M216: Exercise sheet 5

Warmup questions

1. Let V be an inner product space and $u_1, \dots, u_n$ an orthonormal list of vectors (not necessarily a basis).
   For $v \in V$, prove *Bessel's inequality*:

$$\|v\|^2 \geq \sum_{i=1}^{n} |\langle u_i, v\rangle|^2.$$

   Hint: Contemplate $w := v - \sum_{i=1}^{n} \langle u_i, v\rangle u_i$.

2. Find an orthonormal basis for the subspace $U := \{(x_1, x_2, x_3) \mid x_1 + 2x_2 + 3x_3 = 0\}$ of $\mathbb{R}^3$ with respect to dot product.

3. What happens if you apply the Gram–Schmidt procedure to a list of vectors which is already orthonormal?

Homework

4. Find an orthonormal basis of $U := \operatorname{span}\{(1, 1, 0, 0), (1, 1, 1, 2)\} \leq \mathbb{R}^4$ with respect to dot product.

5. Find the QR decomposition of the matrix A given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}.$$

6. Let $U \leq \mathbb{R}^4$ be as in Question 4. Find the $u \in U$ for which $\|u - (1, 2, 3, 4)\|$ is as small as possible.

Additional questions

7. Let V be a finite-dimensional inner product space with subspaces $U, W \leq V$. Let π_U, π_W be the orthogonal projections onto U, W, respectively.
   Show that $\pi_U \circ \pi_W = 0$ if and only if $\langle u, w\rangle = 0$, for all $u \in U$, $w \in W$.

8. Recall that $C^0[-1, 1]$ is an inner product space with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} fg.$$

   Let $P_2 \leq C^0[-1, 1]$ be the subspace of polynomial functions of degree no more than 2.
   We know that the functions $1, x, x^2$ are a basis of P_2.
   Use Gram–Schmidt to find an orthonormal basis of P_2.

Please hand in at 4W level 1 by NOON on Friday 9th November

M216: Exercise sheet 5—Solutions

1. Observe that w is orthogonal to each u_j and so to $u := \sum_{i=1}^{n} \langle u_i, v\rangle u_i$. Thus, by Pythagoras,
$$\|v\|^2 = \|w\|^2 + \|u\|^2.$$
But $\|u\|^2 = \sum_{i=1}^{n} |\langle u_i, v\rangle|^2$ (expand out and use the orthonormality of the u_i) and $\|w\|^2 \geq 0$ so rearranging gives the result.

2. U is two-dimensional with basis v_1, v_2 where $v_1 = (2, -1, 0)$ and $v_2 = (3, 0, -1)$. We apply Gram–Schmidt to these.
First $w_1 = v_1$ so that $\|w_1\|^2 = 5$ while $\langle w_1, v_2\rangle = 6$. Thus $w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\|w_1\|^2} w_1 = v_2 - \frac{6}{5} w_1 = \frac{1}{5}(3, 6, -5)$ and $\|w_2\|^2 = (9 + 36 + 25)/25 = 14/5$. We conclude that $u_1 = v_1/\sqrt{5} = \frac{1}{\sqrt{5}}(2, -1, 0)$ and $u_2 = \sqrt{\frac{5}{14}} w_2 = \frac{1}{\sqrt{70}}(3, 6, -5)$ give an orthonormal basis of U.
Of course, if you started with a different basis you will get a different answer. Whatever u_1, u_2 you end up with, you can check your answer by verifying the following: $u_1, u_2 \in U$ (do the coefficients satisfy $x_1 + 2x_2 + 3x_3 = 0$?); u_1, u_2 orthonormal: do you have $\langle u_1, u_1\rangle = \langle u_2, u_2\rangle = 1$ and $\langle u_1, u_2\rangle = 0$?

3. If $v_1, \dots, v_m$ are already orthonormal, the Gram-Schmidt procedure leaves them unchanged: $u_i = v_i$, $1 \leq i \leq m$.
Indeed, in this case, $\|v_1\| = 1$ so $u_1 = v_1/\|v_1\| = v_1$. For an induction, suppose that $u_i = v_i$ for $i < k$ and contemplate $w_k = v_k - \sum_{i<k} \langle u_i, v_k\rangle u_i$. Since the $u_i = v_i \perp v_k$, we get $v_k = w_k$ so that $u_k = w_k/\|w_k\| = v_k$. Thus, by induction, $u_i = v_i$, for $1 \leq i \leq m$.

4. Call these vectors v_1, v_2 and unleash the Gram–Schmidt machine on them.
First, with $w_1 = v_1$, we have $\|w_1\|^2 = 2$ and $\langle w_1, v_2\rangle = 1 + 1 = 2$ so that $w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\|w_1\|^2} w_1 = (1, 1, 1, 2) - (1, 1, 0, 0) = (0, 0, 1, 2)$. Now $\|w_2\|^2 = 5$ so that $u_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0)$ and $u_2 = \frac{1}{\sqrt{5}}(0, 0, 1, 2)$.
Thus our orthonormal basis is $\frac{1}{\sqrt{2}}(1, 1, 0, 0), \frac{1}{\sqrt{5}}(0, 0, 1, 2)$.

5. The columns of Q are obtained from those of A by Gram–Schmidt. In this case, we rapidly obtain
$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}.$$
Now we get R from the formula $R = Q^T A$. Thus
$$R = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. Let $v = (1, 2, 3, 4)$. We know from Theorem 3.14 that the u we seek is $\pi_U(v)$, the orthogonal projection of v onto U, and, according to Proposition 3.13(4), if u_1, u_2 is an orthonormal basis of U, this is given by $\pi_U(v) = \langle u_1, v\rangle u_1 + \langle u_2, v\rangle u_2$. We use the orthonormal basis we computed in question 4, so that $\langle u_1, v\rangle = \frac{1}{\sqrt{2}}(1 \times 1 + 1 \times 2) = 3/\sqrt{2}$, $\langle u_2, v\rangle = \frac{1}{\sqrt{5}}(1 \times 3 + 2 \times 4) = 11/\sqrt{5}$ so that
$$\pi_U(v) = \tfrac{3}{2}(1, 1, 0, 0) + \tfrac{11}{5}(0, 0, 1, 2) = (\tfrac{3}{2}, \tfrac{3}{2}, \tfrac{11}{5}, \tfrac{22}{5}).$$

7. First remark that $\pi_U \circ \pi_W = 0$ if and only if $\operatorname{im} \pi_W \leq \ker \pi_U$. But we know from lectures (Proposition 3.13) that $\ker \pi_U = U^\perp$ and $\operatorname{im} \pi_W = W$ so that $\pi_U \circ \pi_W = 0$ if and only if $W \leq U^\perp$, that is, $\langle u, w\rangle = 0$ for all $u \in U$ and $w \in W$.

8. Define $v_1, v_2, v_3 \in P_2$ by $v_1(x) = 1$, $v_2(x) = x$, $v_3(x) = x^2$, for all $x \in [-1, 1]$.
Now do Gram–Schmidt: first, with $w_1 = v_1$,
$$\|w_1\|^2 = \int_{-1}^{1} 1 = x|_{-1}^{1} = 2$$

while

$$\langle w_1, v_2\rangle = \int_{-1}^{1} x = \frac{x^2}{2}\Big|_{-1}^{1} = 0$$

so that $w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\|w_1\|^2} w_1 = v_2$. Then

$$\|w_2\|^2 = \int_{-1}^{1} x^2 = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}.$$

For w_3, we need:

$$\langle w_1, v_3\rangle = \int_{-1}^{1} x^2 = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}, \qquad \langle w_2, v_3\rangle = \int_{-1}^{1} x^3 = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

so that $w_3(x) = v_3 - \frac{\langle w_1, v_3\rangle}{\|w_1\|^2} w_1(x) = x^2 - 1/3$ and then

$$\|w_3\|^2 = \int_{-1}^{1} (x^2 - 1/3)^2 = \frac{8}{45},$$

according to my computer.
This yields $u_1 = w_1/\|w_1\|$ is the constant function $1/\sqrt{2}$, $u_2 = \sqrt{3/2}v_2$ and $u_3(x) = \sqrt{45/8}(x^2 - 1/3)$.
Punchline: an orthonormal basis of P_2 is given by $1/\sqrt{2}$, $\sqrt{3/2}x$, $\sqrt{45/8}(x^2 - 1/3)$.