M216: Exercise sheet 4

Warmup questions

1. Define $v_1, v_2, v_3 \in \mathbb{C}^3$ by
$$v_1 = (1, 0, 1) \qquad v_2 = (2 + i, 1, 1 + 2i) \qquad v_3 = (0, 1, i).$$
Compute all nine $\langle v_i, v_j \rangle$, $1 \le i, j \le 3$ where the inner product is the dot product on $\mathbb{C}^3$.

2. Let V be an inner product space and $v, w \in V$.
   (a) Prove the Pythagoras theorem: if $v \perp w$ then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.
   (b) More generally, for any v, w, prove that $\|v + w\|^2 = \|v\|^2 + 2\operatorname{Re}\langle v, w\rangle + \|w\|^2$.
   (c) Prove the parallelogram identity: $\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$.

3. For $n \in \mathbb{N}$, an n-th root of unity is $\omega \in \mathbb{C}$ such that $\omega^n = 1$. Prove:
   (a) $|\omega| = 1$, for any n-th root of unity ω.
   (b) If ω_1, ω_2 are n-th roots of unity then so is $\omega_1\omega_2$ and ω_1^{-1}.
   (c) If ω is an n-th root of unity and $\omega \ne 1$,
$$1 + \omega + \cdots + \omega^{n-1} = 0$$
   Hint: Recall (or prove!) that $x^n - 1 = (x - 1)(1 + x + \cdots + x^{n-1})$.

Homework questions

4. Prove the Apollonius identity:
$$\|u - v\|^2 + \|u - w\|^2 = \tfrac{1}{2}\|v - w\|^2 + 2\|u - \tfrac{1}{2}(v + w)\|^2.$$

5. Let $\omega_k := e^{2\pi i k/n}$, $0 \le k \le n - 1$, be the n n-th roots of unity and define $v_k \in \mathbb{C}^n$ by
$$v_k = (\omega_k^0, \omega_k^1, \dots, \omega_k^{n-1})/\sqrt{n},$$
for $0 \le k \le n - 1$.
Prove that, for all $0 \le k, h \le n - 1$, $\langle v_k, v_h \rangle = 1$ if $k = h$ and 0 otherwise, where the inner product is the dot product.

Extra questions

6. Let a, b, c, d be positive numbers. Show that
$$16 \le (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$
Hint: Learned any good inequalities lately?

7. (For fans of MA10207) Recall that $\ell_2 = \{(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=0}^{\infty} a_n^2 \text{ converges}\}$.
   (a) Show that $\ell_2 \le \mathbb{R}^{\mathbb{N}}$.
   (b) Show that if $a, b \in \ell_2$ then $\sum_{n=0}^{\infty} a_n b_n$ is absolutely convergent.
   (c) Show that $\langle a, b\rangle := \sum_{n=0}^{\infty} a_n b_n$ defines an inner product on ℓ_2.
   Hint: For $x, y \in \mathbb{R}$, $|xy| \le \frac{1}{2}(x^2 + y^2)$ and $(x + y)^2 \le 2(x^2 + y^2)$. Use these and the comparison theorems of MA10207.

Please hand in at 4W level 1 by NOON on Friday 2nd November

Home page: `http://go.bath.ac.uk/ma20216`

M216: Exercise sheet 4—Solutions

1. We compute:

$$\begin{aligned}
\langle v_1, v_2\rangle &= 1\times(2+i)+0\times 1+1\times(1+2i)=3+3i\\
\langle v_1, v_3\rangle &= 1\times 0+0\times 1+1\times i=i\\
\langle v_2, v_3\rangle &= \overline{(2+i)}\times 0+1\times 1+\overline{1+2i}\times i=1+(1-2i)\times i=3+i.
\end{aligned}$$

Now use the conjugate symmetry to get

$$\begin{aligned}
\langle v_2, v_1\rangle &= \overline{3+3i}=3-3i\\
\langle v_3, v_1\rangle &= -i\\
\langle v_3, v_2\rangle &= 3-i.
\end{aligned}$$

Finally,

$$\begin{aligned}
\langle v_1, v_1\rangle &= 1^2+1^2=2\\
\langle v_2, v_2\rangle &= |2+i|^2+1^2+|1+2i|^2=5+1+5=11\\
\langle v_3, v_3\rangle &= 2.
\end{aligned}$$

2. We prove (b) and deduce the (a) from it. Now $\|v+w\|^2=\langle v+w, v+w\rangle$ and we expand this out using sesquilinearity:

$$\langle v+w, v+w\rangle=\langle v,v\rangle+\langle v,w\rangle+\langle w,v\rangle+\langle w,w\rangle.$$

Conjugate symmetry tells us that $\langle w,v\rangle=\overline{\langle v,w\rangle}$ so that $\langle v,w\rangle+\langle w,v\rangle=2\operatorname{Re}\langle v,w\rangle$ and putting the whole thing together yields

$$\|v+w\|^2=\|v\|^2+2\operatorname{Re}\langle v,w\rangle+\|w\|^2$$

as required.
Finally, $v\perp w$ means that $\langle v,w\rangle=0$ and the Pythagoras theorem follows at once.
For (c), we get from (b):

$$\|v+w\|^2=\|v\|^2+2\operatorname{Re}\langle v,w\rangle+\|w\|^2$$

and the same argument (or just replace w by $-w$ in the last equation) gives

$$\|v-w\|^2=\|v\|^2-2\operatorname{Re}\langle v,w\rangle+\|w\|^2.$$

Add these to achieve satori[1].

3. (a) Absolute value is multiplicative so if $\omega^n=1$, $|\omega|^n=|\omega|^n=1^n=1$ so that, since $|\omega|>0$, we have $|\omega|=1$.
   (b) $(\omega_1\omega_2)^n=\omega_1^n\omega_2^n=1\times 1=1$. Similarly, $(1/\omega_1)^n=1/\omega_1^n=1/1=1$.
   (c) The polynomial $x^n-1=(x-1)(1+x+\cdots+x^{n-1})$ (multiply out the right hand side and see most terms cancel). Taking $x=\omega$, this gives

$$0=(\omega-1)(1+\omega+\cdots+\omega^{n-1})$$

whence the result since $\omega-1\neq 0$ by assumption.

[1] A state of sudden enlightenment.

4. Expanding out the left hand side gives:

$$\|u\|^2 - 2\operatorname{Re}\langle u, v\rangle + \|v\|^2 + \|u\|^2 - 2\operatorname{Re}\langle u, w\rangle + \|w\|^2$$
$$= 2\|u\|^2 + \|v\|^2 + \|w\|^2 - 2\operatorname{Re}\langle u, v + w\rangle.$$

Meanwhile, doing the same on the right gives us

$$\tfrac{1}{2}\|v\|^2 - \operatorname{Re}\langle v, w\rangle + \tfrac{1}{2}\|w\|^2 + 2\|u^2\| - 2\operatorname{Re}\langle u, v + w\rangle + \tfrac{1}{2}\|v + w\|^2$$
$$= \tfrac{1}{2}\|v\|^2 - \operatorname{Re}\langle v, w\rangle + \tfrac{1}{2}\|w\|^2 + 2\|u^2\| - 2\operatorname{Re}\langle u, v + w\rangle + \tfrac{1}{2}\|v\|^2 + \operatorname{Re}\langle v, w\rangle + \tfrac{1}{2}\|w\|^2$$

which cancels to give the same as the left side.

5. First, using question 3(a), any $|\omega_k^j| = 1$ so that

$$\langle v_k, v_k\rangle = (\sum_{j=0}^{n-1} |\omega_k^j|)/n = n/n = 1.$$

Next, for $k \neq h$, set $\omega := \overline{\omega_k}\omega_h$. Then $\omega = e^{2\pi i(h-k)/n}$ is also an n-th root of unity, not equal to 1 and
$$\langle v_k, v_h\rangle = (1 + \omega + \cdots + \omega^{n-1})/n = 0$$
by question 3(c).

6. We apply Cauchy–Schwarz to $x = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$ and $y = (1/\sqrt{a}, 1/\sqrt{b}, 1/\sqrt{c}, 1/\sqrt{d}) \in \mathbb{R}^4$ which is an inner product space for dot product. This gives

$$|x \cdot y| = 1 + 1 + 1 + 1 = 4 \leq \|x\|\|y\| = (a + b + c + d)^{\frac{1}{2}}(1/a + 1/b + 1/c + 1/d)^{\frac{1}{2}}$$

and squaring both sides bakes the cake.

7. (a) First ℓ_2 is non-empty since it contains the zero sequence.
If $a, b \in \ell_2$ then $\sum_{n=0}^{\infty} a_n^2$ and $\sum_{n=0}^{\infty} b_n^2$ converge so that, by algebra of series, $\sum_{n=0}^{\infty} 2(a_n^2 + b_n^2)$ converges. Now, for each $n \in \mathbb{N}$, $(a_n + b_n)^2 \leq 2(a_n^2 + b_n^2)$ so the comparison theorem tells us that $\sum_{n=0}^{\infty}(a_n + b_n)^2$ converges. Thus $a + b \in \ell_2$.
Again, if $\lambda \in \mathbb{R}$, then $\sum_{n=0}^{\infty}(\lambda a_n)^2 = \sum_{n=0}^{\infty} \lambda^2 a_n^2$ converges by algebras of series. Thus $\lambda a \in \ell_2$ and ℓ_2 is a subspace.

(b) If $a, b \in \ell_2$, algebra of series tells us that $\sum_{n=0}^{\infty} \frac{1}{2}(a_n^2 + b_n^2)$ converges. On the other hand, each $|a_n b_n| \leq \frac{1}{2}(a_n^2 + b_n^2)$ so that, by comparison, $\sum_{n=0}^{\infty}|a_n b_n|$ converges. Otherwise said, $\sum_{n=0}^{\infty} a_n b_n$ is absolutely convergent.

(c) If $\langle a, b\rangle = \sum_{n=0}^{\infty} a_n b_n$ then algebra of series shows that $\langle\, , \,\rangle$ is linear in the second slot. It is plainly symmetric since each $a_n b_n = b_n a_n$. For definiteness, observe that $\langle a, a\rangle = \sum_{n=0}^{\infty} a_n^2 \geq a_k^2 \geq 0$, for each $k \in \mathbb{N}$: indeed, if $S_N = \sum_{n=0}^{N} a_n^2$ is the N-th partial sum, then $S_N \geq a_k$, for all $N \geq k$, so that $\sum_{n=0}^{\infty} a_n^2 = \lim_{N\to\infty} S_N \geq a_k$. Thus $\langle a, a\rangle \geq 0$ and vanishes if and only if each $a_k = 0$, that is $a = 0$.