M216: Exercise sheet 6

Warmup questions

1. Let $f : X \to X$ be a map of sets. Show that if f is injective then f^k is injective for each $k \in \mathbb{N}$.
2. Let $U_1 \leq U_2 \leq \cdots \leq V$ be an increasing sequence of subspaces of V, so that $U_m \leq U_n$ whenever $m \leq n$.
   Show that $\bigcup_{n\in\mathbb{N}} U_n \leq V$.

Homework

3. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 1 & -1 \\ -10 & -2 & 5 \\ -6 & 2 & 1 \end{pmatrix}.$$

   (a) Compute the characteristic and minimum polynomials of ϕ.

   (b) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

4. Let $\phi = \phi_A \in L(\mathbb{C}^3)$ where A is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

   (a) Compute the characteristic and minimum polynomials of ϕ.

   (b) Find bases for the eigenspaces and generalised eigenspaces of ϕ.

Extra questions

5. Let $\phi \in L(V)$ be an invertible linear operator on a finite-dimensional vector space and λ an eigenvalue of ϕ. Show that $G_\phi(\lambda) = G_{\phi^{-1}}(\lambda^{-1})$.
6. Let $\lambda \in \mathbb{F}$ and define $J(\lambda, n) \in M_n(\mathbb{F})$ by

$$J(\lambda, n) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}.$$

   Set $J_n := J(0, n)$.
   Prove:

   (a) $\ker J_n^k = \operatorname{span}\{e_1, \ldots, e_k\}$.

   (b) $\operatorname{im} J_n^k = \operatorname{span}\{e_1, \ldots, e_{n-k}\}$.

   (c) $m_{J(\lambda,n)} = \pm\Delta_{J(\lambda,n)} = (x - \lambda)^n$.

   (d) λ is the only eigenvalue of $J(\lambda, n)$ and $E_{J(\lambda,n)}(\lambda) = \operatorname{span}\{e_1\}$, $G_{J(\lambda,n)}(\lambda) = \mathbb{F}^n$.

Please hand in at 4W level 1 by NOON on Friday 17th November

M216: Exercise sheet 6—Solutions

1. Let $x, y \in X$ be such that $f^k(x) = f^k(y)$. Thus $f(f^{k-1}(x)) = f(f^{k-1}(y))$ whence, since f is injective, $f^{k-1}(x) = f^{k-1}(y)$. Repeat the argument to eventually conclude that $x = y$ so that f^k is injective. For a more formal argument, induct on k.

2. Let $U = \bigcup_{n\in\mathbb{N}} U_n$ and let $v, w \in U$. Then there are $n, m \in \mathbb{N}$ with $v \in U_n$ and $w \in U_m$. Without loss of generality, assume that $m \leq n$ so that $U_m \leq U_n$ whence $v, w \in U_n$. Since U_n is a subspace, $v + \lambda w \in U_n \subseteq U$, for any $\lambda \in \mathbb{F}$, so that U is indeed a subspace.

3. (a) We compute the characteristic polynomial: $\Delta_\phi = \Delta_A = -x^3 - x^2 + 8x + 12 = (3-x)(x+2)^2$. Consequently, m_ϕ is either $(x-3)(x+2)^2$ or $(x-3)(x+2)$. We try the latter:

$$A - 3I_3 = \begin{pmatrix} -3 & 1 & -1 \\ -10 & -5 & 5 \\ -6 & 2 & -2 \end{pmatrix} \qquad A + 2I_3 = \begin{pmatrix} 2 & 1 & -1 \\ -10 & 0 & 5 \\ -6 & 2 & 3 \end{pmatrix}$$

so that

$$(A - 3I_3)(A + 2I_3) = \begin{pmatrix} -10 & -5 & 5 \\ 0 & 0 & 0 \\ -20 & -10 & 10 \end{pmatrix} \neq 0.$$

Thus $m_\phi = m_A = (x-3)(x+2)^2$.

(b) We deduce that $G_\phi(3) = E_\phi(3) = \ker(A - 3I_3)$ while $E_\phi(-2) = \ker(A+2I_3)$ and $G_\phi(-2) = \ker(A + 2I_3)^2$. We compute these: an eigenvector x with eigenvalue 3 solves

$$\begin{aligned} -3x_1 + x_3 - x_3 &= 0 \\ -2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

which rapidly yields $x_1 = 0$ and $x_2 = x_3$. Thus the 3-eigenspace is spanned by $(0, 1, 1)$.
An eigenvector x with eigenvalue 2 solves

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 0 \\ -2x_1 + 0x_2 + x_3 &= 0 \end{aligned}$$

giving $x_2 = 0$ and $2x_1 = x_3$ so the eigenspace is spanned by $(1, 0, 2)$.
Finally,

$$(A + 2I_3)^2 = \begin{pmatrix} 0 & 0 & 0 \\ -50 & 0 & 25 \\ -50 & 0 & 25 \end{pmatrix}$$

with kernel spanned by $(1, 0, 2)$ and $(0, 1, 0)$.
To summarise:

$$\begin{aligned} E_\phi(3) = G_\phi(3) &= \operatorname{span}\{(0, 1, 1)\} \\ E_\phi(-2) &= \operatorname{span}\{(1, 0, 2)\} \\ G_\phi(-2) &= \operatorname{span}\{(1, 0, 2), (0, 1, 0)\}. \end{aligned}$$

4. (a) Since A is lower triangular, we immediately see that $\Delta_\phi = \Delta_A = x^2(x-5)$. So the only possibilities for $m_\phi = x(x-5)$ and $x^2(x-5)$. However

$$A - 5I_3 = \begin{pmatrix} -5 & 0 & 0 \\ 4 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

$$A(A - 5I_3) = \begin{pmatrix} 0 & 0 & 0 \\ -20 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0.$$

We conclude that $m_\phi = x^2(x-5)$.
Alternatively, A is block diagonal:

$$A = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} \oplus (5)$$

and the summands clearly have minimum polynomials x^2 and $x-5$ respectively. It follows from a previous sheet that $m_\phi = x^2(x-5)$.

(*b*) We have $E_\phi(5) = G_\phi(5) = \operatorname{span}\{(0,0,1)\}$, $E_\phi(0) = \ker A = \operatorname{span}\{(0,1,0)\}$ and finally $G_\phi(0) = \ker A^2 = \operatorname{span}\{(1,0,0),(0,1,0)\}$ since

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

5. Note that $(\phi - \lambda \operatorname{id}_V)^n(v) = 0$ if and only if $\lambda^{-n}\phi^{-n}(\phi - \lambda^n \operatorname{id}_V)^n(v) = 0$, that is $(\lambda^{-1}\operatorname{id}_V - \phi)^n(v) = 0$. Thus $G_\phi(\lambda) = G_{\phi^{-1}}(\lambda^{-1})$.
Here, of course, we need $\lambda \neq 0$ but, since ϕ is invertible, zero is not an eigenvalue.

6. Note that $\phi_{J_n}(x) = (x_2, \ldots, x_n, 0)$ so that $\phi^k_{J_n}(x) = (x_{k+1}, \ldots, x_n, 0, \ldots, 0)$, $k < n$ and $\phi^n_{J_n} = 0$.
   (*a*) It is clear from the above that $\ker J_n^k = \{x \in \mathbb{F}^n \mid x_{k+1} = \cdots = x_n = 0\} = \operatorname{span}\{e_1, \ldots, e_k\}$.
   (*b*) Similarly, $\operatorname{im} J_n^k = \{y \in \mathbb{F}^n \mid y_{n-k+1} = \cdots = y_n = 0\} = \operatorname{span}\{e_1, \ldots, e_{n-k}\}$.
   (*c*) $J(\lambda, n)$ is upper triangular so that $\Delta_{J(\lambda,n)} = (\lambda - x)^n$. Therefore $m_{J(\lambda,n)} = (x-\lambda)^s$, for some $s \leq n$. However $(J(\lambda,n) - \lambda I_n)^k = J_n^k \neq 0$, for $k < n$, so that $m_{J(\lambda,n)} = (x-\lambda)^n$.
   (*d*) Finally, it is clear that λ is the only eigenvalue and the eigenspace is $\ker(J(\lambda,n) - \lambda I_n) = \ker J_n = \operatorname{span}\{e_1\}$ by part (a). Similarly, $G_{J(\lambda,n)}(\lambda) = \ker J_n^n = \mathbb{F}^n$.