Series expansion for the sound field of rotating sources

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A method is derived for the fast, exact prediction of acoustic fields around rotating sources by using a series expansion which generalizes a previously published method for a circular piston. The technique gives exact predictions for the field outside the sphere containing the rotor in a computational time two orders of magnitude less than that required for direct numerical evaluation of the acoustic integrals. Its use is demonstrated by application to two sample problems characteristic of real aircraft propellers. © *2006 Acoustical Society of America.*
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I. INTRODUCTION

This paper presents an efficient, exact method for the prediction of noise from rotating sources, such as aircraft propellers. The problem to be considered is the evaluation of the integral for the sound radiated by a sinusoidally varying source distribution on a disk. A rotor of radius a has a geometry which can be decomposed into Fourier components in azimuth θ. For a given azimuthal order n the source on the disk is $s(r_1,\theta_1)=\epsilon s_n(r_1)\exp jn\theta_1$, where the coordinate system is that shown in Fig. 1, the subscript 1 refers to a variable of integration and $\epsilon \ll 1$.

We begin with the special case of $s_n\equiv 1$. In this case, the radiated acoustic field can be written as a Rayleigh integral[1]

$$p(\mathbf{x},\omega) = -\epsilon M_t^2 n^2 e^{jn\theta} I_n(a,r,z,\omega), \tag{1}$$

$$I_n(a,r,z,\omega) = \int_0^a \int_0^{2\pi} \frac{e^{j(kR'+n\theta_1)}}{4\pi R'} r_1 d\theta_1 dr_1,$$

$$R' = [r^2 + r_1^2 - 2rr_1 \cos\theta_1 + z^2]^{1/2}, \tag{2}$$

where ω is the source frequency, c is the speed of sound, and $k=\omega/c$. For a source which rotates at angular velocity Ω, $\omega=n\Omega$ and $M_t=\Omega a/c=ka/n$ is the tip rotational Mach number.

A number of methods have been devised for the evaluation of I_n. In particular, previous work has recognized its relationship to the standard Rayleigh integral with $n\equiv 0$. Oberhettinger's numerical approach for transient radiation,[2] which reduces I_0 to a one-dimensional integral, has been extended to the case of $n\neq 0$ (Refs. 1 and 3–5) and used to study the physical structure of rotating sound fields. In this paper, a similar approach is used and the exact series expansion of Mast and Yu[6] is applied to the problem of rotating sources to yield a very efficient, exact, general method of computing the noise from a rotor. Mast and Yu's approach to the problem of radiation from a piston is the most recent in a series of papers which use series expansions to evaluate the field. The earliest such work in English appears to be that of Spence,[7] although Mast and Yu cite two papers in German, those of Backhaus (1930) and Stenzel (1935). Further developments of the series expansion approach have included Carter and Williams' expansion,[8] the work of Wittman and Yaghjian[9] who introduced methods from electromagnetism and, in particular, the results of Hasegawa, Inoue and Matsuzawa[10] which formed the basis of Mast and Yu's recent work.

II. THEORY

In a recent paper,[6] a series expansion was developed for the calculation of harmonic radiation from a circular piston. The method of this paper is now used to develop a similar expansion for a disk-shaped source with azimuthal variation in source strength. The result presented, Eq. (6), gives the acoustic field around a rotor at distances greater than the rotor radius. It has not, so far, proven possible to derive an equivalent expansion for the region within the sphere enclosing the source.

A. Exact series

The expansion for a rotor noise field can be derived from the integral of Eq. (2) using the summation theorem for Bessel functions[11]

$$\frac{e^{jkR'}}{R'} = \frac{j\pi}{2(\rho S)^{1/2}} \sum_{m=0}^{\infty} (2m+1) J_{m+1/2}(k\rho) H_{m+1/2}^{(1)}(kS) \times P_m(\cos\theta_1 \sin\beta), \tag{3}$$

where

$$R' = [\rho^2 + S^2 - 2S\rho\cos\theta_1 \sin\beta]^{1/2},$$

$$r^2 + r_1^2 + z^2 = \rho^2 + S^2, \quad \sin\beta = r_1 r/\rho S,$$

J_ν is the Bessel function of the first kind and order ν, $H_\nu^{(1)}$ is the Hankel function of order ν, and P_m is the Legendre polynomial of order m.

In order to perform the integration over θ_1, we require the addition theorem for Legendre polynomials[11]

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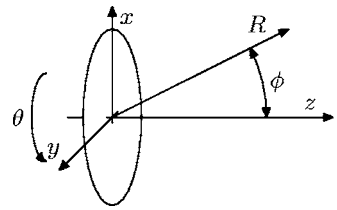

FIG. 1. Coordinate system for the disk source.

$$P_m(\cos\theta_1 \sin\beta) = P_m(\cos\beta)P_m(0) + 2\sum_{q=1}^{\infty}\frac{\Gamma(m-q+1)}{\Gamma(m+q+1)}P_m^q(\cos\beta) \times P_m^q(0)\cos q\theta_1,$$

where P_m^q is the associated Legendre function which is zero for $q>m$. Inserting this into

$$\int_0^{2\pi} P_m(\cos\theta_1 \sin\beta)e^{jn\theta_1}d\theta_1,$$

gives, for $n\neq 0$:

$$\int_0^{2\pi} P_m(\cos\theta_1 \sin\beta)e^{jn\theta_1}d\theta_1 = 2\pi\frac{\Gamma(m-n+1)}{\Gamma(m+n+1)}P_m^n(\cos\beta)P_m^n(0), \quad (4)$$

which, upon insertion into Eqs. (2) and (3), gives

$$I_n = j\pi^{3/2}2^{n-2}\sum_{m=0}^{\infty}\frac{(4m+2n+1)\Gamma(2m+1)}{\Gamma(2m+2n+1)\Gamma(m+1)\Gamma(1/2-m)} \times \int_0^a \frac{1}{(\rho S)^{1/2}}J_{n+2m+1/2}(k\rho)H^{(1)}_{n+2m+1/2}(kS) \times P^n_{n+2m}(\cos\beta)r_1dr_1, \quad (5)$$

where use has been made of the fact that $P^n_{n+2m+1}(0)\equiv 0$.

A series expansion can now be derived by making a suitable choice of coordinates. The choice of Mast and Yu,[6] $\rho=r_1$, $S=R=(r^2+z^2)^{1/2}$ with $\sin\beta=\sin\phi$ gives an expansion valid for $R>a$. With these coordinates, the integral can be evaluated[12] and

$$I_n = (\pi a^2)^{1/2}\frac{(-ka)^{n+1/2}}{(kR)^{1/2}}\sum_{m=0}^{\infty}A_mH^{(1)}_{n+2m+1/2}(kR) \times \mathrm{unP}^n_{n+2m}(\cos\phi) \times {}_1F_2\left[\frac{n+2m+2}{2};\frac{n+2m+4}{2},n+2m+3/2;-\left(\frac{ka}{2}\right)^2\right]\left(\frac{ka}{2}\right)^{2m+1/2},$$

$$A_m = (-1)^m\frac{2^{2m-1}}{n+2m+2}\frac{(2m-1)!!}{(2n+2m)!!(2n+4m-1)!!}, \quad (6)$$

where use has been made of the functional relations for the gamma function[11] and ${}_1F_2(\cdot)$ is a generalized hypergeometric function[13]

$${}_1F_2(a;b,c;x) = \sum_{n=0}^{\infty}B_nx^n,$$

$$B_n = \frac{(a)_n}{(b)_n(c)_n}\frac{1}{n!}, \quad (7)$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ is Pochammer's symbol.[11]

The series expansion in Eq. (6) is the main result of this paper. It gives an exact expansion for the acoustic field around a rotating source in terms of three special functions. It is worth noting that the special functions are "uncoupled:" for given k and n, if the polar angle ϕ is fixed, only the Hankel functions need be computed more than once per term so that the field can be calculated very quickly for a large number of points on a given radius.

B. General source distributions

The expansion of Eq. (6) can be extended to cover the general case where $s_n(r_1)$ varies with radius. The integral is rewritten

$$K_n = \int_0^a s_n(r_1)\int_0^{2\pi}\frac{e^{j(kR+n\theta_1)}}{4\pi R}r_1d\theta_1dr_1, \quad (8)$$

which upon integration by parts becomes

$$K_n = s_n(a)I_n(a,r,z,\omega) - \int_0^a \frac{ds_n}{dr_1}I_n(r_1,r,z,\omega)dr_1. \quad (9)$$

On the assumption that $s_n(a)=0$, i.e., that the source vanishes at a rotor blade tip, Eq. (9) can be integrated termwise to give

$$K_n = -\pi^{1/2}\frac{(-k)^{n+1/2}}{(kR)^{1/2}}\sum_{m=0}^{\infty}A_mH^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi) \times\left(\frac{k}{2}\right)^{2m+1/2}\times\int_0^a\frac{ds_n}{dr_1}r_1^{n+2m+2}{}_1F_2\left[-\left(\frac{kr_1}{2}\right)^2\right]dr_1, \quad (10)$$

where the parameters of the hypergeometric function have been dropped for brevity.

In computing the acoustic field of a real source, the integrals of Eq. (10) would have to be calculated numerically, although this would only have to be done once, since the integrals depend only on the source and are the same no matter what the observer position. In order to find an analytical formula, we use a source distribution which has been applied in asymptotic studies[14] $s_n=(a-r_1)^{\nu}$. Then the integral is readily evaluated

$$\int_0^a {}_1F_2[-(\alpha x)^2](a-x)^{\nu}x^ndx = a^{n+\nu+1}\sum_{q=0}^{\infty}B_q\frac{(2q+n)!}{(\nu+1)_{2q+n+1}} \times(-\alpha a)^{2q}, \quad (11)$$

and the acoustic field integral is

$$K_n = j\pi^{1/2}\frac{(-k)^n}{R^{1/2}}\sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi)$$
$$\times\left(\frac{k}{2}\right)^{2m+1/2}\times\int_0^a \nu(a-r_1)^{\nu-1}r_1^{n+2m+2}$$
$$\times{}_1F_2\left[-\left(\frac{kr_1}{2}\right)^2\right]\mathrm{d}r_1. \qquad (12)$$

III. RESULTS

Results are presented to demonstrate the efficiency and accuracy of the series expansion. For comparison, I_n and K_n are also computed by direct two-dimensional integration using Gaussian quadrature. The number of terms in the quadrature was adjusted to the minimum which gave a converged result. In evaluating the expansion, the Hankel functions were computed using the finite series[11]

$$H^{(1)}_{n-1/2}(kR) = j^{-n}\left(\frac{2}{\pi kR}\right)^{1/2} e^{jkR}$$
$$\times\sum_{q=0}^{n-1}(-1)^q\frac{(n+q-1)!}{q!(n-q-1)!}\frac{1}{(j2kr)^q}, \qquad (13)$$

the associated Legendre functions were computed using the implementation in the GNU Scientific Library[15] and the generalized hypergeometric function and its integral were evaluated by direct summation to machine precision (tolerance 10^{-15}). In summing the series, the convergence criterion was that the magnitude of the last term added be less than 10^{-6}. All calculations were performed on a GNU/Linux personal computer using code written in C using the GNU C compiler and the GNU Scientific Library.[15]

A. Numerical performance

The first test was a check on the accuracy and efficiency of the series expansion compared to direct integration. Figure 2 compares the real part of K_n computed using the two methods for $n=32$, $M_t=1.0$, $ka=32$ at a polar angle $\phi=\pi/2$, parameters characteristic of a high speed propeller of diameter 2.74 m rotating at 1200 rpm, similar to those used in a NASA study on high speed propellers.[16] As a check on the integration scheme, a variation in the radial source distribution was introduced by setting $\nu=1/2$. The direct integration was performed using 128 quadrature points in radius and azimuth. The series expansion converged after nine terms and, as can be seen, its accuracy is very good. To examine the computational effort, the calculation was repeated for a piston (i.e., $n=0$) with the same value of ka and the same convergence criterion. This required 39 terms for convergence, more than four times as many as in the rotor case.

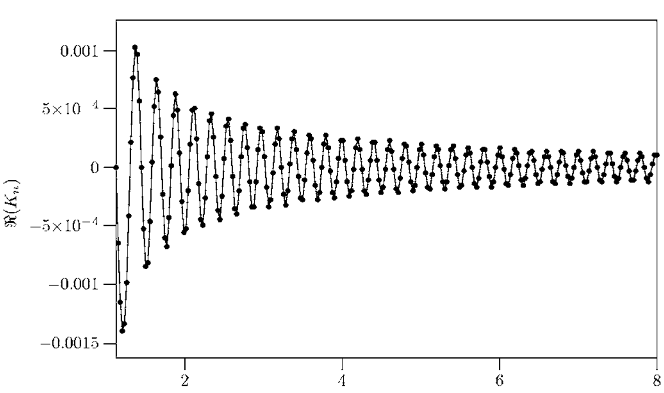

FIG. 2. Real part of acoustic integral I_n for $n=32$, $\nu=1/2$, and $\phi=\pi/2$; by direct integration and series evaluation: series solution solid; direct integration dots.

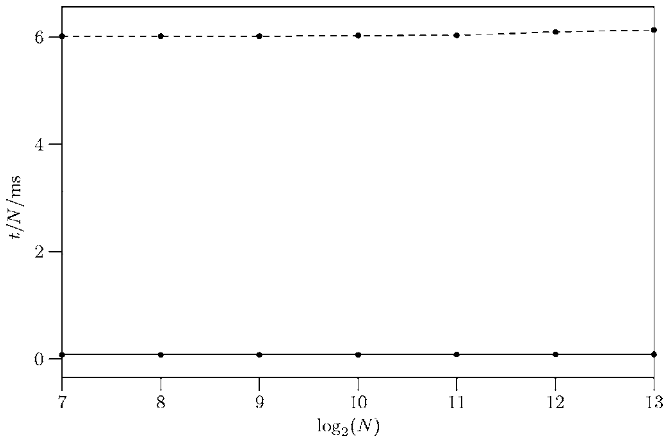

FIG. 3. Computational time per point t/N vs number of points N: solid line: series expansion; dashed line: direct integration.

Figure 3 shows the computational effort for the calculation as computational time per field point against $\log_2(N)$ where N is the number of field points. Over a wide range of N, the computation time per point for the series expansion is two orders of magnitude less than that required for direct integration, even though the result is no less accurate. The mean time per point for the series evaluation was 0.08 ms while that for direct integration was 6.0 ms.

B. Acoustic fields

Sample results are presented for the acoustic field around a rotating source, taking parameters representative of

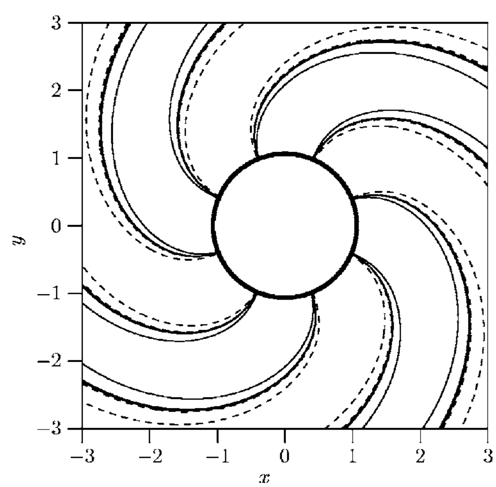

FIG. 4. Real part of acoustic pressure in the plane $z=0$ with $M_t=0.7$, $n=4$, and $\nu=1/4$. Contour levels $\pm 10^{-3}$, 10^{-4}, 10^{-5}, positive levels solid, negative levels dashed.

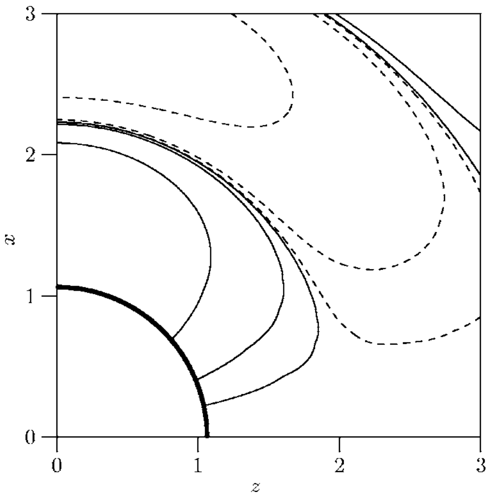

FIG. 5. Real part of acoustic pressure in the plane $y=0$ with $M_t=0.7$, $n=4$, and $\nu=1/4$. Contour levels $\pm 10^{-3}$, 10^{-4}, 10^{-5}, positive levels solid, negative levels dashed.

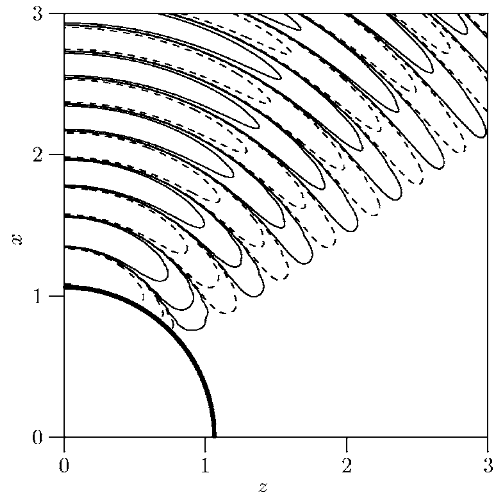

FIG. 7. Real part of acoustic pressure in the plane $y=0$ with $M_t=1.125$, $n=16$, and $\nu=1/4$. Contour levels $\pm 10^{-3}$, 10^{-4}, 10^{-5}, positive levels solid, negative levels dashed.

conventional rotors and of high speed propellers. The structure of rotating sound fields has been presented in detail in previous work[1,3–5] and so the examples shown here serve to illustrate application of the method.

The first results are calculated for $M_t=0.7$, $n=4$, $ka=2.8$, and $\nu=1/4$, equivalent to a 2.74 m propeller rotating at 830 rpm. Figure 4 shows the acoustic field $[\Re(K_n)]$ in the plane $z=0$ while Fig. 5 shows equivalent results for the plane $y=0$. A maximum of 28 terms were required in evaluating the series. Data were computed outside the sphere $R \geqslant 1.0625a$ with $a=1$, the inner radius being shown as a heavy line in each plot. The field has the structure described in previous work,[1] being composed of segments like those of an orange with the field decaying exponentially over the "peel" around the sonic radius $1/M_t=1.43$. Since the rotor lies completely inside the sonic radius, it radiates only weakly into the far field.

Figures 6 and 7 show similar results for a high speed rotor (diameter 2.74 m rotating at 1334 rpm) with $M_t=1.125$, $n=16$, $ka=18$, and $\nu=1/4$. A maximum of 23 terms

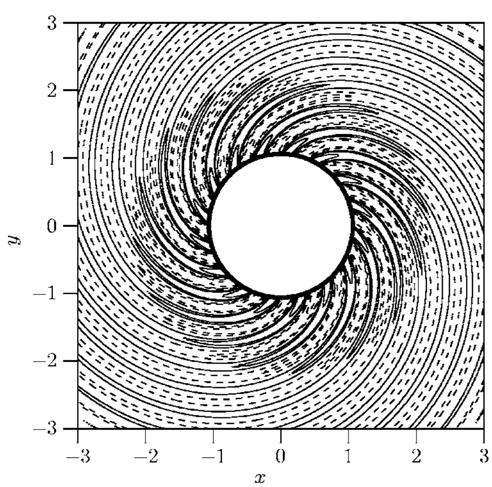

FIG. 6. Real part of acoustic pressure in the plane $z=0$ with $M_t=1.125$, $n=16$, and $\nu=1/4$. Contour levels $\pm 10^{-2}$, 5×10^{-3}, 2.5×10^{-3}, positive levels solid, negative levels dashed.

was needed in evaluating the series. In this case, part of the source lies outside the sonic radius $1/M_t=0.89$ and it can radiate strongly into the far field, without losing energy in the transition to the radiation zone. This strong radiation shows up as the long "swirls" spiraling out of the source disk in Fig. 6 and as the slow decay of the field on radial lines in Fig. 7. This plot also shows the sharp demarcation between the "quiet zone" of a supersonic rotor, near the z axis, and the "loud zone" where field points are subject to a source approaching at a Mach number of unity.

IV. CONCLUSIONS

A fast, exact method for the prediction of acoustic fields around rotating sources has been developed as a generalization of a technique for the prediction of time-harmonic fields around circular pistons.[6] The method uses an exact series expansion valid for general radial source distributions which gives a converged solution in a time two orders of magnitude smaller than that required for evaluation by direct two-dimensional integration. The accuracy and efficiency of the method have been demonstrated by computing the field around sample sources characteristic of actually existing propellers.

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