## M216: Exercise sheet 3

### Warmup questions

1. Let $U \leq V$. Show that congruence modulo $U$ is an equivalence relation.
2. Recall (from Algebra 1A) that if $f : X \to Y$ is a map of sets and $A \subset Y$, the **inverse image of $A$ by $f$** is the subset $f^{-1}(A) \subset X$ given by
$$f^{-1}(A) := \{x \in X \mid f(x) \in A\}.$$
Now let $\phi : V \to W$ be a linear map of vector spaces and $U \leq W$. Prove that $\ker \phi \leq \phi^{-1}(U) \leq V$.
3. Let $U \leq V$ and $v \in V$. Let $v_1, \ldots, v_k \in v + U$.
Show that $\lambda_1 v_1 + \cdots + \lambda_k v_k \in v + U$ whenever $\lambda_1 + \cdots + \lambda_k = 1$.

### Homework

4. Let $v_1, \ldots, v_k \in V$. The **affine span of** $v_1, \ldots, v_k$ is the subset
$$A(v_1, \ldots, v_k) := \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_1 + \cdots + \lambda_k = 1\} \subset V.$$
(*a*) Show that each $v_i \in A(v_1, \ldots, v_k)$.
(*b*) Show that $A(v_1, \ldots, v_k)$ is a coset of some subspace $U \leq V$.
**Hint**: If this was true, each $v_i - v_1 \in U$: use this to define $U$.
Together with question 3, this shows that the affine span is the smallest affine subspace containing $v_1, \ldots, v_k$.
5. Let $U, W \leq V$. Define a linear map $\phi : U \to (U + W)/W$ by $\phi(u) = u + W$.
(*a*) Use the first isomorphism theorem, applied to $\phi$, to prove the second isomorphism theorem:
$$U/(U \cap W) \cong (U + W)/W.$$
(*b*) Deduce that, when $V$ is finite-dimensional,
$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

### Extra questions

6. Let $U \leq V$ and $q : V \to V/U$ the quotient map. Let $W$ be a complement to $U$.
Show that $q_{|W} : W \to V/U$ is an isomorphism.
7. Let $U \leq V$ and suppose that $V/U$ is finite-dimensional. Show that $U$ has a complement.

## Please hand in at 4W level 1 by NOON on Friday 27th October

## M216: Exercise sheet 3—Solutions

1. **Reflexive** $v - v = 0 \in U$ so $v \equiv v \mod U$.
   **Symmetric** If $v \equiv w \mod U$ then $v - w \in U$ so that $w - v = -(v - w) \in U$ and $w \equiv v \mod U$.
   **Transitive** If $v \equiv w \mod U$ and $w \equiv u \mod U$, then $v - w, w - u \in U$ whence $v - u = (v - w) + (w - u) \in U$ and so $v \equiv u \mod U$.

2. First, if $v \in \ker \phi$ then $\phi(v) = 0 \in U$ so $v \in \phi^{-1}(U)$. Thus $\ker \phi \subset \phi^{-1}(U)$ and, in particular, $\phi^{-1}(U)$ is non-empty.
   Next we see that $\phi^{-1}(U)$ is closed under addition and scalar multiplication: if $v, w \in \phi^{-1}(U)$ and $\lambda \in \mathbb{F}$, then $\phi(u), \phi(w) \in U$ so that $\phi(v + w) = \phi(v) + \phi(w) \in U$ and $\phi(\lambda v) = \lambda\phi(v) \in U$, since $U$ is closed under addition and scalar multiplication. Otherwise said, $v + w, \lambda v \in \phi^{-1}(U)$ and we are done.

3. Since $v_i \in v + U$, there is $u_i \in U$ so that $v_i = v + u_i$. Now, if $\lambda_1 + \dots + \lambda_k = 1$, we have

$$\begin{aligned}\lambda_1 v_1 + \dots + \lambda_k v_k &= \lambda_1(v + u_1) + \dots + \lambda_k(v + u_k) \\ &= (\lambda_1 + \dots + \lambda_k)v + (\lambda_1 u_1 + \dots + \lambda_k u_k) = v + (\lambda_1 u_1 + \dots + \lambda_k u_k) \in v + U.\end{aligned}$$

4. (*a*) Take $\lambda_i = 1$ and $\lambda_j = 0$, for $i \neq j$, to get $v_i = \lambda_1 v_1 + \dots + \lambda_k v_k$ with $\lambda_1 + \dots + \lambda_k = 1$ so that $v_i \in A(v_1, \dots, v_k)$.
   (*b*) If $A(v_1, \dots, v_k)$ was a coset of a subspace $U$, we would have to have

$$A(v_1, \dots, v_k) = v_1 + U$$

   and then each $v_i - v_1$, $2 \leq i \leq k$, would lie in $U$. So set

$$U := \operatorname{span}\{v_i - v_1 \mid 2 \leq i \leq k\}.$$

   Then each $v_i - v_1 \in U$ so that $v_i \in v_1 + U$. Now question 3 tells us that $A(v_1, \dots, v_k) \subset v + U$.
   Conversely, any element of $v_1 + U$ can be written

$$v_1 + \sum_{i \geq 2} \mu_i(v_i - v_1) = \lambda_1 v_1 + \dots + \lambda_k v_k$$

   with $\lambda_1 = 1 - \sum_{i \geq 2} \mu_i$ and $\lambda_j = \mu_j$, for $j \geq 2$. Thus $\lambda_1 + \dots + \lambda_k = 1$ and we conclude that $v + U \subset A(v_1, \dots, v_k)$.

5. (*a*) Let $q : U + W \to (U + W)/W$ be the quotient map. Then $\phi$ is simply the restriction $q_{|U}$ of $q$ to $U$ and so is linear. Moreover, $\ker \phi = U \cap \ker q = U \cap W$.
   Finally, if $q(u + w) \in (U + W)/W$, then, since $q(w) = 0$,

$$q(u + w) = q(u) + q(w) = q(u) = \phi(u)$$

   so that $\phi$ is onto. The first isomorphism theorem now reads

$$U/(U \cap W) = U/\ker \phi \cong \operatorname{im} \phi = (U + W)/W.$$

(*b*) When $V$ is finite-dimensional, we have

$$\dim U - \dim(U \cap W) = \dim U/(U \cap W) = \dim(U + W)/W = \dim(U + W) - \dim W$$

and rearranging this gives the result.

6. Let $v \in V$. Since $V = U + W$, we write $v = u + w$ with $u \in U$ and $w \in W$. Then, since $\ker q = U$, $q(v) = q(u + w) = q(w)$ so that $\operatorname{im} q_{|W} = \operatorname{im} q = V/U$. Thus $q_{|W}$ surjects.
Further, $\ker q_{|W} = \ker q \cap W = U \cap W = \{0\}$ since $\ker q = U$. Thus $q_{|W}$ has trivial kernel and so injects.

7. Let $q : V \to V/U$ be the quotient map and $q(v_1), \dots, q(v_k)$ be a basis for $V/U$, for some $v_1, \dots, v_k \in V.$ Set $W = \operatorname{span}\{v_1, \dots, v_k\} \leq V$. I claim that $W$ is a complement to $U$. So let $v \in V$. Then there are $\lambda_1, \dots, \lambda_k \in \mathbb{F}$ so that

$$q(v) = \lambda_1 q(v_1) + \cdots + \lambda_k q(v_k) = q(\lambda_1 v_1 + \cdots + \lambda_k v_k).$$

Otherwise said, $v - (\lambda_1 v_1 + \cdots + \lambda_k v_k) \in \ker q = U$ so that

$$v = \lambda_1 v_1 + \cdots + \lambda_k v_k + u,$$

for some $u \in U$ and we have $V = U + W$.
Now suppose $v \in U \cap W$. Then we can write $v = \lambda_1 v_1 + \cdots + \lambda_k v_k$ since $v \in W$ but $v \in \ker q$ so that

$$0 = q(v) = \lambda_1 q(v_1) + \cdots + \lambda_k q(v_k).$$

Since the $q(v_i)$ are linearly independent, we get that each $\lambda_i = 0$ and so $v = 0$.