M216: Exercise sheet 6

Warmup questions

1. Let V be an inner product space and suppose $\phi, \psi \in L(V)$ have adjoints. Show that $\psi \circ \phi$; $\phi + \lambda\psi$, $\lambda \in \mathbb{F}$; ϕ^* and id_V all have adjoints given by:
   (1) $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ (note the change of order here!).
   (2) $(\phi + \lambda\psi)^* = \phi^* + \overline{\lambda}\psi^*$.
   (3) $(\phi^*)^* = \phi$.
   (4) $\mathrm{id}_V^* = \mathrm{id}_V$.

2. Let V be an inner product space with $u, w \in V$. Define a map $\phi_{u,w} : V \to V$ by
$$\phi_{u,w}(v) = \langle u, v\rangle w.$$
   Show that $\phi_{u,w}$ is a linear map and $(\phi_{u,w})^* = \phi_{w,u}$.

3. Let V be a *complex* inner product space and $\phi \in L(V)$ a linear operator.
   Show that ϕ is self-adjoint if and only if $\sqrt{-1}\phi$ is skew-adjoint.

Homework

4. Let $U, W \leq V$ an inner product space.
   (a) Show that $(U + W)^\perp = U^\perp \cap W^\perp$.
   (b) Show that $U^\perp + W^\perp \leq (U \cap W)^\perp$ with equality if V is finite-dimensional.

5. Let V be an inner product space and $\phi \in L(V)$ a linear operator with adjoint ϕ^*.
   (a) Show that $\ker\phi = (\mathrm{im}\,\phi^*)^\perp$.
   Deduce that if V is finite-dimensional, $\mathrm{rank}\,\phi = \mathrm{rank}\,\phi^*$.
   (b) Show that $\mathrm{im}\,\phi \leq (\ker\phi^*)^\perp$ with equality if V is finite-dimensional.

Extra questions

6. Let u be a unit vector in a (real or complex) inner product space. Define $R_u : V \to V$ by
$$R_u(v) = v - 2\langle u, v\rangle u,$$
   for all $v \in V$.
   (a) Show that R_u is orthogonal/unitary.
   (b) Describe R_u geometrically when $V = \mathbb{R}^n$.

7. Let V be an inner product space and $\pi \in L(V)$ a projection: $\pi \circ \pi = \pi$. Recall that, with $U := \ker\pi$ and $W := \mathrm{im}\,\pi$, we then have $V = U \oplus W$.
   Show that U and W are orthogonal complements if and only if π is self-adjoint.

8. Define $Z : \ell_2 \to \ell_2$ by $Z(a) = (0, a_0, a_1, \dots)$.
   Show that Z is a linear isometry which does not surject.

Please hand in at 4W level 1 by NOON on Friday 16th November

Home page: `http://go.bath.ac.uk/ma20216`

M216: Exercise sheet 6—Solutions

1. The method for all of these is the same: given $\chi, \eta \in L(V)$, χ is an adjoint of η exactly when $\langle \chi(w), v\rangle = \langle w, \eta(v)\rangle$, for all $v, w \in V$. With this in mind:
(1) $\langle w, (\psi \circ \phi)(v)\rangle = \langle w, \psi(\phi(v))\rangle = \langle \psi^*(w), \phi(v)\rangle = \langle \phi^*(\psi^*(w)), v\rangle = \langle (\phi^* \circ \psi^*)(w), v\rangle$.
(2) We have

$$\begin{aligned}\langle w, (\phi + \lambda\psi)(v)\rangle &= \langle w, \phi(v) + \lambda\psi(v)\rangle = \langle w, \phi(v)\rangle + \lambda\langle w, \psi(v)\rangle \\ &= \langle \phi^*(w), v\rangle + \lambda\langle \psi^*(w), v\rangle = \langle \phi^*(w) + \overline{\lambda}\psi^*(w), v\rangle = \langle (\phi^* + \overline{\lambda}\psi^*)(w), v\rangle.\end{aligned}$$

(3) The complex conjugate of $\langle \phi^*(v), w\rangle = \langle v, \phi(w)\rangle$ reads $\langle w, \phi^*(v)\rangle = \langle \phi(w), v\rangle$ so that $((\phi)^*)^* = \phi$.
(4) $\langle \mathrm{id}_V(w), v\rangle = \langle w, v\rangle = \langle w, \mathrm{id}_V(v)\rangle$.

2. Linearity of $\phi_{u,w}$ comes straight from second slot linearity of the inner product:

$$\phi_{u,w}(v_1 + \lambda v_2) = \langle u, v_1 + \lambda v_2\rangle w = \langle u, v_1\rangle w + \lambda\langle u, v_2\rangle w = \phi_{u,w}(v_1) + \lambda\phi_{u,w}(v_2).$$

As for the adjoint:

$$\langle x, \phi_{u,w}(v)\rangle = \langle x, \langle u, v\rangle w\rangle = \langle u, v\rangle\langle x, w\rangle = \langle\langle w, x\rangle u, v\rangle = \langle \phi_{w,u}(x), v\rangle,$$

where the third equality comes from anti-linearity in the first slot and $\overline{\langle x, w\rangle} = \langle w, x\rangle$.

3. From question 1(2), we know that $(\sqrt{-1}\phi)^* = \overline{\sqrt{-1}}\phi^* = -\sqrt{-1}\phi^*$. So $\phi = \phi^*$ if and only if $\sqrt{-1}\phi = -(\sqrt{-1}\phi)^*$.

4. (a) Let $v \in (U + W)^\perp$. Then, in particular, since $U, W \leq U + W$, we have $\langle u, v\rangle = \langle w, v\rangle = 0$, for all $u \in U$, $w \in W$. Thus $v \in U^\perp$ and $v \in W^\perp$. That is $(U + W)^\perp \leq U^\perp \cap W^\perp$.
Conversely, if $x \in U+W$, we can write $x = u+w$ with $u \in U$, $w \in W$. Now if $v \in U^\perp \cap W^\perp$ then

$$\langle x, v\rangle = \langle u, v\rangle + \langle w, v\rangle = 0 + 0 = 0,$$

since $v \in U^\perp$ and $v \in W^\perp$. Thus $v \in (U + W)^\perp$ whence $U^\perp \cap W^\perp \leq (U + W)^\perp$.

(b) Let $v \in U^\perp + W^\perp$ so that $v = x + y$ with $x \in U^\perp$ and $y \in W^\perp$. Then for $u \in U \cap W$, we have

$$\langle u, v\rangle = \langle u, x\rangle + \langle u, y\rangle = 0 + 0 = 0,$$

since $u \perp x$ ($u \in U$) and $u \perp y$ ($u \in W$). Thus $U^\perp + W^\perp \leq (U \cap W)^\perp$.
For equality when V is finite-dimensional, we show that both spaces have the same dimension using the dimension formula: if $A, B \leq V$ then

$$\dim(A + B) = \dim A + \dim B - \dim(A \cap B).$$

In the case at hand, we get

$$\begin{aligned}\dim(U^\perp + W^\perp) &= \dim U^\perp + \dim W^\perp - \dim(U^\perp \cap W^\perp) \\ &= \dim U^\perp + \dim W^\perp - \dim(U + W)^\perp\end{aligned}$$

using part (a)

$$= \dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U + W))$$

since $\dim A^\perp = \dim V - \dim A$,

$$\begin{aligned}&= \dim V - (\dim U + \dim W - \dim(U + W)) \\ &= \dim V - \dim(U \cap W) = \dim(U \cap W)^\perp.\end{aligned}$$

5. (a) Let $v \in V$. Then $v \in \ker \phi$ if and only if $\phi(v) = 0$ if and only if $\langle \phi(v), w \rangle = 0$, for all $w \in V$, by the nondegeneracy lemma, if and only if $\langle v, \phi^*(w) \rangle = 0$, for all $w \in V$. But this last is exactly the same as $v \in (\operatorname{im} \phi^*)^\perp$. Thus $\ker \phi = (\operatorname{im} \phi^*)^\perp$.
If V is finite-dimensional, we get

$$\dim \ker \phi = \dim V - \dim \operatorname{im} \phi^*.$$

Rearrange this and use rank-nullity to get

$$\dim \operatorname{im} \phi = \dim V - \dim \ker \phi = \dim \operatorname{im} \phi^*.$$

Otherwise said, $\operatorname{rank} \phi = \operatorname{rank} \phi^*$.

(b) Let $v \in \operatorname{im} \phi$ so that $v = \phi(u)$, for some $u \in V$ and let $w \in \ker \phi^*$. Then

$$\langle w, v \rangle = \langle w, \phi(u) \rangle = \langle \phi^*(w), u \rangle = \langle 0, u \rangle = 0,$$

so that $v \in (\ker \phi^*)^\perp$. Thus $\operatorname{im} \phi \leq (\ker \phi^*)^\perp$.
For equality when V is finite-dimensional, show that both subspaces have the same dimension: we already know that $\dim \operatorname{im} \phi = \dim \operatorname{im} \phi^*$ from part (a) and, this, with rank-nullity says that

$$\dim \operatorname{im} \phi = \dim V - \dim \ker \phi^* = \dim(\ker \phi^*)^\perp$$

as required.

6. (a) Notice that $R_u = \mathrm{id}_V - 2\phi_{u,u}$, in the notation of question 2 so that R_u is easily seen to be self-adjoint. We must therefore prove that $R_u \circ R_u = \mathrm{id}_V$. So let $v \in V$ then

$$R_u(R_u(v)) = R_u(v) - 2\langle u, R_u(v) \rangle u = R_u(v) - 2\langle R_u(u), v \rangle u,$$

since R_u is self-adjoint. Now

$$R_u(u) = u - 2\|u\|^2 u = -u$$

since u has unit length so that

$$R_u(R_u(v)) = R_u(v) + 2\langle u, v \rangle u = v.$$

Thus $R_u \circ R_u = \mathrm{id}_v$ and we are done.

(b) If $v \perp u$, then $R_u(v) = v$ while we have just seen that $R_u(u) = -u$. Thus R_u is reflection in the subspace $\operatorname{span}\{u\}^\perp$.

7. If π is self-adjoint, we learn from question 5 that

$$U = \ker \pi = (\operatorname{im} \pi)^\perp = W^\perp$$

so that U and W are orthogonal complements.
Conversely, if U and W are orthogonal complements and $v_1, v_2 \in V$, we write $v_i = u_i + w_i$ with $u_i \in U$ and $w_i \in W$. Then

$$\langle \pi(v_1), v_2 \rangle = \langle w_1, u_2 + w_2 \rangle = \langle w_1, w_2 \rangle = \langle u_1 + w_1, w_2 \rangle = \langle v_1, \pi(v_2) \rangle$$

so that π is self-adjoint.

8. Z is clearly linear and non-surjective: $(1, 0, 0, \dots)$ is not in $\operatorname{im} Z$. It is also a linear isometry: if $a, b \in \ell_2$ then $Z(a)_n = a_{n-1}$, for $n \geq 1$ and similarly for $Z(b)$ so that

$$\langle Z(a), Z(b) \rangle = \sum_{n \in \mathbb{N}} Z(a)_n Z(b)_n = 0 + \sum_{n \geq 1} a_{n-1} b_{n-1} = \sum_{n \in \mathbb{N}} a_n b_n = \langle a, b \rangle.$$