Chapter 1

Linear algebra: concepts and examples

1.1 Vector spaces

Definition. A vector space $V$ over a field $F$ is a set $V$ with two operations:

- **addition** $V \times V \rightarrow V : (v, w) \mapsto v + w$ with respect to which $V$ is an abelian group:
  - $v + w = w + v$, for all $v, w \in V$;
  - $u + (v + w) = (u + v) + w$, for all $u, v, w \in V$;
  - there is a zero element $0 \in V$ for which $v + 0 = v = 0 + v$, for all $v \in V$;
  - each element $v \in V$ has an additive inverse $-v \in V$ for which $v + (-v) = 0 = (-v) + v$.

- **scalar multiplication** $F \times V \rightarrow V : (\lambda, v) \mapsto \lambda v$ such that
  - $(\lambda + \mu)v = \lambda v + \mu v$, for all $v \in V$, $\lambda, \mu \in F$.
  - $\lambda(v + w) = \lambda v + \lambda w$, for all $v, w \in V$, $\lambda \in F$.
  - $(\lambda \mu)v = \lambda(\mu v)$, for all $v \in V$, $\lambda, \mu \in F$.
  - $1v = v$, for all $v \in V$.

We call the elements of $F$ *scalars* and those of $V$ *vectors*.

1.2 Subspaces

Definition. A vector (or linear) subspace of a vector space $V$ over $F$ is a non-empty subset $U \subseteq V$ which is closed under addition and scalar multiplication: whenever $u, u_1, u_2 \in U$ and $\lambda \in F$, then $u_1 + u_2 \in U$ and $\lambda u \in U$.

In this case, we write $U \leq V$.

Say that $U$ is trivial if $U = \{0\}$ and proper if $U \neq V$.

1.3 Bases

Definitions. Let $v_1, \ldots, v_n$ be a list of vectors in a vector space $V$. 

1. The span of \(v_1, \ldots, v_n\) is
   \[
   \text{span}\{v_1, \ldots, v_n\} := \{\lambda_1 v_1 + \cdots + \lambda_n v_n \mid \lambda_i \in \mathbb{F}, 1 \leq i \leq n\} \leq V.
   \]

2. \(v_1, \ldots, v_n\) span \(V\) (or are a spanning list for \(V\)) if \(\text{span}\{v_1, \ldots, v_n\} = V\).

3. \(v_1, \ldots, v_n\) are linearly independent if, whenever \(\lambda_1 v_1 + \cdots + \lambda_n v_n = 0\), then each \(\lambda_i = 0, 1 \leq i \leq n\), and linearly dependent otherwise.

4. \(v_1, \ldots, v_n\) is a basis for \(V\) if they are linearly independent and span \(V\).

**Definition.** A vector space is finite-dimensional if it admits a finite list of vectors as basis and infinite-dimensional otherwise.

If \(V\) is finite-dimensional, the dimension of \(V\), \(\dim V\), is the number of vectors in a (any) basis of \(V\).

**Proposition 1.1** (Algebra 1B, Section 2.4, Proposition 5). \(v_1, \ldots, v_n\) is a basis for \(V\) if and only if any \(v \in V\) can be written in the form
   \[
   v = \lambda_1 v_1 + \cdots + \lambda_n v_n
   \]
   (1.1)
   for unique \(\lambda_1, \ldots, \lambda_n \in \mathbb{F}\). In this case, \((\lambda_1, \ldots, \lambda_n)\) is called the coordinate vector of \(v\) with respect to \(v_1, \ldots, v_n\).

### 1.3.1 Standard bases

**Proposition 1.2.** For \(I\) a set and \(i \in I\), define \(e_i \in \mathbb{F}^I\) by
   \[
e_i(j) = \begin{cases} 
   1 & \text{if } i = j \\
   0 & \text{if } i \neq j
   \end{cases}
   \]
   for all \(j \in I\).

If \(I\) is finite then \((e_i)_{i \in I}\) is a basis, called the standard basis, of \(\mathbb{F}^I\).

In particular, \(\dim \mathbb{F}^I = |I|\).

### 1.3.2 Useful facts

**Proposition 1.3** (Algebra 1B, Chapter 3, Theorem 6(b)). Any linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis.

**Lemma 1.4** (Algebra 1B, Chapter 3, Theorem 5). Let \(V\) be a finite-dimensional vector space and \(U \leq V\). Then
   \[
   \dim U \leq \dim V
   \]
   with equality if and only if \(U = V\).

### 1.4 Linear maps

**Definitions.** A map \(\phi : V \to W\) of vector spaces over \(F\) is a linear map (or, in older books, linear transformation) if
   \[
   \phi(v + w) = \phi(v) + \phi(w)
   \]
   \[
   \phi(\lambda v) = \lambda \phi(v),
   \]
   for all \(v, w \in V, \lambda \in \mathbb{F}\).

The kernel of \(\phi\) is \(\ker \phi := \{v \in V \mid \phi(v) = 0\} \leq V\).

The image of \(\phi\) is \(\text{im} \phi := \{\phi(v) \mid v \in V\} \leq W\).
**Definition.** A linear map \( \phi : V \to W \) is a (linear) isomorphism if there is a linear map \( \psi : W \to V \) such that
\[
\psi \circ \phi = \text{id}_V, \quad \phi \circ \psi = \text{id}_W.
\]
If there is an isomorphism \( V \to W \), say that \( V \) and \( W \) are isomorphic and write \( V \cong W \).

**Lemma 1.5.** \( \phi : V \to W \) is an isomorphism if and only if \( \phi \) is a linear bijection (and then \( \psi = \phi^{-1} \)).

1.4.1 Vector spaces of linear maps

**Notation.** For vector spaces \( V, W \) over \( F \), denote by \( L_F(V, W) \) (or simply \( L(V, W) \)) the set \( \{ \phi : V \to W \mid \phi \) is linear\} of linear maps from \( V \) to \( W \).

**Theorem 1.6** (Linearity is a linear condition). \( L(V, W) \) is a vector space under pointwise addition and scalar multiplication. Otherwise said, \( L(V, W) \leq W^V \).

1.4.2 Linear maps and matrices

**Definition.** Let \( V, W \) be finite-dimensional vector spaces over \( F \) with bases \( B : v_1, \ldots, v_n \) and \( B' : w_1, \ldots, w_m \) respectively. Let \( \phi \in L(V, W) \). The matrix of \( \phi \) with respect to \( B, B' \) is the matrix \( A = (A_{ij}) \in M_{m \times n}(F) \) defined by:
\[
\phi(v_j) = \sum_{i=1}^{m} A_{ij} w_i, \quad (1.2)
\]
for all \( 1 \leq j \leq n \).

In the special case where \( V = W \) and \( B = B' \), we call \( A \) the matrix of \( \phi \) with respect to \( B \).

1.4.3 Extension by linearity

**Proposition 1.7** (Extension by linearity). Let \( V, W \) be vector spaces over \( F \). Let \( v_1, \ldots, v_n \) be a basis of \( V \) and \( w_1, \ldots, w_n \) any vectors in \( W \).

Then there is a unique \( \phi \in L(V, W) \) such that
\[
\phi(v_i) = w_i, \quad 1 \leq i \leq n. \quad (1.3)
\]

1.4.4 The rank-nullity theorem

**Theorem 1.8** (Rank-nullity). Let \( \phi : V \to W \) be linear with \( V \) finite-dimensional. Then
\[
\dim \ker \phi \leq \dim V.
\]

**Proposition 1.9.** Let \( \phi : V \to W \) be linear with \( V, W \) finite-dimensional vector spaces of the same dimension: \( \dim V = \dim W \).

Then the following are equivalent:
1. \( \phi \) is injective.
2. \( \phi \) is surjective.
3. \( \phi \) is an isomorphism.
Chapter 2

Sums and quotients

Convention. In this chapter, all vector spaces are over the same field \( F \) unless we say otherwise.

2.1 Sums of subspaces

Definition. Let \( V_1, \ldots, V_k \leq V \). The sum \( V_1 + \cdots + V_k \) is the set
\[
V_1 + \cdots + V_k := \{ v_1 + \cdots + v_k \mid v_i \in V_i, 1 \leq i \leq k \}.
\]

Proposition 2.1. Let \( V_1, \ldots, V_k \leq V \). Then
\begin{enumerate}
    \item \( V_1 + \cdots + V_k \leq V \).
    \item If \( W \leq V \) and \( V_1, \ldots, V_k \leq W \) then \( V_1, \ldots, V_k \leq V_1 + \cdots + V_k \leq W \).
\end{enumerate}

2.2 Direct sums

Definition. Let \( V_1, \ldots, V_k \leq V \). The sum \( V_1 + \cdots + V_k \) is direct if each \( v \in V_1 + \cdots + V_k \) can be written
\[
v = v_1 + \cdots + v_k
\]
in only one way, that is, for unique \( v_i \in V_i, 1 \leq i \leq k \).

In this case, we write \( V_1 \oplus \cdots \oplus V_k \) instead of \( V_1 + \cdots + V_k \).

Proposition 2.2. Let \( V_1, V_2 \leq V \). Then \( V_1 + V_2 \) is direct if and only if \( V_1 \cap V_2 = \{0\} \).

Definition. Let \( V_1, V_2 \leq V \). \( V \) is the (internal) direct sum of \( V_1 \) and \( V_2 \) if \( V = V_1 \oplus V_2 \).

In this case, say that \( V_2 \) is a complement of \( V_1 \) (and \( V_1 \) is a complement of \( V_2 \)).

Proposition 2.3. Let \( V_1, \ldots, V_k \leq V, k \geq 2 \). Then the sum \( V_1 + \cdots + V_k \) is direct if and only if for each \( 1 \leq i \leq k \), \( V_i \cap (\sum_{j \neq i} V_j) = \{0\} \).

2.2.1 Direct sums and projections

Definition. Let \( V \) be a vector space. A linear map \( \pi : V \to V \) is a projection if \( \pi \circ \pi = \pi \).

Proposition 2.4. Let \( V_1, V_2 \leq V \) with \( V = V_1 \oplus V_2 \). Then there are projections \( \pi_1, \pi_2 : V \to V \) such that:
\begin{enumerate}
    \item \( \text{im} \pi_i = V_i, i = 1, 2 \);
\end{enumerate}
(b) \( \ker \pi_1 = V_2, \ker \pi_2 = V_1 \);
(c) \( v = \pi_1(v) + \pi_2(v) \), for all \( v \in V \). Otherwise said, \( \text{id}_V = \pi_1 + \pi_2 \).

**Proposition 2.5.** Let \( V = V_1 \oplus V_2 \) with \( V \) finite-dimensional. Then
\[
\dim V = \dim V_1 + \dim V_2.
\]

### 2.2.2 Induction from two summands

**Lemma 2.6.** Let \( V_1, \ldots, V_k \leq V \). Then \( V_1 + \cdots + V_k \) is direct if and only if \( V_1 + \cdots + V_{k-1} \) is direct and \( (V_1 + \cdots + V_{k-1}) + V_k \) (two summands) is direct.

**Corollary 2.7.** Let \( V_1, \ldots, V_k \leq V \) be subspaces of a finite-dimensional vector space \( V \) with \( V_1 + \cdots + V_k \) direct. Then
\[
\dim V_1 + \cdots + V_k = \dim V_1 + \cdots + \dim V_k.
\]

### 2.2.3 Direct sums and bases

**Proposition 2.8.** Let \( V_1, V_2 \leq V \) be finite-dimensional subspaces with bases \( B_1 : v_1, \ldots, v_k \) and \( B_2 : w_1, \ldots, w_l \). Then \( V_1 + V_2 \) is direct if and only if the concatenation\(^1\) \( B_1 B_2 : v_1, \ldots, v_k, w_1, \ldots, w_l \) is a basis of \( V_1 + V_2 \).

**Corollary 2.9.** Let \( V_1, \ldots, V_k \leq V \) be finite-dimensional subspaces with \( B_i \) a basis of \( V_i \), \( 1 \leq i \leq k \). Then \( V_1 + \cdots + V_k \) is direct if and only if the concatenation \( B_1 \cdots B_k \) is a basis for \( V_1 + \cdots + V_k \).

### 2.2.4 Complements

**Proposition 2.10** (Complements exist). Let \( U \leq V \), a finite-dimensional vector space. Then there is a complement to \( U \).

**Proposition 2.11** (Extension of linear maps). Let \( V, W \) be vector spaces with \( V \) finite-dimensional. Let \( U \leq V \) be a subspace and \( \phi : U \to W \) a linear map. Then there is a linear map \( \Phi : V \to W \) such that the restriction\(^2\) of \( \Phi \) to \( U \) is \( \phi \): \( \Phi|_U = \phi \). Otherwise said: for all \( u \in U \)
\[
\Phi(u) = \phi(u).
\]

### 2.3 Quotients

**Definition.** Let \( U \leq V \). Say that \( v, w \in V \) are *congruent modulo* \( U \) if \( v - w \in U \). In this case, we write \( v \equiv w \mod U \).

**Lemma 2.12.** Congruence modulo \( U \) is an equivalence relation.

**Definition.** For \( v \in V, U \leq V \), the set \( v + U := \{ v + u \mid u \in U \} \subseteq V \) is called a *coset of* \( U \) and \( v \) is called a *coset representative* of \( v + U \).

**Definition.** Let \( U \leq V \). The *quotient space* \( V/U \) of \( V \) by \( U \) is the set \( V/U \), pronounced “\( V \) mod \( U \)”, of cosets of \( U \):
\[
V/U := \{ v + U \mid v \in V \}.
\]
This is a subset of the *power set*\(^3\) \( \mathcal{P}(V) \) of \( V \).

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\(^1\)The concatenation of two lists is simply the list obtained by adjoining all entries in the second list to the first.

\(^2\)Recall that if \( f : X \to Y \) is a map of sets and \( A \subseteq X \) then the restriction of \( f \) to \( A \) is the map \( f|_A : A \to Y \) given by \( f|_A(a) = f(a) \), for all \( a \in A \).

\(^3\)Recall from Algebra 1A that the power set of a set \( A \) is the set of all subsets of \( A \).
The quotient map \( q : V \to V/U \) is defined by

\[ q(v) = v + U. \]

**Theorem 2.13.** Let \( U \leq V \). Then, for \( v, w \in V \), \( \lambda \in F \),

\[
(v + U) + (w + U) := (v + w) + U \\
\lambda(v + U) := (\lambda v) + U
\]

give well-defined operations of addition and scalar multiplication on \( V/U \) with respect to which \( V/U \) is a vector space and \( q : V \to V/U \) is a linear map.

Moreover, \( \ker q = U \) and \( \text{im } q = V/U \).

**Corollary 2.14.** Let \( U \leq V \). If \( V \) is finite-dimensional then so is \( V/U \) and

\[
\dim V/U = \dim V - \dim U.
\]

**Theorem 2.15** (First Isomorphism Theorem). Let \( \phi : V \to W \) be a linear map of vector spaces.

Then \( V/\ker \phi \cong \text{im } \phi \).

In fact, define \( \bar{\phi} : V/\ker \phi \to \text{im } \phi \) by

\[
\bar{\phi}(q(v)) = \phi(v),
\]

where \( q : V \to V/\ker \phi \) is the quotient map.

Then \( \bar{\phi} \) is a well-defined linear isomorphism.
Chapter 3

Inner product spaces

Convention. In this chapter, we take the field $\mathbb{F}$ of scalars to be either $\mathbb{R}$ or $\mathbb{C}$.

3.1 Inner products

3.1.1 Definition and examples

Definition. Let $V$ be a vector space of $\mathbb{F}$ (which is $\mathbb{R}$ or $\mathbb{C}$). An inner product on $V$ is a map $V \times V \to \mathbb{F}$: $(v, w) \mapsto \langle v, w \rangle$ which is:

1. (conjugate) symmetric: $\langle w, v \rangle = \overline{\langle v, w \rangle}$, for all $v, w \in V$. In particular $\langle v, v \rangle = \langle v, v \rangle$ and so is real.
2. linear in the second slot:

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$
$$\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle,$$
for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$.
3. positive definite: For all $v \in V$, $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

A vector space with an inner product is called an inner product space.

Definition. A map $\phi : V \to W$ of complex vector spaces is conjugate linear (or anti-linear) if

$$\phi(v + w) = \phi(v) + \phi(w)$$
$$\phi(\lambda v) = \overline{\lambda} \phi(v),$$
for all $v, w \in V$ and $\lambda \in \mathbb{F}$.

Definition. Let $V$ be an inner product space.

1. The norm of $v \in V$ is $\|v\| := \sqrt{\langle v, v \rangle} \geq 0$.
2. Say $v, w \in V$ are orthogonal or perpendicular if $\langle v, w \rangle = 0$. In this case, we write $v \perp w$.

3.1.2 Cauchy–Schwarz inequality

Theorem 3.1 (Cauchy–Schwarz inequality). Let $V$ be an inner product space. For $v, w \in V$,

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

(3.1)
with equality if and only if $v, w$ are linearly dependent, that is, either $v = 0$ or $w = \lambda v$, for some $\lambda \in \mathbb{F}$.
Proposition 3.2. Let \( V \) be an inner product space and \( v, w \in V \).

1. **Pythagoras Theorem**: If \( v \perp w \) then
   \[
   \|v + w\|^2 = \|v\|^2 + \|w\|^2.
   \] (3.2)

2. **Triangle inequality**: \( \|v + w\| \leq \|v\| + \|w\| \) with equality if and only if \( v = 0 \) or \( w = \lambda v \) with \( \lambda \geq 0 \).

3. **Parallelogram identity**: \( \|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2) \).

### 3.2 Orthogonality

#### 3.2.1 Orthonormal bases

**Definition.** A list of vectors \( u_1, \ldots, u_k \) in an inner product space \( V \) is **orthonormal** if, for all \( 1 \leq i, j \leq k \),

\[
\langle u_i, u_j \rangle = \delta_{ij} := \begin{cases} 
1 & \text{if } i = j; \\
0 & \text{if } i \neq j.
\end{cases}
\]

If \( u_1, \ldots, u_k \) is also a basis, we call it an **orthonormal basis**.

**Lemma 3.3.** Let \( V \) be an inner product space with orthonormal basis \( u_1, \ldots, u_n \) and let \( v \in V \). Then

\[
v = \sum_{i=1}^{n} \langle u_i, v \rangle u_i.
\]

**Lemma 3.4.** Any orthonormal list of vectors \( u_1, \ldots, u_k \) is linearly independent.

**Proposition 3.5.** Let \( u_1, \ldots, u_n \) be an orthonormal basis of an inner product space \( V \). Let \( v = x_1 u_1 + \cdots + x_n u_n \) and \( w = y_1 u_1 + \cdots + y_n u_n \). Then

\[
\langle v, w \rangle = \sum_{i=1}^{n} \bar{x_i} y_i = x \cdot y.
\]

Thus the inner product of two vectors is the dot product of their coordinates with respect to an orthonormal basis.

**Proposition 3.6.** Let \( u_1, \ldots, u_n \) be an orthonormal basis of an inner product space \( V \) and \( v, w \in V \). Then:

1. **Parseval’s identity**: \( \langle v, w \rangle = \sum_{i=1}^{n} \langle v, u_i \rangle \langle u_i, w \rangle \).
2. **Bessel’s equality**: \( \|v\|^2 = \sum_{i=1}^{n} |\langle v, u_i \rangle|^2 \).

**Theorem 3.7 (Gram–Schmidt orthogonalisation).** Let \( v_1, \ldots, v_m \) be linearly independent vectors in an inner product space \( V \).

Then there is an orthonormal list \( u_1, \ldots, u_m \) such that

\[
\text{span}\{u_1, \ldots, u_k\} = \text{span}\{v_1, \ldots, v_k\},
\]

for all \( 1 \leq k \leq m \), defined inductively by:

\[
u_k := \frac{w_k}{\|w_k\|}
\]

where,

\[
w_1 := v_1
\]
and, for $k > 1$,

$$w_k := v_k - \sum_{j=1}^{k-1} (u_j, v_k) u_j = v_k - \sum_{j=1}^{k-1} \frac{(w_j, v_k)}{\|w_j\|^2} w_j.$$ 

**Corollary 3.8.** Any finite-dimensional inner product space $V$ has an orthonormal basis.

**Definition.** A matrix $Q \in M_{n \times n}(\mathbb{R})$ is orthogonal if

$$Q^T Q = I_n,$$

or, equivalently, $Q$ has orthonormal columns with respect to the dot product. Here $I_n$ is the $n \times n$ identity matrix.

**Theorem 3.9 (QR decomposition).** Let $A \in M_{n \times n}(\mathbb{R})$ be an invertible matrix. Then we can write

$$A = QR,$$

where $Q$ is orthogonal and $R$ is upper triangular ($R_{ij} = 0$ if $i > j$) with positive entries on the diagonal.

### 3.2.2 Orthogonal complements and orthogonal projection

**Definition.** Let $V$ be an inner product space and $U \leq V$. The orthogonal complement $U^\perp$ of $U$ (in $V$) is given by

$$U^\perp := \{v \in V \mid (u, v) = 0, \text{ for all } u \in U\}.$$ 

**Proposition 3.10.** Let $V$ be an inner product space and $U \leq V$. Then

1. $U^\perp \leq V$;
2. $U \cap U^\perp = \{0\}$;
3. $U \leq (U^\perp)^\perp$.

**Theorem 3.11.** Let $U$ be a finite-dimensional subspace of an inner product space $V$. Then $V$ is an internal direct sum:

$$V = U \oplus U^\perp.$$ 

**Corollary 3.12.** Let $V$ be a finite-dimensional inner product space and $U \leq V$. Then

1. $\dim U^\perp = \dim V - \dim U$.
2. $U = (U^\perp)^\perp$.

**Definition.** Let $V$ be an inner product space and $U \leq V$ such that $V = U \oplus U^\perp$. The projection $\pi_U : V \to V$ with image $U$ and kernel $U^\perp$ is called the orthogonal projection onto $U$.

**Lemma 3.13.** Let $V$ be an inner product space and $U \leq V$ a finite-dimensional subspace with orthonormal basis $u_1, \ldots, u_k$ then, for all $v \in V$,

$$\pi_U(v) = \sum_{i=1}^k (u_i, v) u_i.$$ 

**Theorem 3.14.** Let $V$ be an inner product space and $U \leq V$ such that $V = U \oplus U^\perp$.

For $v \in V$, $\pi_U(v)$ is the nearest point of $U$ to $v$: for all $u \in U$,

$$\|v - \pi_U(v)\| \leq \|v - u\|.$$ 

9
Chapter 4

Linear operators on inner product spaces

Convention. In this chapter, we once again take the field \( F \) of scalars to be either \( \mathbb{R} \) or \( \mathbb{C} \).

4.1 Linear operators and their adjoints

4.1.1 Linear operators and matrices

Definition. Let \( V \) be a vector space over \( F \). A linear operator on \( V \) is a linear map \( \phi : V \to V \).

The vector space of linear operators on \( V \) is denoted \( L(V) \) (instead of \( L(V,V) \)).

4.1.2 Adjoints

Lemma 4.1 (Nondegeneracy Lemma). Let \( V \) be an inner product space and \( v \in V \). Then \( \langle v, w \rangle = 0 \), for all \( w \in V \), if and only if \( v = 0 \).

Definition. Let \( V \) be an inner product space and \( \phi \in L(V) \). An adjoint to \( \phi \) is a linear operator \( \phi^* \in L(V) \) such that, for all \( v, w \in V \), we have

\[
\langle \phi^*(v), w \rangle = \langle v, \phi(w) \rangle
\]

or, equivalently, by conjugate symmetry,

\[
\langle w, \phi^*(v) \rangle = \langle \phi(w), v \rangle.
\]

Proposition 4.2. Let \( V \) be an inner product space and suppose \( \phi, \psi \in L(V) \) have adjoints. Then \( \phi \circ \psi; \phi + \lambda \psi, \lambda \in \mathbb{F} \; \text{and} \; \text{id}_V \) all have adjoints given by:

1. \( (\phi \circ \psi)^* = \psi^* \circ \phi^* \) (note the change of order here!).
2. \( (\phi + \lambda \psi)^* = \phi^* + \bar{\lambda} \psi^* \).
3. \( (\phi^*)^* = \phi \).
4. \( \text{id}_V^* = \text{id}_V \).

Proposition 4.3. Let \( V \) be a finite-dimensional inner product space and \( \phi \in L(V) \) a linear operator. Then
(1) \( \phi \) has a unique adjoint \( \phi^* \).

(2) Let \( u_1, \ldots, u_n \) be an orthonormal basis of \( V \) with respect to which \( \phi \) has matrix \( A \). Then \( \phi^* \) has matrix \( A^\dagger := \overline{A^T} \) (which is \( A^T \) when \( F = \mathbb{R} \)).

Definitions.

1. Let \( V \) be an inner product space and \( \phi \in L(V) \).
   Say that \( \phi \) is self-adjoint if \( \phi^* = \phi \), or, equivalently, for all \( v, w \in V \),
   \[
   (\phi(v), w) = (v, \phi(w)).
   \]
   Say \( \phi \) is skew-adjoint if \( \phi^* = -\phi \), or, equivalently, for all \( v, w \in V \),
   \[
   (\phi(v), w) = -(v, \phi(w)).
   \]

2. Let \( A \in M_{n \times n}(F) \).
   (a) If \( F = \mathbb{C} \), say that \( A \) is Hermitian if \( A^\dagger = A \) and skew-Hermitian if \( A^\dagger = -A \).
   (b) If \( F = \mathbb{R} \), say that \( A \) is symmetric if \( A^T = A \) and skew-symmetric if \( A^T = -A \).

### 4.1.3 Linear isometries

**Definition.** Let \( V, W \) be inner product spaces with inner products \( \langle , \rangle_V \) and \( \langle , \rangle_W \) respectively. A linear map \( \phi : V \to W \) is a linear isometry if, for all \( v_1, v_2 \in V \),
   \[
   \langle \phi(v_1), \phi(v_2) \rangle_W = \langle v_1, v_2 \rangle_V.
   \]

**Proposition 4.4.** Let \( V \) be a finite-dimensional inner product space and \( \phi \in L(V) \). Then \( \phi \) is a linear isometry if and only if \( \phi \) is an isomorphism with \( \phi^{-1} = \phi^* \) (equivalently, \( \phi^* \circ \phi = \text{id}_V = \phi \circ \phi^* \)).

**Definitions.** Let \( V \) be an inner product space over \( F \) and \( \phi \in L(V) \). If \( \phi \) is an isomorphism with \( \phi^{-1} = \phi^* \), then say \( \phi \) is:
   - an orthogonal transformation if \( F = \mathbb{R} \);
   - a unitary transformation if \( F = \mathbb{C} \).

The set of all orthogonal, resp. unitary transformations is denoted \( O(V) \), resp. \( U(V) \).

Let \( A \in M_{n \times n}(F) \).
   - \( A \) is orthogonal if \( F = \mathbb{R} \) and \( A^T A = I \);
   - \( A \) is unitary if \( F = \mathbb{C} \) and \( A^\dagger A = I \).

The set of all \( n \times n \) orthogonal, resp. unitary matrices is denoted \( O(n) \), resp. \( U(n) \).

**Definitions.** Let \( V \) be a vector space. The general linear group of \( V \), denoted \( \text{GL}(V) \), is:
   \[
   \text{GL}(V) := \{ \phi \in L(V) \mid \phi \text{ is an isomorphism} \}.
   \]

Similarly, the general linear group of \( n \times n \) matrices over \( F \), denoted \( \text{GL}(n, F) \), is:
   \[
   \text{GL}(n, F) := \{ A \in M_{n \times n}(F) \mid A \text{ is invertible} \}.
   \]

**Proposition 4.5.**

1. Let \( V \) be a vector space. Then \( \text{GL}(V) \) is a group under composition: \( \psi \phi := \psi \circ \phi \).

2. If \( V \) is an inner product space, then \( O(V) \), resp. \( U(V) \), is a subgroup of \( \text{GL}(V) \), when \( F = \mathbb{R} \), resp. \( \mathbb{C} \).
Theorem 4.6 (Classification of rigid motions). Let $V$ be a real inner product space. Recall that the distance between $v, w \in V$ is $d(v, w) := \|v - w\|.$

A map $f : V \to V$ (not necessarily linear) is distance-preserving or a rigid motion if $d(f(v), f(w)) = d(v, w),$ for all $v, w \in V.$

$f$ is distance-preserving if and only if there is a $v_0 \in V$ and $\phi \in L(V)$ a linear isometry such that

$$f(v) = \phi(v) + v_0,$$

(4.1)

for all $v \in V.$

4.2 The spectral theorem

4.2.1 Eigenvalues and eigenvectors

Definitions. Let $V$ be a vector space over $\mathbb{F}$ and $\phi \in L(V)$.

An eigenvalue of $\phi$ is a scalar $\lambda \in \mathbb{F}$ such that there is a non-zero $v \in V$ with

$$\phi(v) = \lambda v.$$

Such a vector $v$ is called an eigenvector of $\phi$ with eigenvalue $\lambda$.

The $\lambda$-eigenspace $E_\phi(\lambda)$ of $\phi$ is given by

$$E_\phi(\lambda) := \ker(\phi - \lambda \text{id}_V) \leq V.$$

Definition. Let $V$ be a finite-dimensional vector space over $\mathbb{F}$ and $\phi \in L(V)$.

The characteristic polynomial $\Delta_\phi$ of $\phi$ is given by

$$\Delta_\phi(\lambda) := \det(\phi - \lambda \text{id}_V) = \det(A - \lambda I),$$

where $A$ is the matrix of $\phi$ with respect to some (any!) basis of $V$.

Lemma 4.7. A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of $\phi$ if and only if $\Delta_\phi(\lambda) = 0,$ that is, $\lambda$ is a root of $\Delta_\phi$.

Corollary 4.8. Let $\phi$ be a linear operator on a finite-dimensional complex vector space $V$. Then $\phi$ has an eigenvalue.

4.2.2 Invariant subspaces and adjoints

Definition. Let $V$ be a vector space and $\phi \in L(V)$.

A subspace $U \leq V$ is $\phi$-invariant if $\phi(U) \leq U,$ that is, $\phi(u) \in U,$ for all $u \in U$.

Lemma 4.9. Let $\phi, \psi \in L(V)$ and suppose that

- $\psi \circ \phi = \phi \circ \psi$ (say that $\phi$ and $\psi$ commute).
- $U = E_\phi(\lambda)$ is an eigenspace of $\phi$.

Then $U$ is $\psi$-invariant.

Lemma 4.10. Let $V$ be a finite-dimensional¹ inner product space and $\phi \in L(V)$.

Let $U \leq V$ be a $\phi$-invariant subspace. Then $U^\perp$ is $\phi^*$-invariant.

Definition. Let $V$ be a finite-dimensional inner product space. A linear operator $\phi \in L(V)$ is normal if it commutes with its adjoint: $\phi^* \circ \phi = \phi \circ \phi^*$.

¹We only need this hypothesis to ensure that $\phi^*$ exists.
Proposition 4.11. Let $V$ be a finite-dimensional inner product space and $\phi \in L(V)$. Suppose that:
- $\phi$ is normal;
- $U \leq V$ is an eigenspace of $\phi$.
Then $U^\perp$ is $\phi$-invariant.

4.2.3 The spectral theorem for normal operators

Definition. Let $V$ be a finite-dimensional vector space. A linear operator $\phi \in L(V)$ is diagonalisable if $V$ has a basis of eigenvectors of $\phi$.

Definition. Let $V$ be a finite-dimensional inner product space. A linear operator $\phi \in L(V)$ is orthogonally diagonalisable if $V$ has an orthonormal basis of eigenvectors.

Proposition 4.12. Let $V$ be a finite-dimensional inner product space over $\mathbb{F}$ and $\phi \in L(V)$ an orthogonally diagonalisable linear operator. Then:
1. If $\mathbb{F} = \mathbb{C}$, $\phi$ is normal.
2. If $\mathbb{F} = \mathbb{R}$, $\phi$ is self-adjoint.

Theorem 4.13 (Spectral theorem for normal operators). Let $V$ be a finite-dimensional complex inner product space and $\phi \in L(V)$ a linear operator. Then $\phi$ is orthogonally diagonalisable if and only if $\phi$ is normal.

4.2.4 The spectral theorem for real self-adjoint operators

Lemma 4.14. Let $V$ be an inner product space and $\phi \in L(V)$ be self-adjoint.
1. Any eigenvalue of $\phi$ is real.
2. If $v, w \in V$ are eigenvectors of $\phi$ with eigenvalues $\lambda \neq \mu$ then $v \perp w$.

Proposition 4.15. A self-adjoint operator $\phi$ on a real, finite-dimensional inner product space $V$ has an eigenvalue.

Theorem 4.16 (Spectral theorem for real self-adjoint operators). Let $V$ be a real, finite-dimensional inner product space and $\phi \in L(V)$ a linear operator. Then $\phi$ is orthogonally diagonalisable if and only if $\phi$ is self-adjoint.

4.2.5 The spectral theorem for symmetric and Hermitian matrices

Theorem 4.17 (Spectral theorem for symmetric/hermitian matrices).
1. Let $A \in M_{n \times n}(\mathbb{R})$ be symmetric. Then there is an orthogonal matrix $P \in O(n)$ such that $P^{-1}AP$ is diagonal.
2. Let $A \in M_{n \times n}(\mathbb{C})$ be Hermitian. Then there is an unitary matrix $P \in U(n)$ such that $P^{-1}AP$ is diagonal.

4.2.6 Singular value decomposition

Lemma 4.18. Let $V$ be a finite-dimensional inner product space and $\phi \in L(V)$. Then:
1. All eigenvalues of $\phi^* \circ \phi$ are non-negative.

\footnote{We do not demand that $V$ be finite-dimensional.}
(2) \( \ker(\phi^* \circ \phi) = \ker \phi \).

**Definition.** Let \( V \) be a finite-dimensional inner product space and \( \phi \in L(V) \). The singular values of \( \phi \) are \( \sigma_1, \ldots, \sigma_n \) where \( \sigma_i = \sqrt{\mu_i} \geq 0 \) and \( \mu_1, \ldots, \mu_n \) are the eigenvalues of \( \phi^* \circ \phi \) listed with multiplicity (thus each distinct \( \mu \) appears \( \dim E_{\phi^* \circ \phi}(\mu) \) times).

**Theorem 4.19** (Singular value decomposition). Let \( V \) be a finite-dimensional inner product space and \( \phi \in L(V) \) a linear operator with singular values \( \sigma_1, \ldots, \sigma_n \).

Then there are orthonormal bases \( u_1, \ldots, u_n \) and \( w_1, \ldots, w_n \) of \( V \) such that

\[
\phi(v) = \sum_{i=1}^{n} \sigma_i \langle u_i, v \rangle w_i, \tag{4.2}
\]

for all \( v \in V \).
Chapter 5

Duality

5.1 Dual spaces

Definition. Let $V$ be a vector space over $F$. The dual space $V^*$ of $V$ is

$$V^* := L(V,F) = \{ \alpha : V \to F \mid \alpha \text{ is linear} \}.$$  

Elements of $V^*$ are called linear functionals or (less often) linear forms.

Proposition 5.1. Let $V$ be a finite-dimensional vector space with basis $v_1, \ldots, v_n$.

Define $v_1^* , \ldots , v_n^* \in V^*$ by setting

$$v_i^*(v_j) = \delta_{ij} \in F$$

and extending by linearity (thus applying Proposition 1.7).

Then $v_1^*, \ldots, v_n^*$ is a basis of $V^*$ called the dual basis to $v_1, \ldots, v_n$.

Corollary 5.2. If $V$ is finite-dimensional then $\dim V = \dim V^*$.

Theorem 5.3 (Riesz Representation Theorem). Let $V$ be a finite-dimensional inner product space and $\alpha \in V^*$. Then there is a unique $w \in V$ such that

$$\alpha(v) = \langle w,v \rangle,$$

for all $v \in V$. Thus $\alpha = \alpha_w$.

Indeed, if $u_1 , \ldots , u_n$ is an orthonormal basis of $V$ then

$$w = \sum_{i=1}^n \overline{\alpha(u_i)} u_i. \quad (5.1)$$

Theorem 5.4 (Sufficiency principle). Let $V$ be a vector space and $v \in V$. Then $\alpha(v) = 0$, for all $\alpha \in V^*$, if and only if $v = 0$.

Proposition 5.5. Let $V$ be a finite-dimensional vector space and $\alpha_1 , \ldots , \alpha_n$ a basis of $V^*$. Then there is a basis $v_1, \ldots, v_n$ of $V$ such that

$$\alpha_i(v_j) = \delta_{ij}.$$ 

Thus $\alpha_i = v_i^*$, for $1 \leq i \leq n$.

Theorem 5.6. If $V$ is a finite-dimensional vector space then $ev : V \to V^{**}$ is an isomorphism.
5.2 Solution sets and annihilators

Definition. Let $E \leq V^\ast$. The solution set of $E$ is

$$\text{sol }E := \{v \in V \mid \alpha(v) = 0, \text{ for all } \alpha \in E\} = \bigcap_{\alpha \in E} \ker \alpha \leq V.$$

Proposition 5.7. If $V$ is finite-dimensional and $E \leq V^\ast$ then

$$\dim \text{sol }E = \dim V - \dim E.$$

We say that $E$ and $\text{sol }E$ have complementary dimension.

Corollary 5.8. Let $V$ have dimension $n$ and suppose that $\alpha_1, \ldots, \alpha_n \in V^\ast$ are such that

$$\bigcap_{i=1}^n \ker \alpha_i = \{0\}.$$

Then $\alpha_1, \ldots, \alpha_n$ is a basis of $V^\ast$.

Proposition 5.9. Let $E, F \leq V^\ast$. Then

1. If $E \leq F$ then $\text{sol }F \leq \text{sol }E$.
2. $\text{sol}$ swaps sums and intersections:

$$\begin{align*}
\text{sol}(E + F) &= (\text{sol }E) \cap (\text{sol }F) \\
(\text{sol }E) + (\text{sol }F) &\leq \text{sol}(E \cap F)
\end{align*}$$

with equality if $V$ is finite-dimensional.

Definition. Let $U \leq V$. The annihilator of $U$, denoted $\text{ann }U$ or $U^\circ$, is given by:

$$\text{ann }U := \{\alpha \in V^\ast \mid \alpha|_U = 0\} = \{\alpha \in V^\ast \mid \alpha(u) = 0, \text{ for all } u \in U\}.$$

Proposition 5.10. Let $V$ be finite-dimensional and $U \leq V$. Then

$$\dim \text{ann }U = \dim V - \dim U.$$

Proposition 5.11. Let $U, W \leq V$. Then

1. If $U \leq W$ then $\text{ann }W \leq \text{ann }U$.
2. $\text{ann}$ swaps sums and intersections:

$$\begin{align*}
\text{ann}(U + W) &= (\text{ann }U) \cap (\text{ann }W) \\
(\text{ann }U) + (\text{ann }W) &\leq \text{ann}(U \cap W)
\end{align*}$$

with equality if $V$ is finite-dimensional.

Lemma 5.12. Let $U \leq V$ and $E \leq V^\ast$ then $U \leq \text{sol }E$ if and only if $E \leq \text{ann }U$.

Theorem 5.13. Let $U \leq V$ and $E \leq V^\ast$. Then

$$\begin{align*}
U &\leq \text{sol}(\text{ann }U) \\
E &\leq \text{ann}(\text{sol }E),
\end{align*}$$

with equality if $V$ is finite-dimensional.

5.3 Transposes

Definition. Let $\phi \in L(V, W)$ be a linear map of vector spaces. The transpose $\phi^T$ of $\phi$ is the map $\phi^T : W^\ast \rightarrow V^\ast$ given by

$$\phi^T(\alpha) := \alpha \circ \phi,$$

for all $\alpha \in W^\ast$.  

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Lemma 5.14. \( \phi^T : W^* \to V^* \) is also a linear map.

Proposition 5.15. Let \( V,W \) be finite-dimensional vector spaces and \( \phi \in L(V,W) \) with matrix \( A \in M_{m \times n}(F) \) with respect to bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) of \( V \) and \( W \). Then \( \phi^T \) has matrix \( A^T \) with respect to the dual bases \( w^*_1, \ldots, w^*_m \) and \( v^*_1, \ldots, v^*_n \) of \( W^* \) and \( V^* \).

Theorem 5.16. Let \( \phi \in L(V,W) \) be a linear map of vector spaces. Then

(1) \[ \ker \phi = \text{sol}(\text{im} \, \phi^T) \]
\[ \text{im} \phi \leq \text{sol}(\ker \phi^T) \]

with equality if \( V,W \) are finite-dimensional.

(2) \[ \ker \phi^T = \text{ann}(\text{im} \, \phi) \]
\[ \text{im} \phi^T \leq \text{ann}(\ker \phi) \]

with equality if \( V,W \) are finite-dimensional.

Corollary 5.17. Let \( \phi \in L(V,W) \) be a linear map of finite-dimensional vector spaces. Then
\[ \text{rank} \phi = \text{rank} \phi^T. \]

Proposition 5.18. Let \( \phi \in L(V,W) \) be a linear map of finite-dimensional vector spaces. Then

(1) \( \phi \) injects if and only if \( \phi^T \) surjects.

(2) \( \phi^T \) injects if and only if \( \phi \) surjects.
Chapter 6

Bilinearity

6.1 Bilinear maps

6.1.1 Definitions and examples

**Definition.** Let \( U, V, W \) be vector spaces over a field \( F \). A map \( B : U \times V \to W \) is *bilinear* if it is linear in each slot separately:

\[
B(\lambda u_1 + u_2, v) = \lambda B(u_1, v) + B(u_2, v)
\]

\[
B(u, \lambda v_1 + v_2) = \lambda B(u, v_1) + B(u, v_2),
\]

for all \( u, u_1, u_2 \in U, v, v_1, v_2 \in V \) and \( \lambda \in F \).

A bilinear map \( U \times V \to F \) is called a *bilinear pairing*.

A bilinear map \( V \times V \to F \) is called a *bilinear form on \( V \).*

**Notation.** We let \( \text{Bil}(U, V; W) \) denote the set of bilinear maps \( U \times V \to W \).

6.1.2 Bilinear forms and matrices

**Definition.** Let \( V \) be a vector space over \( F \) with basis \( \mathcal{B} = v_1, \ldots, v_n \) and let \( B : V \times V \to F \) be a bilinear form. The *matrix of* \( B \) *with respect to* \( \mathcal{B} \) is \( A \in M_n(F) \) given by

\[
A_{ij} = B(v_i, v_j),
\]

for \( 1 \leq i, j \leq n \).

**Proposition 6.1.** Let \( B : V \times V \to F \) be a bilinear form with matrix \( A \) with respect to \( \mathcal{B} = v_1, \ldots, v_n \). Then \( B \) is completely determined by \( A \): if \( v = \sum_{i=1}^{n} x_i v_i \) and \( w = \sum_{j=1}^{n} y_j v_j \) then

\[
B(v, w) = \sum_{i,j=1}^{n} x_i y_j A_{ij},
\]

or, equivalently, for all \( x, y \in F^n \),

\[
B(\phi_{\mathcal{B}}(x), \phi_{\mathcal{B}}(y)) = B_A(x, y) = x^T A y.
\]

**Proposition 6.2.** Let \( B : V \times V \to F \) be a bilinear form with matrices \( A \) and \( A' \) with respect to bases \( \mathcal{B} \) and \( \mathcal{B}' \) of \( V \). Then

\[
A' = P^T A P
\]

where \( P \) is the change of basis matrix\(^1 \) from \( \mathcal{B} \) to \( \mathcal{B}' \): thus \( \phi_P = \phi_{\mathcal{B}'}^{-1} \circ \phi_{\mathcal{B}} \).

\(^1\)See Definition 1 in Section 2.6 of Algebra 1B.
Definition. We say that matrices $A, B \in M_{n \times n} (\mathbb{F})$ are congruent if there is $P \in \text{GL}(n, \mathbb{F})$ such that $B = P^T A$.

### 6.2 Symmetric bilinear forms

**Definition.** A bilinear form $B : V \times V \to \mathbb{F}$ is symmetric if, for all $v, w \in V$,

$$B(v, w) = B(w, v)$$

#### 6.2.1 Rank and radical

**Definitions.** Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form.

The **radical** $\text{rad} B$ of $B$ is given by

$$\text{rad} B := \{ v \in V \mid B(v, w) = 0, \text{ for all } w \in V \}.$$  

We shall shortly see that $\text{rad} B \leq V$.

We say that $B$ is **non-degenerate** if $\text{rad} B = \{0\}$.

If $V$ is finite-dimensional, the **rank** of $B$ is $\dim V - \dim \text{rad} B$ (so that $B$ is non-degenerate if and only if rank $B = \dim V$).

**Lemma 6.3.** Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form on a finite-dimensional vector space $V$ with matrix $A$ with respect to some basis of $V$. Then

$$\text{rank} B = \text{rank} A.$$  

In particular, $B$ is non-degenerate if and only if $\det A \neq 0$.

#### 6.2.2 Classification of symmetric bilinear forms

**Convention.** In this section, we work with a field $\mathbb{F}$ where $1 + 1 \neq 0$ so that $\frac{1}{2} = (1 + 1)^{-1}$ makes sense. This excludes, for example, the 2-element field $\mathbb{Z}_2$.

**Lemma 6.4.** Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form such that $B(v, v) = 0$, for all $v \in V$. Then $B \equiv 0$.

**Theorem 6.5** (Diagonalisation Theorem). Let $B$ be a symmetric bilinear form on a finite-dimensional vector space over $\mathbb{F}$. Then there is a basis $v_1, \ldots, v_n$ of $V$ with respect to which the matrix of $B$ is diagonal:

$$B(v_i, v_j) = 0,$$

for all $1 \leq i \neq j \leq n$. We call $v_1, \ldots, v_n$ a diagonalising basis for $B$.

**Corollary 6.6.** Let $A \in M_{n \times n}(\mathbb{F})$ be symmetric. Then there is an invertible matrix $P \in \text{GL}(n, \mathbb{F})$ such that $P^T A P$ is diagonal.

#### 6.2.3 Sylvester’s Theorem

**Definitions.** Let $B$ be a symmetric bilinear form on a real vector space $V$.

Say that $B$ is **positive definite** if $B(v, v) > 0$, for all $v \in V \setminus \{0\}$.

Say that $B$ is **negative definite** if $-B$ is positive definite.
If $V$ is finite-dimensional, the signature of $B$ is the pair $(p,q)$ where

\[ p = \max \{ \dim U \mid U \leq V \text{ with } B|_{U \times U} \text{ positive definite} \} \]
\[ q = \max \{ \dim W \mid W \leq V \text{ with } B|_{W \times W} \text{ negative definite} \}. \]

**Theorem 6.7** (Sylvester’s Law of Inertia). Let $B$ be a symmetric bilinear form of signature $(p,q)$ on a finite-dimensional real vector space. Then:

- $p + q = \text{rank } B$;
- any diagonal matrix representing $B$ has $p$ positive entries and $q$ negative entries (necessarily on the diagonal!).

### 6.3 Application: Quadratic forms

**Convention.** We continue working with a field $\mathbb{F}$ where $1 + 1 \neq 0$.

**Definition.** A quadratic form on a vector space $V$ over $\mathbb{F}$ is a function $Q : V \to \mathbb{F}$ of the form

\[ Q(v) = B(v,v), \]

for all $v \in V$, where $B : V \times V \to \mathbb{F}$ is a symmetric bilinear form.

**Lemma 6.8.** Let $Q : V \to \mathbb{F}$ be a quadratic form with $Q(v) = B(v,v)$ for a symmetric bilinear form $B$. Then

\[ B(v,w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w)), \]

for all $v, w \in V$.

$B$ is called the polarisation of $Q$.

**Definitions.** Let $Q$ be a quadratic form on a finite-dimensional vector space $V$ over $\mathbb{F}$.

The **rank** of $Q$ is the rank of its polarisation.

If $\mathbb{F} = \mathbb{R}$, the **signature** of $Q$ is the signature of its polarisation.

**Theorem 6.9.** Let $Q$ be a quadratic form with rank $r$ polarisation on a finite-dimensional vector space over $\mathbb{F}$.

1. When $\mathbb{F} = \mathbb{C}$, there is a basis $v_1, \ldots, v_n$ of $V$ such that

\[ Q(\sum_{i=1}^{n} x_i v_i) = x_1^2 + \cdots + x_r^2. \]

2. When $\mathbb{F} = \mathbb{R}$ and $Q$ has signature $(p,q)$, there is a basis $v_1, \ldots, v_n$ of $V$ such that

\[ Q(\sum_{i=1}^{n} x_i v_i) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_r^2. \]