Chapter 1

Linear algebra: concepts and examples

1.1 Vector spaces

Definition. A *vector space V over a field* $\mathbb{F}$ is a set V with two operations:

addition $V \times V \to V : (v, w) \mapsto v + w$ with respect to which V is an abelian group:

- $v + w = w + v$, for all $v, w \in V$;
- $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$;
- there is a *zero element* $0 \in V$ for which $v + 0 = v = 0 + v$, for all $v \in V$;
- each element $v \in V$ has an *additive inverse* $-v \in V$ for which $v + (-v) = 0 = (-v) + v$.

scalar multiplication $\mathbb{F} \times V \to V : (\lambda, v) \mapsto \lambda v$ such that

- $(\lambda + \mu)v = \lambda v + \mu v$, for all $v \in V$, $\lambda, \mu \in \mathbb{F}$.
- $\lambda(v + w) = \lambda v + \lambda w$, for all $v, w \in V$, $\lambda \in \mathbb{F}$.
- $(\lambda\mu)v = \lambda(\mu v)$, for all $v \in V$, $\lambda, \mu \in \mathbb{F}$.
- $1v = v$, for all $v \in V$.

We call the elements of $\mathbb{F}$ *scalars* and those of V *vectors*.

1.2 Subspaces

Definition. A *vector* (or *linear*) *subspace* of a vector space V over $\mathbb{F}$ is a non-empty subset $U \subseteq V$ which is closed under addition and scalar multiplication: whenever $u, u_1, u_2 \in U$ and $\lambda \in \mathbb{F}$, then $u_1 + u_2 \in U$ and $\lambda u \in U$.

In this case, we write $U \leq V$.

Say that U is *trivial* if $U = \{0\}$ and *proper* if $U \neq V$.

1.3 Bases

Definitions. Let $v_1, \ldots, v_n$ be a list of vectors in a vector space V.

1. The *span* of $v_1, \ldots, v_n$ is

$$\operatorname{span}\{v_1, \ldots, v_n\} := \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_i \in \mathbb{F}, 1 \leq i \leq n\} \leq V.$$

2. $v_1, \ldots, v_n$ *span* V (or *are a spanning list for* V) if $\operatorname{span}\{v_1, \ldots, v_n\} = V$.
3. $v_1, \ldots, v_n$ are *linearly independent* if, whenever $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$, then each $\lambda_i = 0$, $1 \leq i \leq n$, and *linearly dependent* otherwise.
4. $v_1, \ldots, v_n$ is a *basis* for V if they are linearly independent and span V.

Definition. A vector space is *finite-dimensional* if it admits a finite list of vectors as basis and *infinite-dimensional* otherwise.

If V is finite-dimensional, the *dimension* of V, $\dim V$, is the number of vectors in a (any) basis of V.

Proposition 1.1 (Algebra 1B, Chapter 2, Proposition 4)**.** *$v_1, \ldots, v_n$ is a basis for V if and only if any $v \in V$ can be written in the form*

$$v = \lambda_1 v_1 + \cdots + \lambda_n v_n \tag{1.1}$$

for unique $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$*. In this case, $(\lambda_1, \ldots, \lambda_n)$ is called the* coordinate vector *of v with respect to $v_1, \ldots, v_n$.*

1.3.1 Standard bases

Proposition 1.2. *For $\mathcal{I}$ a set and $i \in \mathcal{I}$, define $e_i \in \mathbb{F}^{\mathcal{I}}$ by*

$$e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $j \in \mathcal{I}$.

If $\mathcal{I}$ is finite then $(e_i)_{i \in \mathcal{I}}$ is a basis, called the standard basis, *of* $\mathbb{F}^{\mathcal{I}}$.

In particular, $\dim \mathbb{F}^{\mathcal{I}} = |\mathcal{I}|$.

1.3.2 Useful facts

Proposition 1.3 (Algebra 1B, Chapter 3, Theorem 6(b))**.** *Any linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis.*

Lemma 1.4 (Algebra 1B, Chapter 3, Theorem 5)**.** *Let V be a finite-dimensional vector space and $U \leq V$. Then*

$$\dim U \leq \dim V$$

with equality if and only if $U = V$.

1.4 Linear maps

Definitions. A map $\phi : V \to W$ of vector spaces over $\mathbb{F}$ is a *linear map* (or, in older books, *linear transformation*) if

$$\begin{aligned} \phi(v + w) &= \phi(v) + \phi(w) \\ \phi(\lambda v) &= \lambda \phi(v), \end{aligned}$$

for all $v, w \in V$, $\lambda \in \mathbb{F}$.

The *kernel* of ϕ is $\ker \phi := \{v \in V \mid \phi(v) = 0\} \leq V$.

The *image* of ϕ is $\operatorname{im} \phi := \{\phi(v) \mid v \in V\} \leq W$.

Definition. A linear map $\phi : V \to W$ is a *(linear) isomorphism* if there is a linear map $\psi : W \to V$ such that

$$\psi \circ \phi = \mathrm{id}_V, \qquad \phi \circ \psi = \mathrm{id}_W .$$

If there is an isomorphism $V \to W$, say that V and W are isomorphic and write $V \cong W$.

Lemma 1.5. *$\phi : V \to W$ is an isomorphism if and only if ϕ is a linear bijection (and then $\psi = \phi^{-1}$).*

1.4.1 Vector spaces of linear maps

Notation. For vector spaces V, W over $\mathbb{F}$, denote by $L_{\mathbb{F}}(V, W)$ (or simply $L(V, W)$) the set $\{\phi : V \to W \mid \phi \text{ is linear}\}$ of linear maps from V to W.

Theorem 1.6 (Linearity is a linear condition)**.** *$L(V, W)$ is a vector space under pointwise addition and scalar multiplication. Otherwise said, $L(V, W) \leq W^V$.*

1.4.2 Linear maps and matrices

Definition. Let V, W be finite-dimensional vector spaces over $\mathbb{F}$ with bases $\mathcal{B} : v_1, \dots, v_n$ and $\mathcal{B}' : w_1, \dots, w_m$ respectively. Let $\phi \in L(V, W)$. The *matrix of ϕ with respect to $\mathcal{B}, \mathcal{B}'$* is the matrix $A = (A_{ij}) \in M_{m \times n}(\mathbb{F})$ defined by:

$$\phi(v_j) = \sum_{i=1}^{m} A_{ij} w_i, \tag{1.2}$$

for all $1 \leq j \leq n$.

In the special case where $V = W$ *and* $\mathcal{B} = \mathcal{B}'$, we call A the *matrix of ϕ with respect to $\mathcal{B}$.*

1.4.3 Extension by linearity

Proposition 1.7 (Extension by linearity)**.** *Let V, W be vector spaces over $\mathbb{F}$. Let $v_1, \dots, v_n$ be a basis of V and $w_1, \dots, w_n$* any *vectors in W.*

Then there is a unique *$\phi \in L(V, W)$ such that*

$$\phi(v_i) = w_i, \qquad 1 \leq i \leq n. \tag{1.3}$$

1.4.4 The rank-nullity theorem

Theorem 1.8 (Rank-nullity)**.** *Let $\phi : V \to W$ be linear with V finite-dimensional. Then*

$$\dim \operatorname{im} \phi + \dim \ker \phi = \dim V.$$

Proposition 1.9. *Let $\phi : V \to W$ be linear with V, W finite-dimensional vector spaces of the same dimension: $\dim V = \dim W$.*

Then the following are equivalent:

1. *ϕ is injective.*
2. *ϕ is surjective.*
3. *ϕ is an isomorphism.*

Chapter 2

Sums and quotients

Convention. In this chapter, all vector spaces are over the same field $\mathbb{F}$ unless we say otherwise.

2.1 Sums of subspaces

Definition. Let $V_1, \ldots, V_k \leq V$. The *sum* $V_1 + \cdots + V_k$ is the set

$$V_1 + \cdots + V_k := \{v_1 + \cdots + v_k \mid v_i \in V_i, 1 \leq i \leq k\}.$$

Proposition 2.1. *Let $V_1, \ldots, V_k \leq V$. Then*

(1) $V_1 + \cdots + V_k \leq V$.

(2) If $W \leq V$ and $V_1, \ldots, V_k \leq W$ then $V_1, \ldots, V_k \leq V_1 + \cdots + V_k \leq W$.

2.2 Direct sums

Definition. Let $V_1, \ldots, V_k \leq V$. The sum $V_1 + \cdots + V_k$ is *direct* if each $v \in V_1 + \cdots + V_k$ can be written

$$v = v_1 + \cdots + v_k$$

in only one way, that is, for unique $v_i \in V_i$, $1 \leq i \leq k$.

In this case, we write $V_1 \oplus \cdots \oplus V_k$ instead of $V_1 + \cdots + V_k$.

Proposition 2.2. *Let $V_1, V_2 \leq V$. Then $V_1 + V_2$ is direct if and only if $V_1 \cap V_2 = \{0\}$.*

Definition. Let $V_1, V_2 \leq V$. V is the *(internal) direct sum of* V_1 *and* V_2 if $V = V_1 \oplus V_2$.

In this case, say that V_2 is a *complement* of V_1 (and V_1 is a complement of V_2).

Proposition 2.3. *Let $V_1, \ldots, V_k \leq V$, $k \geq 2$. Then the sum $V_1 + \cdots + V_k$ is direct if and only if for each $1 \leq i \leq k$, $V_i \cap (\sum_{j \neq i} V_j) = \{0\}$.*

2.2.1 Direct sums and projections

Definition. Let V be a vector space. A linear map $\pi : V \to V$ is a *projection* if $\pi \circ \pi = \pi$.

Proposition 2.4. *Let $V_1, V_2 \leq V$ with $V = V_1 \oplus V_2$. Then there are projections $\pi_1, \pi_2 : V \to V$ such that:*

(a) $\operatorname{im} \pi_i = V_i$, $i = 1, 2$;

(b) $\ker \pi_1 = V_2$, $\ker \pi_2 = V_1$*;*

(c) $v = \pi_1(v) + \pi_2(v)$*, for all* $v \in V$*. Otherwise said,* $\mathrm{id}_V = \pi_1 + \pi_2$*.*

Proposition 2.5. *Let* $V = V_1 \oplus V_2$ *with* V *finite-dimensional. Then*

$$\dim V = \dim V_1 + \dim V_2.$$

2.2.2 Induction from two summands

Lemma 2.6. *Let* $V_1, \dots, V_k \leq V$*. Then* $V_1 + \dots + V_k$ *is direct if and only if* $V_1 + \dots + V_{k-1}$ *is direct and* $(V_1 + \dots + V_{k-1}) + V_k$ *(two summands) is direct.*

Corollary 2.7. *Let* $V_1, \dots, V_k \leq V$ *be subspaces of a finite-dimensional vector space* V *with* $V_1 + \dots + V_k$ *direct. Then*

$$\dim V_1 \oplus \dots \oplus V_k = \dim V_1 + \dots + \dim V_k.$$

2.2.3 Direct sums and bases

Proposition 2.8. *Let* $V_1, V_2 \leq V$ *be finite-dimensional subspaces with bases* $\mathcal{B}_1 : v_1, \dots, v_k$ *and* $\mathcal{B}_2 : w_1, \dots, w_l$*. Then* $V_1 + V_2$ *is direct if and only if the* concatenation[1] $\mathcal{B}_1\mathcal{B}_2 : v_1, \dots, v_k, w_1, \dots, w_l$ *is a basis of* $V_1 + V_2$*.*

Corollary 2.9. *Let* $V_1, \dots, V_k \leq V$ *be finite-dimensional subspaces with* $\mathcal{B}_i$ *a basis of* V_i*,* $1 \leq i \leq k$*. Then* $V_1 + \dots + V_k$ *is direct if and only if the concatenation* $\mathcal{B}_1 \dots \mathcal{B}_k$ *is a basis for* $V_1 + \dots + V_k$*.*

2.2.4 Complements

Proposition 2.10 (Complements exist)**.** *Let* $U \leq V$*, a finite-dimensional vector space. Then there is a complement to* U*.*

Proposition 2.11 (Extension of linear maps)**.** *Let* V, W *be vector spaces with* V *finite-dimensional. Let* $U \leq V$ *be a subspace and* $\phi : U \to W$ *a linear map. Then there is a linear map* $\Phi : V \to W$ *such that the restriction*[2] *of* Φ *to* U *is* ϕ*:* $\Phi_{|U} = \phi$*. Otherwise said: for all* $u \in U$

$$\Phi(u) = \phi(u).$$

2.3 Quotients

Definition. Let $U \leq V$. Say that $v, w \in V$ are *congruent modulo* U if $v - w \in U$. In this case, we write $v \equiv w \mod U$.

Lemma 2.12. *Congruence modulo* U *is an equivalence relation.*

Definition. For $v \in V$, $U \leq V$, the set $v + U := \{v + u \mid u \in U\} \subseteq V$ is called a *coset of* U and v is called a *coset representative* of $v + U$.

Definition. Let $U \leq V$. The *quotient space* V/U *of* V *by* U is the set V/U, pronounced "V mod U", of cosets of U:

$$V/U := \{v + U \mid v \in V\}.$$

This is a subset of the *power set*[3] $\mathcal{P}(V)$ of V.

[1] The concatenation of two lists is simply the list obtained by adjoining all entries in the second list to the first.

[2] Recall that if $f : X \to Y$ is a map of sets and $A \subseteq X$ then the *restriction* of f to A is the map $f_{|A} : A \to Y$ given by $f_{|A}(a) = f(a)$, for all $a \in A$.

[3] Recall from Algebra 1A that the power set of a set A is the set of all subsets of A.

The *quotient map* $q : V \to V/U$ is defined by

$$q(v) = v + U.$$

Theorem 2.13. *Let $U \leq V$. Then, for $v, w \in V$, $\lambda \in \mathbb{F}$,*

$$\begin{aligned}(v + U) + (w + U) &:= (v + w) + U \\ \lambda(v + U) &:= (\lambda v) + U\end{aligned}$$

give well-defined operations of addition and scalar multiplication on V/U with respect to which V/U is a vector space and $q : V \to V/U$ is a linear map.

Moreover, $\ker q = U$ *and* $\operatorname{im} q = V/U$*.*

Corollary 2.14. *Let $U \leq V$. If V is finite-dimensional then so is V/U and*

$$\dim V/U = \dim V - \dim U.$$

Theorem 2.15 (First Isomorphism Theorem)**.** *Let $\phi : V \to W$ be a linear map of vector spaces.*

Define $\bar{\phi} : V/\ker\phi \to \operatorname{im}\phi$ by

$$\bar{\phi}(q(v)) = \phi(v),$$

where $q : V \to V/\ker\phi$ is the quotient map.

Then $\bar{\phi}$ is a well-defined linear isomorphism.

In particular, $V/\ker\phi \cong \operatorname{im}\phi$.

Chapter 3

Inner product spaces

Convention. In this chapter, we take the field $\mathbb{F}$ of scalars to be either $\mathbb{R}$ or $\mathbb{C}$.

3.1 Inner products

3.1.1 Definition and examples

Definition. Let V be a vector space of $\mathbb{F}$ (which is $\mathbb{R}$ or $\mathbb{C}$).

An *inner product* on V is a map $V \times V \to \mathbb{F} : (v, w) \mapsto \langle v, w \rangle$ which is:

(1) *(conjugate) symmetric*: $\langle w, v \rangle = \overline{\langle v, w \rangle}$, for all $v, w \in V$. In particular $\langle v, v \rangle = \overline{\langle v, v \rangle}$ and so is real.

(2) *linear in the second slot*:

$$\begin{aligned} \langle u, v + w \rangle &= \langle u, v \rangle + \langle u, w \rangle \\ \langle u, \lambda v \rangle &= \lambda \langle u, v \rangle, \end{aligned}$$

for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$.

(3) *positive definite:* For all $v \in V$, $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

A vector space with an inner product is called an *inner product space.*

Definition. A map $\phi : V \to W$ of complex vector spaces is *conjugate linear* (or *anti-linear*) if

$$\begin{aligned} \phi(v + w) &= \phi(v) + \phi(w) \\ \phi(\lambda v) &= \bar{\lambda} \phi(v), \end{aligned}$$

for all $v, w \in V$ and $\lambda \in \mathbb{F}$.

Definition. Let V be an inner product space.

1. The *norm* of $v \in V$ is $\|v\| := \sqrt{\langle v, v \rangle} \geq 0$.
2. Say $v, w \in V$ are *orthogonal* or *perpendicular* if $\langle v, w \rangle = 0$. In this case, we write $v \perp w$.

3.1.2 Cauchy–Schwarz inequality

Theorem 3.1 (Cauchy–Schwarz inequality)**.** *Let V be an inner product space. For $v, w \in V$,*

$$|\langle v, w \rangle| \leq \|v\| \|w\| \tag{3.1}$$

with equality if and only if v, w are linearly dependent, that is, either $v = 0$ or $w = \lambda v$, for some $\lambda \in \mathbb{F}$.

Proposition 3.2. *Let V be an inner product space and $v, w \in V$.*

1. ***Pythagoras Theorem:*** *If $v \perp w$ then*
$$\|v + w\|^2 = \|v\|^2 + \|w\|^2. \tag{3.2}$$
2. ***Triangle inequality:*** *$\|v + w\| \leq \|v\| + \|w\|$ with equality if and only if $v = 0$ or $w = \lambda v$ with $\lambda \geq 0$.*
3. ***Parallelogram identity:*** *$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2)$.*

3.2 Orthogonality

3.2.1 Orthonormal bases

Definition. A list of vectors $u_1, \dots, u_k$ in an inner product space V is *orthonormal* if, for all $1 \leq i, j \leq k$,
$$\langle u_i, u_j \rangle = \delta_{ij} := \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$
If $u_1, \dots, u_k$ is also a basis, we call it an *orthonormal basis.*

Lemma 3.3. *Let V be an inner product space with orthonormal basis $u_1, \dots, u_n$ and let $v \in V$. Then*
$$v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i.$$

Lemma 3.4. *Any orthonormal list of vectors $u_1, \dots, u_k$ is linearly independent.*

Proposition 3.5. *Let $u_1, \dots, u_n$ be an orthonormal basis of an inner product space V.*

Let $v = x_1 u_1 + \cdots + x_n u_n$ and $w = y_1 u_1 + \cdots + y_n u_n$. Then
$$\langle v, w \rangle = \sum_{i=1}^{n} \bar{x}_i y_i = x \cdot y.$$
Thus the inner product of two vectors is the dot product of their coordinates with respect to an orthonormal basis.

Proposition 3.6. *Let $u_1, \dots, u_n$ be an orthonormal basis of an inner product space V and $v, w \in V$. Then:*

(1) ***Parseval's identity:*** *$\langle v, w \rangle = \sum_{i=1}^{n} \langle v, u_i \rangle \langle u_i, w \rangle$.*
(2) ***Bessel's equality:*** *$\|v\|^2 = \sum_{i=1}^{n} |\langle v, u_i \rangle|^2$.*

Theorem 3.7 (Gram–Schmidt orthogonalisation)**.** *Let $v_1, \dots, v_m$ be linearly independent vectors in an inner product space V.*

Then there is an orthonormal list $u_1, \dots, u_m$ such that
$$\operatorname{span}\{u_1, \dots, u_k\} = \operatorname{span}\{v_1, \dots, v_k\},$$
for all $1 \leq k \leq m$, defined inductively by:
$$u_k := w_k / \|w_k\|$$
where,
$$w_1 := v_1$$
and, for $k > 1$,
$$w_k := v_k - \sum_{j=1}^{k-1} \langle u_j, v_k \rangle u_j = v_k - \sum_{j=1}^{k-1} \frac{\langle w_j, v_k \rangle}{\|w_j\|^2} w_j.$$

Corollary 3.8. *Any finite-dimensional inner product space V has an orthonormal basis.*

Definition. A matrix $Q \in M_{n\times n}(\mathbb{R})$ is *orthogonal* if

$$Q^T Q = \mathrm{I}_n,$$

or, equivalently, Q has orthonormal columns with respect to the dot product. Here I_n is the $n \times n$ identity matrix.

Theorem 3.9 (QR decomposition)**.** *Let $A \in M_{n\times n}(\mathbb{R})$ be an invertible matrix. Then we can write*

$$A = QR,$$

where Q is orthogonal and R is upper triangular ($R_{ij} = 0$ if $i > j$) with positive entries on the diagonal.

3.2.2 Orthogonal complements and orthogonal projection

Definition. Let V be an inner product space and $U \leq V$. The *orthogonal complement $U^\perp$ of U (in V)* is given by

$$U^\perp := \{v \in V \mid \langle u, v\rangle = 0, \text{ for all } u \in U\}.$$

Proposition 3.10. *Let V be an inner product space and $U \leq V$. Then*

(1) $U^\perp \leq V$;

(2) $U \cap U^\perp = \{0\}$;

(3) $U \leq (U^\perp)^\perp$.

Theorem 3.11. *Let U be a finite-dimensional subspace of an inner product space V. Then V is an internal direct sum:*

$$V = U \oplus U^\perp.$$

Corollary 3.12. *Let V be a finite-dimensional inner product space and $U \leq V$. Then*

(1) $\dim U^\perp = \dim V - \dim U$.

(2) $U = (U^\perp)^\perp$.

Definition. Let V be an inner product space and $U \leq V$ such that $V = U \oplus U^\perp$. The projection $\pi_U : V \to V$ with image U and kernel $U^\perp$ is called the *orthogonal projection onto U*.

Lemma 3.13. *Let V be an inner product space and $U \leq V$ a finite-dimensional subspace with orthonormal basis $u_1, \dots, u_k$ then, for all $v \in V$,*

$$\pi_U(v) = \sum_{i=1}^{k} \langle u_i, v\rangle u_i.$$

Theorem 3.14. *Let V be an inner product space and $U \leq V$ such that $V = U \oplus U^\perp$.*

For $v \in V$, $\pi_U(v)$ is the nearest point of U to v: for all $u \in U$,

$$\|v - \pi_U(v)\| \leq \|v - u\|.$$

Chapter 4

Linear operators on inner product spaces

Convention. In this chapter, we once again take the field $\mathbb{F}$ of scalars to be either $\mathbb{R}$ or $\mathbb{C}$.

4.1 Linear operators and their adjoints

4.1.1 Linear operators and matrices

Definition. Let V be a vector space over $\mathbb{F}$. A *linear operator on* V is a linear map $\phi : V \to V$.

The vector space of linear operators on V is denoted $L(V)$ (instead of $L(V, V)$).

4.1.2 Adjoints

Lemma 4.1 (Nondegeneracy Lemma)**.** *Let V be an inner product space and $v \in V$. Then $\langle v, w\rangle = 0$, for all $w \in V$, if and only if $v = 0$.*

Definition. Let V be an inner product space and $\phi \in L(V)$. An *adjoint to* ϕ is a linear operator $\phi^* \in L(V)$ such that, for all $v, w \in V$, we have

$$\langle \phi^*(v), w\rangle = \langle v, \phi(w)\rangle$$

or, equivalently, by conjugate symmetry,

$$\langle w, \phi^*(v)\rangle = \langle \phi(w), v\rangle.$$

Proposition 4.2. *Let V be an inner product space and suppose $\phi, \psi \in L(V)$ have adjoints. Then $\phi \circ \psi$; $\phi + \lambda\psi$, $\lambda \in \mathbb{F}$; ϕ^* and id_V all have adjoints given by:*

(1) $(\phi \circ \psi)^* = \psi^* \circ \phi^*$ (note the change of order here!).

(2) $(\phi + \lambda\psi)^* = \phi^* + \bar{\lambda}\psi^*$.

(3) $(\phi^*)^* = \phi$.

(4) $\mathrm{id}_V^* = \mathrm{id}_V$.

Proposition 4.3. *Let V be a* finite-dimensional *inner product space and $\phi \in L(V)$ a linear operator. Then*

(1) ϕ has a unique adjoint ϕ^*.

(2) *Let $u_1, \ldots, u_n$ be an orthonormal basis of V with respect to which ϕ has matrix A. Then ϕ^* has matrix $A^\dagger := \overline{A}^T$ (which is A^T when $\mathbb{F} = \mathbb{R}$).*

Definitions.

1. Let V be an inner product space and $\phi \in L(V)$.
   Say that ϕ is *self-adjoint* if $\phi^* = \phi$, or, equivalently, for all $v, w \in V$,
   $$\langle \phi(v), w \rangle = \langle v, \phi(w) \rangle.$$
   Say ϕ is *skew-adjoint* if $\phi^* = -\phi$ or, equivalently, for all $v, w \in V$,
   $$\langle \phi(v), w \rangle = -\langle v, \phi(w) \rangle.$$
2. Let $A \in M_{n \times n}(\mathbb{F})$.
   (a) If $\mathbb{F} = \mathbb{C}$, say that A is *Hermitian* if $A^\dagger = A$ and *skew-Hermitian* if $A^\dagger = -A$.
   (b) If $\mathbb{F} = \mathbb{R}$, say that A is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

4.1.3 Linear isometries

Definition. Let V, W be inner product spaces with inner products $\langle \, , \, \rangle_V$ and $\langle \, , \, \rangle_W$ respectively. A linear map $\phi : V \to W$ is a *linear isometry* if, for all $v_1, v_2 \in V$,

$$\langle \phi(v_1), \phi(v_2) \rangle_W = \langle v_1, v_2 \rangle_V.$$

Proposition 4.4. *Let V be a finite-dimensional inner product space and $\phi \in L(V)$. Then ϕ is a linear isometry if and only if ϕ is an isomorphism with $\phi^{-1} = \phi^*$ (equivalently, $\phi^* \circ \phi = \mathrm{id}_V = \phi \circ \phi^*$).*

Definitions. Let V be an inner product space over $\mathbb{F}$ and $\phi \in L(V)$. If ϕ is an isomorphism with $\phi^{-1} = \phi^*$, then say ϕ is:

- an *orthogonal transformation* if $\mathbb{F} = \mathbb{R}$;
- a *unitary transformation* if $\mathbb{F} = \mathbb{C}$.

The set of all orthogonal, resp. unitary transformations is denoted $\mathrm{O}(V)$, resp. $\mathrm{U}(V)$.

Let $A \in M_{n \times n}(\mathbb{F})$.

- A is *orthogonal* if $\mathbb{F} = \mathbb{R}$ and $A^T A = \mathrm{I}$;
- A is *unitary* if $\mathbb{F} = \mathbb{C}$ and $A^\dagger A = \mathrm{I}$.

The set of all $n \times n$ orthogonal, resp. unitary matrices is denoted $\mathrm{O}(n)$, resp. $\mathrm{U}(n)$.

Definitions. Let V be a vector space. The *general linear group of V*, denoted $\mathrm{GL}(V)$, is:

$$\mathrm{GL}(V) := \{\phi \in L(V) \mid \phi \text{ is an isomorphism}\}.$$

Similarly, the *general linear group of $n \times n$ matrices over $\mathbb{F}$*, denoted $\mathrm{GL}(n, \mathbb{F})$, is:

$$\mathrm{GL}(n, \mathbb{F}) := \{A \in M_{n \times n}(\mathbb{F}) \mid A \text{ is invertible}\}.$$

Proposition 4.5.

(1) *Let V be a vector space. Then $\mathrm{GL}(V)$ is a group under composition: $\psi\phi := \psi \circ \phi$.*

(2) *If V is an inner product space, then $\mathrm{O}(V)$, resp. $\mathrm{U}(V)$, is a subgroup of $\mathrm{GL}(V)$, when $\mathbb{F} = \mathbb{R}$, resp. $\mathbb{C}$.*

Theorem 4.6 (Classification of rigid motions)**.** *Let V be a* real *inner product space. Recall that the* distance *between $v, w \in V$ is $d(v, w) := \|v - w\|$.*

A map $f : V \to V$ (not necessarily linear) is distance-preserving *or a* rigid motion *if* $d(f(v), f(w)) = d(v, w)$*, for all* $v, w \in V$*.*

f is distance-preserving if and only if there is a $v_0 \in V$ and $\phi \in L(V)$ a linear isometry such that

$$f(v) = \phi(v) + v_0, \tag{4.1}$$

for all $v \in V$.

4.2 The spectral theorem

4.2.1 Eigenvalues and eigenvectors

Definitions. Let V be a vector space over $\mathbb{F}$ and $\phi \in L(V)$.

An *eigenvalue* of ϕ is a scalar $\lambda \in \mathbb{F}$ such that there is a *non-zero* $v \in V$ with

$$\phi(v) = \lambda v.$$

Such a vector v is called an *eigenvector of ϕ with eigenvalue λ*.

The λ-*eigenspace $E_\phi(\lambda)$ of ϕ* is given by

$$E_\phi(\lambda) := \ker(\phi - \lambda \operatorname{id}_V) \leq V.$$

Definition. Let V be a finite-dimensional vector space over $\mathbb{F}$ and $\phi \in L(V)$.

The *characteristic polynomial Δ_ϕ of ϕ* is given by

$$\Delta_\phi(\lambda) := \det(\phi - \lambda \operatorname{id}_V) = \det(A - \lambda \mathrm{I}),$$

where A is the matrix of ϕ with respect to some (any!) basis of V.

Lemma 4.7. *A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of ϕ if and only if $\Delta_\phi(\lambda) = 0$, that is, λ is a root of Δ_ϕ.*

Corollary 4.8. *Let ϕ be a linear operator on a finite-dimensional* complex *vector space V. Then ϕ has an eigenvalue.*

4.2.2 Invariant subspaces and adjoints

Definition. Let V be a vector space and $\phi \in L(V)$.

A subspace $U \leq V$ is ϕ-*invariant* if $\phi(U) \leq U$, that is, $\phi(u) \in U$, for all $u \in U$.

Lemma 4.9. *Let $\phi, \psi \in L(V)$ and suppose that*

- *$\psi \circ \phi = \phi \circ \psi$ (say that ϕ and ψ* commute*).*
- *$U = E_\phi(\lambda)$ is an eigenspace of ϕ.*

Then U is ψ-invariant.

Lemma 4.10. *Let V be a finite-dimensional*[1] *inner product space and $\phi \in L(V)$.*

Let $U \leq V$ be a ϕ-invariant subspace. Then $U^\perp$ is ϕ^*-invariant.

Definition. Let V be a finite-dimensional inner product space. A linear operator $\phi \in L(V)$ is *normal* if it commutes with its adjoint: $\phi^* \circ \phi = \phi \circ \phi^*$.

Proposition 4.11. *Let V be a finite-dimensional inner product space and $\phi \in L(V)$.*

Suppose that:

[1] We only need this hypothesis to ensure that ϕ^* exists.

- *ϕ is normal;*
- *$U \leq V$ is an eigenspace of ϕ.*

Then $U^\perp$ is ϕ-invariant.

4.2.3 The spectral theorem for normal operators

Definition. Let V be a finite-dimensional vector space. A linear operator $\phi \in L(V)$ is *diagonalisable* if V has a basis of eigenvectors of ϕ.

Definition. Let V be a finite-dimensional inner product space. A linear operator $\phi \in L(V)$ is *orthogonally diagonalisable* if V has an *orthonormal* basis of eigenvectors.

Proposition 4.12. *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and $\phi \in L(V)$ an orthogonally diagonalisable linear operator. Then:*

(1) If $\mathbb{F} = \mathbb{C}$, ϕ is normal.

(2) If $\mathbb{F} = \mathbb{R}$, ϕ is self-adjoint.

Theorem 4.13 (Spectral theorem for normal operators)**.** *Let V be a finite-dimensional* complex *inner product space and $\phi \in L(V)$ a linear operator. Then ϕ is orthogonally diagonalisable if and only if ϕ is normal.*

4.2.4 The spectral theorem for real self-adjoint operators

Lemma 4.14. *Let V be an inner product space*[2] *and $\phi \in L(V)$ be self-adjoint.*

(1) Any eigenvalue of ϕ is real.

(2) If $v, w \in V$ are eigenvectors of ϕ with eigenvalues $\lambda \neq \mu$ then $v \perp w$.

Proposition 4.15. *A self-adjoint operator ϕ on a real, finite-dimensional inner product space V has an eigenvalue.*

Theorem 4.16 (Spectral theorem for real self-adjoint operators)**.** *Let V be a real, finite-dimensional inner product space and $\phi \in L(V)$ a linear operator. Then ϕ is orthogonally diagonalisable if and only if ϕ is self-adjoint.*

4.2.5 The spectral theorem for symmetric and Hermitian matrices

Theorem 4.17 (Spectral theorem for symmetric/hermitian matrices)**.**

(1) Let $A \in M_{n \times n}(\mathbb{R})$ be symmetric. Then there is an orthogonal matrix $P \in \mathrm{O}(n)$ such that $P^{-1}AP$ is diagonal.

(2) Let $A \in M_{n \times n}(\mathbb{C})$ be Hermitian. Then there is an unitary matrix $P \in \mathrm{U}(n)$ such that $P^{-1}AP$ is diagonal.

4.2.6 Singular value decomposition

Lemma 4.18. *Let V be a finite-dimensional inner product space and $\phi \in L(V)$. Then:*

(1) All eigenvalues of $\phi^* \circ \phi$ are non-negative.

(2) $\ker(\phi^* \circ \phi) = \ker \phi$.

[2] We do *not* demand that V be finite-dimensional.

Definition. Let V be a finite-dimensional inner product space and $\phi \in L(V)$. The *singular values of* ϕ are $\sigma_1, \dots, \sigma_n$ where $\sigma_i = \sqrt{\mu_i} \geq 0$ and $\mu_1, \dots, \mu_n$ are the eigenvalues of $\phi^* \circ \phi$ listed with multiplicity (thus each distinct μ appears $\dim E_{\phi^* \circ \phi}(\mu)$ times).

Theorem 4.19 (Singular value decomposition)**.** *Let V be a finite-dimensional inner product space and $\phi \in L(V)$ a linear operator with singular values $\sigma_1, \dots, \sigma_n$.*

Then there are orthonormal bases $u_1, \dots, u_n$ and $w_1, \dots, w_n$ of V such that

$$\phi(v) = \sum_{i=1}^{n} \sigma_i \langle u_i, v \rangle w_i, \tag{4.2}$$

for all $v \in V$.

Chapter 5

Duality

5.1 Dual spaces

Definition. Let V be a vector space over $\mathbb{F}$. The *dual space V^* of V* is

$$V^* := L(V, \mathbb{F}) = \{\alpha : V \to \mathbb{F} \mid \alpha \text{ is linear}\}.$$

Elements of V^* are called *linear functionals* or (less often) *linear forms*.

Proposition 5.1. *Let V be a finite-dimensional vector space with basis $v_1, \ldots, v_n$.*

Define $v_1^*, \ldots, v_n^* \in V^*$ by setting

$$v_i^*(v_j) = \delta_{ij} \in \mathbb{F}$$

and extending by linearity (thus applying Proposition 1.7).

Then $v_1^*, \ldots, v_n^*$ is a basis of V^* called the dual basis *to $v_1, \ldots, v_n$.*

Corollary 5.2. *If V is finite-dimensional then $\dim V = \dim V^*$.*

Theorem 5.3 (Riesz Representation Theorem)**.** *Let V be a finite-dimensional inner product space and $\alpha \in V^*$. Then there is a unique $w \in V$ such that*

$$\alpha(v) = \langle w, v \rangle,$$

for all $v \in V$. Thus $\alpha = \alpha_w$.

Theorem 5.4 (Sufficiency principle)**.** *Let V be a vector space and $v \in V$. Then $\alpha(v) = 0$, for all $\alpha \in V^*$, if and only if $v = 0$.*

Proposition 5.5. *Let V be a finite-dimensional vector space and $\alpha_1, \ldots, \alpha_n$ a basis of V^*. Then there is a basis $v_1, \ldots, v_n$ of V such that*

$$\alpha_i(v_j) = \delta_{ij}.$$

Thus $\alpha_i = v_i^*$, for $1 \leq i \leq n$.

Theorem 5.6. *If V is a finite-dimensional vector space then $\mathrm{ev} : V \to V^{**}$ is an isomorphism.*

5.2 Solution sets and annihilators

Definition. Let $E \leq V^*$. The *solution set of E* is

$$\mathrm{sol}\, E := \{v \in V \mid \alpha(v) = 0, \text{ for all } \alpha \in E\} = \bigcap_{\alpha \in E} \ker \alpha \leq V.$$

Proposition 5.7. *If V is finite-dimensional and $E \le V^*$ then*

$$\dim \operatorname{sol} E = \dim V - \dim E.$$

We say that E and $\operatorname{sol} E$ *have* complementary dimension*.*

Corollary 5.8. *Let V have dimension n and suppose that $\alpha_1, \dots, \alpha_n \in V^*$ are such that*

$$\bigcap_{i=1}^{n} \ker \alpha_i = \{0\}.$$

Then $\alpha_1, \dots, \alpha_n$ is a basis of V^*.

Proposition 5.9. *Let $E, F \le V^*$. Then*

(1) If $E \le F$ then $\operatorname{sol} F \le \operatorname{sol} E$*.*

(2) sol *swaps sums and intersections:*

$$\operatorname{sol}(E+F) = (\operatorname{sol} E) \cap (\operatorname{sol} F)$$
$$(\operatorname{sol} E) + (\operatorname{sol} F) \le \operatorname{sol}(E \cap F)$$

with equality if V is finite-dimensional.

Definition. Let $U \le V$. The *annihilator of U, denoted* $\operatorname{ann} U$ *or* U°, is given by:

$$\operatorname{ann} U := \{\alpha \in V^* \mid \alpha_{|U} = 0\} = \{\alpha \in V^* \mid \alpha(u) = 0, \text{ for all } u \in U\}.$$

Proposition 5.10. *Let V be finite-dimensional and $U \le V$. Then*

$$\dim \operatorname{ann} U = \dim V - \dim U.$$

Proposition 5.11. *Let $U, W \le V$. Then*

(1) If $U \le W$ then $\operatorname{ann} W \le \operatorname{ann} U$*.*

(2)
$$\operatorname{ann}(U+W) = (\operatorname{ann} U) \cap (\operatorname{ann} W)$$
$$(\operatorname{ann} U) + (\operatorname{ann} W) \le \operatorname{ann}(U \cap W)$$

with equality if V is finite-dimensional.

Lemma 5.12. *Let $U \le V$ and $E \le V^*$ then $U \le \operatorname{sol} E$ if and only if $E \le \operatorname{ann} U$.*

Theorem 5.13. *Let $U \le V$ and $E \le V^*$. Then*

$$U \le \operatorname{sol}(\operatorname{ann} U)$$
$$E \le \operatorname{ann}(\operatorname{sol} E),$$

with equality if V is finite-dimensional.

5.3 Transposes

Definition. Let $\phi \in L(V, W)$ be a linear map of vector spaces. The *transpose ϕ^T of ϕ* is the map $\phi^T : W^* \to V^*$ given by

$$\phi^T(\alpha) := \alpha \circ \phi,$$

for all $\alpha \in W^*$.

Lemma 5.14. $\phi^T : W^* \to V^*$ *is also a linear map.*

Proposition 5.15. *Let V, W be finite-dimensional vector spaces and $\phi \in L(V, W)$ with matrix $A \in M_{m \times n}(\mathbb{F})$ with respect to bases $v_1, \dots, v_n$ and $w_1, \dots, w_m$ of V and W.*

Then ϕ^T has matrix A^T with respect to the dual bases $w_1^*, \dots, w_m^*$ and $v_1^*, \dots, v_n^*$ of W^* and V^*.

Theorem 5.16. *Let $\phi \in L(V,W)$ be a linear map of vector spaces. Then*

(1)
$$\ker\phi = \operatorname{sol}(\operatorname{im}\phi^T)$$
$$\operatorname{im}\phi \le \operatorname{sol}(\ker\phi^T)$$

with equality if V,W are finite-dimensional.

(2)
$$\ker\phi^T = \operatorname{ann}(\operatorname{im}\phi)$$
$$\operatorname{im}\phi^T \le \operatorname{ann}(\ker\phi)$$

with equality if V,W are finite-dimensional.

Corollary 5.17. *Let $\phi \in L(V,W)$ be a linear map of finite-dimensional vector spaces. Then*

$$\operatorname{rank}\phi = \operatorname{rank}\phi^T.$$

Proposition 5.18. *Let $\phi \in L(V,W)$ be a linear map of finite-dimensional vector spaces. Then*

(1) ϕ injects if and only if ϕ^T surjects.

(2) ϕ^T injects if and only if ϕ surjects.

Chapter 6

Bilinearity

6.1 Bilinear maps

Definition. Let U, V, W be vector spaces over a field $\mathbb{F}$. A map $B : U \times V \to W$ is *bilinear* if it is linear in each slot separately:

$$\begin{aligned} B(\lambda u_1 + u_2, v) &= \lambda B(u_1, v) + B(u_2, v) \\ B(u, \lambda v_1 + v_2) &= \lambda B(u, v_1) + B(u, v_2), \end{aligned}$$

for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$ and $\lambda \in \mathbb{F}$.

A bilinear map $U \times V \to \mathbb{F}$ is called a *bilinear pairing.*

A bilinear map $V \times V \to \mathbb{F}$ is called a *bilinear form on V.*

Notation. We let $\operatorname{Bil}(U, V; W)$ denote the set of bilinear maps $U \times V \to W$.

6.2 Bilinear forms and quadratic forms

6.2.1 Bilinear forms and matrices

Definition. Let V be a vector space over $\mathbb{F}$ with basis $\mathcal{B} = v_1, \dots, v_n$ and let $B : V \times V \to \mathbb{F}$ be a bilinear form. The *matrix of B with respect to $\mathcal{B}$* is $A \in M_{n\times n}(\mathbb{F})$ given by

$$A_{ij} = B(v_i, v_j),$$

for $1 \leq i, j \leq n$.

Proposition 6.1. *Let $B : V \times V \to \mathbb{F}$ be a bilinear form with matrix A with respect to $\mathcal{B} = v_1, \dots, v_n$. Then B is completely determined by A: if $v = \sum_{i=1}^n x_i v_i$ and $w = \sum_{j=1}^n y_j v_j$ then*

$$B(v, w) = \sum_{i,j=1}^n x_i y_j A_{ij},$$

or, equivalently, for all $x, y \in \mathbb{F}^n$,

$$B(\phi_{\mathcal{B}}(x), \phi_{\mathcal{B}}(y)) = B_A(x, y) = \mathbf{x}^T A \mathbf{y}.$$

Proposition 6.2. *Let $B : V \times V \to \mathbb{F}$ be a bilinear form with matrices A and A' with respect to bases $\mathcal{B}$ and $\mathcal{B}'$ of V. Then*

$$A' = P^T A P$$

where P is the change of basis matrix[1] *from $\mathcal{B}$ to $\mathcal{B}'$: thus $\phi_P = \phi_{\mathcal{B}}^{-1} \circ \phi_{\mathcal{B}'}$.*

Definition. We say that matrices $A, B \in M_{n\times n}(\mathbb{F})$ are *congruent* if there is $P \in \mathrm{GL}(n, \mathbb{F})$ such that

$$B = P^T AP$$

6.2.2 Symmetric bilinear forms

Definition. A bilinear form $B : V \times V \to \mathbb{F}$ is *symmetric* if, for all $v, w \in V$,

$$B(v, w) = B(w, v)$$

Definitions. Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form.

The *radical* $\operatorname{rad} B$ *of* B is given by

$$\operatorname{rad} B := \{v \in V \mid B(v, w) = 0, \text{ for all } w \in V\}.$$

We shall shortly see that $\operatorname{rad} B \leq V$.

We say that B is *non-degenerate* if $\operatorname{rad} B = \{0\}$.

If V is finite-dimensional, the *rank* of B is $\dim V - \dim \operatorname{rad} B$ (so that B is non-degenerate if and only if $\operatorname{rank} B = \dim V$).

Lemma 6.3. *Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form on a finite-dimensional vector space V with matrix A with respect to some basis of V. Then*

$$\operatorname{rank} B = \operatorname{rank} A.$$

In particular, B is non-degenerate if and only if $\det A \neq 0$.

6.2.3 Quadratic forms

Convention. In this section, we work with a field $\mathbb{F}$ where $1 + 1 \neq 0$ so that $\frac{1}{2} = (1 + 1)^{-1}$ makes sense. This excludes, for example, the 2-element field $\mathbb{Z}_2$.

Definition. A *quadratic form* on a vector space V over $\mathbb{F}$ is a function $Q : V \to \mathbb{F}$ of the form

$$Q(v) = B(v, v),$$

for all $v \in V$, where $B : V \times V \to \mathbb{F}$ is a symmetric bilinear form.

Lemma 6.4. *Let $Q : V \to \mathbb{F}$ be a quadratic form with $Q(v) = B(v, v)$ for a symmetric bilinear form B. Then*

$$B(v, w) = \tfrac{1}{2}\big(Q(v + w) - Q(v) - Q(w)\big),$$

for all $v, w \in V$.

B is called the polarisation of Q.

6.2.4 Classification of symmetric bilinear and quadratic forms

Convention. We retain our assumption that $1 + 1 \neq 0$ in $\mathbb{F}$.

Theorem 6.5 (Diagonalisation Theorem)**.** *Let B be a symmetric bilinear form on a finite-dimensional vector space over $\mathbb{F}$. Then there is a basis $v_1, \dots, v_n$ of V with respect to which the matrix of B is diagonal:*

$$B(v_i, v_j) = 0,$$

for all $1 \leq i \neq j \leq n$. We call $v_1, \dots, v_n$ a diagonalising basis *for B.*

[1] See Definition 2 in Chapter 5 of Algebra 1B.

Definitions. Let Q be a quadratic form on a *real* vector space V.

Say that Q is *positive definite* if $Q(v) \geq 0$, for all $v \in V$, with equality if and only if $v = 0$.

Say that Q is *negative definite* if $-Q$ is positive definite.

If V is finite-dimensional, the *signature* of Q (or its polarisation B) is the pair (p, q) where

$$\begin{aligned} p &= \max\{\dim U \mid U \leq V \text{ with } Q_{|U} \text{ positive definite}\} \\ q &= \max\{\dim W \mid W \leq V \text{ with } Q_{|W} \text{ negative definite}\}. \end{aligned}$$

Theorem 6.6 (Sylvester's Law of Inertia)**.** *Let Q be a quadratic form of signature (p, q) on a finite-dimensional real vector space and B its polarisation. Then:*

- $p + q = \operatorname{rank} B$*;*
- *any* diagonal *matrix representing B has p positive entries on the diagonal and q negative entries.*

Theorem 6.7. *Let Q be a quadratic form with rank r polarisation on a finite-dimensional vector space over $\mathbb{F}$.*

(1) When $\mathbb{F} = \mathbb{C}$, there is a basis $v_1, \dots, v_n$ of V such that

$$Q\left(\sum_{i=1}^{n} x_i v_i\right) = x_1^2 + \cdots + x_r^2.$$

(2) When $\mathbb{F} = \mathbb{R}$ and Q has signature (p, q), there is a basis $v_1, \dots, v_n$ of V such that

$$Q\left(\sum_{i=1}^{n} x_i v_i\right) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2.$$