be an orthonormal basis of eigenvectors of ϕ with eigenvalues $\lambda_1, \dots, \lambda_n$. For $v \in V$,

$$\phi^*(v) = \sum_{i=1}^{n}\langle u_i, \phi^*(v)\rangle u_i = \sum_{i=1}^{n}\langle \phi(u_i), v\rangle u_i = \sum_{i=1}^{n}\langle \lambda_i u_i, v\rangle u_i = \sum_{i=1}^{n}\overline{\lambda}_i\langle u_i, v\rangle u_i.$$

To summarise:

$$\phi^*(v) = \sum_{i=1}^{n}\overline{\lambda}_i\langle u_i, v\rangle u_i.$$

(c) We know that if ϕ is self-adjoint then it is normal and all eigenvalues are real. For the converse, if ϕ is normal and each $\lambda_i = \overline{\lambda}_i$ then comparing our formulae for $\phi(v)$ and $\phi^*(v)$, we conclude that $\phi = \phi^*$ so that ϕ is self-adjoint.

4. Let $u_1, \ldots, u_n$ be an orthonormal basis of eigenvectors with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. By Parseval's identity,

$$\langle v, \phi(v)\rangle = \sum_{i=1}^{n} \langle v, u_i\rangle\langle u_i, \phi(v)\rangle = \sum_{i=1}^{n} \langle v, u_i\rangle\langle \phi(u_i), v\rangle = \sum_{i=1}^{n} \lambda_i \langle v, u_i\rangle\langle u_i, v\rangle = \sum_{i=1}^{n} \lambda_i |\langle u_i, v\rangle|^2,$$

where we have used that ϕ is self-adjoint and the λ_i are real.
Now Bessel's equality says that $\|v\|^2 = \sum_{i=1}^{n} |\langle u_i, v\rangle|^2$ so we have

$$\lambda_1 \|v\|^2 = \sum_{i=1}^{n} \lambda_1 |\langle u_i, v\rangle|^2 \leq \sum_{i=1}^{n} \lambda_i |\langle u_i, v\rangle|^2 \leq \sum_{i=1}^{n} \lambda_n |\langle u_i, v\rangle|^2 = \lambda_n \|v\|^2.$$

Otherwise said $\lambda_1 \|v\|^2 \leq \langle v, \phi(v)\rangle \leq \lambda_n \|v\|^2$. Moreover, take $v = u_1$ to get equality in the first inequality and $v = u_n$ to get equality in the second. Dividing by $\|v\|^2$ now bakes the cake.

5. Let $u_1, \ldots, u_n$ be an orthonormal basis of eigenvectors of ϕ with eigenvalues $\lambda_1, \ldots, \lambda_n$. For each i, let μ_i be a square root of λ_i and define $\psi : V \to V$ by setting $\psi(u_i) = \mu_i u_i$ and extending by linearity.
Then $\psi \circ \psi(u_i) = \mu_i \psi(u_i) = \mu_i^2 u_i = \lambda_i u_i = \phi(u_i)$ so that $\psi \circ \psi$ and ϕ agree on the u_i and so everywhere.
Notice that what we have really proved here is that any diagonalisable operator (orthogonally diagonalisable or not) has a square root.

6. (a) We know that $\phi \circ \phi^* - \phi^* \circ \phi = 0$ so we evaluate this at $v \in V$ and take the inner product with v to get

$$\begin{aligned} 0 &= \langle \phi(\phi^*(v)) - \phi^*(\phi(v)), v\rangle = \langle \phi(\phi^*(v)), v\rangle - \langle \phi^*(\phi(v)), v\rangle \\ &= \langle \phi^*(v), \phi^*(v)\rangle - \langle \phi(v), \phi(v)\rangle = \|\phi^*(v)\|^2 - \|\phi(v)\|^2. \end{aligned}$$

(b) We follow the hint and set $\psi := \phi - \lambda \,\mathrm{id}_V$. From the notes, we know that $\psi^* = \phi^* - \overline{\lambda}\,\mathrm{id}_V^* = \phi^* - \overline{\lambda}\,\mathrm{id}_V$. Then

$$\psi \circ \psi^* = \phi \circ \phi^* - \lambda\phi^* - \overline{\lambda}\phi + |\lambda|^2 \,\mathrm{id}_V = \phi^* \circ \phi - \lambda\phi^* - \overline{\lambda}\phi + |\lambda|^2 \,\mathrm{id}_V = \psi^* \circ \psi$$

so that ψ is normal too. If v is an eigenvector of ϕ with eigenvalue λ then $\psi(v) = 0$ so that part (a), applied to ψ, gives

$$0 = \|\psi(v)\| = \|\psi^*(v)\|.$$

Thus $\psi^*(v) = 0$. Otherwise said, $\phi^*(v) = \overline{\lambda} v$ and we are done.