Series expansion for the sound field of a ring source

Michael J. Carley[a)]
Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, England

(Received 15 April 2010; revised 7 July 2010; accepted 16 September 2010)

An exact series expansion for the field radiated by a monopole ring source with angular variation in source strength is derived from a previously developed expression for the field from a finite disk. The derived series can be used throughout the field, via the use of a reciprocity relation, and can be readily integrated to find the field radiated by arbitrary circular sources of finite extent, and differentiated to find the field due to higher order sources such as dipoles and quadrupoles.
 [DOI: 10.1121/1.3500678]

PACS number(s): 43.28.Ra, 43.40.Rj, 43.20.Rz [AH] Pages: 3375–3380

I. INTRODUCTION

Many problems in acoustics are related to the sound radiated or scattered by systems with axial symmetry. These systems include rotors such as cooling fans and aircraft propellers, circular ducts, vibrating bodies such as baffled speakers, and bodies of rotation. In each case, to predict the radiated noise, there is a requirement to compute the field produced by an elemental ring source of given radius, frequency, and angular dependence. This calculation is an essential part of many noise prediction methods and there is a need for efficient techniques to perform it.

A second motivation for the study of ring sources is their use as a model problem for propeller and rotor noise. There are a number of approximations to the ring source field, developed, in the main, to examine the nature of the field and its variation with operating parameters,[1] or to give a far-field approximation for use in noise prediction.[2] These approximations have proven useful for industrial noise prediction, such as in aircraft propeller noise, and form the basis of many practical prediction techniques.

An approach which does not seem to have found much favor is the use of exact series expansions for the field of the ring source. There are numerous such expansions for disk sources of finite extent with examples covering a number of different configurations.[3–6] The number of published expansions for a ring source is quite small, however. One is the method of Matviyenko[7] which gives a five term recursion for the ring source of a given azimuthal order. A second, very recent paper, is that of Conway and Cohl[8] which gives series expansions for the ring source, in terms of Bessel and Hankel functions and associated Legendre functions. These series are accurate and easily implemented but they are expressed in terms of modified variables, of the type used in elliptic integral solutions of ring potential problems, or of toroidal type. The variables used in the series are, in the notation of this paper,

$$\left[\frac{4ar}{(a+r)^2+z^2}\right]^{1/2} \quad \text{and} \quad k\left[(a+r)^2+z^2\right]^{1/2},$$

where a is the source radius, k is the wavenumber, and (r, z) is the field point in cylindrical coordinates, as defined in Sec. II. The first difficulty in applying the expansions of Ref. 8 lies in the requirement to differentiate the expansions in order to find the field due to a dipole source. Second, it is difficult to integrate the terms of the series in order to find expansions for the field radiated from a finite disk. Both of these difficulties arise because the source radius and the observer coordinates do not appear explicitly in the expansions but are buried inside functions of variables of the form shown above.

In this paper, we take a previously published, quite simple, series for a finite disk source[4] and use it to derive an expansion for the field from a ring source. The derivation depends on routine use of mathematical tables and yields an expansion expressed in physical variables which can, if necessary, be integrated to give a series for the field of a finite source with arbitrary radial variation in source strength.

II. ANALYSIS

The problem to be considered is shown in Fig. 1. In cylindrical coordinates (r, θ, z), we require the field radiated by a ring monopole source at radius a in the plane $z = 0$, with source strength $\exp[j(n\theta_1 - \omega t)]$. Inserting the Green's function for the Helmholtz equation, the radiated field is given by

$$e^{-j\omega t}\int_0^{2\pi}\frac{e^{j(kR'+n\theta_1)}}{4\pi R'}d\theta_1,$$

with

$$R' = [r^2 + a^2 - 2ra\cos(\theta-\theta_1) + z^2]^{1/2},$$

and wavenumber $k = \omega/c$.

Suppressing the time dependence, the radiated field can then be written $\exp[jn\theta]R_n(k, a, r, z)$,

$$R_n(k,a,r,z) = \int_0^{2\pi}\frac{e^{j(kR'+n\theta_1)}}{4\pi R'}d\theta_1,$$
$$R' = [r^2 + a^2 - 2ra\cos\theta_1 + z^2]^{1/2}. \qquad (1)$$

We note that there is a reciprocity relation such that R_n is unchanged if a and r are switched.

[a)]Author to whom correspondence should be addressed. Electronic mail: m.j.carley@bath.ac.uk

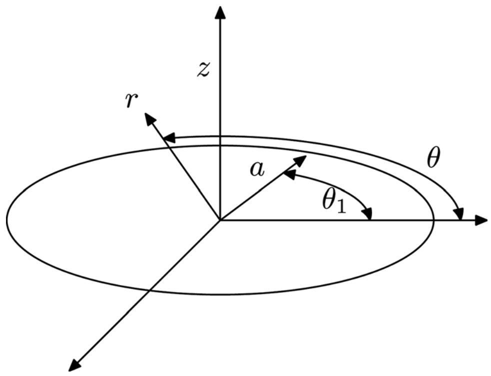

FIG. 1. Coordinate system for calculations.

A simple, exact series expansion for R_n can now be derived using previously developed results for a finite disk source. The starting point is the integral expression for the field radiated by a source of unit strength on the disk $r_1 \leq a$, $z = 0$,

$$I_n(k, a, r, z) = \int_0^a R_n(k, r_1, r, z) r_1 dr_1, \tag{2}$$

which has an exact series expansion,[4]

$$\begin{aligned} I_n &= (\pi a^2)^{1/2} \frac{(-ka)^{n+1/2}}{(kR)^{1/2}} \\ &\quad \times \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P^n_{n+2m}(\cos\phi) \\ &\quad \times {}_1F_2\left[\frac{n+2m+2}{2}; \frac{n+2m+4}{2}, n+2m+3/2; -\left(\frac{ka}{2}\right)^2\right] \left(\frac{ka}{2}\right)^{2m+1/2}, \\ A_m &= (-1)^m \frac{2^{2m-1}}{n+2m+2} \frac{(2m-1)!!}{(2n+2m)!!(2n+4m-1)!!}, \end{aligned} \tag{3}$$

where $R = [r^2 + z^2]$ is the distance of the observer from the origin, $\phi = \cos^{-1} z/R$ is the polar angle of the observer, $H^{(1)}_\nu$ is the Hankel function of the first kind of order ν, P^q_m is the associated Legendre function, ${}_1F_2(\cdot)$ is a generalized hypergeometric function,[9]

$$\begin{aligned} {}_1F_2(a; b, c; x) &= \sum_{n=0}^{\infty} B_n x^n, \\ B_n &= \frac{(a)_n}{(b)_n (c)_n} \frac{1}{n!}, \end{aligned} \tag{4}$$

and $(a)_n = \Gamma(a + n)/\Gamma(a)$ is Pochhammer's symbol.[10] This expansion for a finite disk source is valid outside the sphere containing the source, i.e., for $R > a$; the reciprocity relation which allows swapping of r and a will be used to evaluate R_n for $R < a$.

Differentiating with respect to a gives an expression for the field radiated by a ring source of radius a,

$$\frac{1}{a}\frac{\partial I_n}{\partial a} = R_n(k, a, r, z). \tag{5}$$

Likewise, differentiating Eq. (3),

$$\begin{aligned} \frac{\partial I_n}{\partial a} &= j^{2n+1} \frac{\pi^{1/2}}{(kR)^{1/2}} \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P^n_{n+2m}(\cos\phi) \\ &\quad \times \frac{(ka)^{2m+n+1}}{2^{2m-1/2}} \left\{ \frac{n+2m+2}{2} {}_1F_2\left[\frac{n+2m+2}{2}; \frac{n+2m+4}{2}, n+2m+3/2; -\left(\frac{ka}{2}\right)^2\right] \right. \\ &\quad \left. - \left(\frac{ka}{2}\right)^2 {}_1F_2'\left[\frac{n+2m+2}{2}; \frac{n+2m+4}{2}, n+2m+3/2; -\left(\frac{ka}{2}\right)^2\right] \right\}, \end{aligned} \tag{6}$$

where the derivative ${}_1F_2'$ is taken with respect to the argument. Using the relation

$$\begin{aligned} &{}_1F_2(a_1; b_1, b_2; x) a_1 + {}_1F_2'(a_1; b_1, b_2; x) x \\ &\quad = {}_1F_2(a_1 + 1; b_1, b_2; x) a_1, \end{aligned}$$

Eq. (6) can be rewritten

$$\begin{aligned} \frac{\partial I_n}{\partial a} &= j^{2n+1} \frac{\pi^{1/2}}{(kR)^{1/2}} \sum_{m=0}^{\infty} A_m H^{(1)}_{n+2m+1/2}(kR) P^n_{n+2m} \\ &\quad \times (\cos\phi) \frac{(ka)^{2m+n+1}}{2^{2m+1/2}} (n+2m+2) {}_1F_2 \\ &\quad \times \left[\frac{n+2m+4}{2}; \frac{n+2m+4}{2}, n+2m+3/2; -\left(\frac{ka}{2}\right)^2\right], \end{aligned}$$

noting the change in the first parameter of the hypergeometric function.

Using the cancellation property of hypergeometric functions

$${}_1F_2(a_1; a_1, b_2; x) = {}_0F_1(; b_2; x)$$

and the relation[11]

$${}_0F_1\left[; \nu+1; -\left(\frac{z}{2}\right)^2\right] = \Gamma(\nu+1)\left(\frac{2}{z}\right)^\nu J_\nu(z),$$

where J_ν is a Bessel function of the first kind,

$$\begin{aligned} &{}_1F_2\left[\frac{n+2m+4}{2}; \frac{n+2m+4}{2}, n+2m+3/2; -\left(\frac{ka}{2}\right)^2\right] \\ &\quad = \pi^{1/2} \frac{(2n+4m+1)!!}{2^{n+2m+1}} \left(\frac{2}{ka}\right)^{n+2m+1/2} J_{n+2m+1/2}(ka), \end{aligned} \tag{7}$$

we find an expansion for the field radiated by a ring source,

$$R_n(k,a,r,z) = \frac{1}{a}\frac{\partial I_n}{\partial a} = j^{2n+1}\frac{\pi}{4}\frac{1}{(aR)^{1/2}} \times \sum_{m=0}^{\infty}(-1)^m \frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!} \times H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\varphi) \times J_{n+2m+1/2}(ka). \quad (8)$$

Equation (8) is the main result of the paper. It is an exact series expansion for the field radiated by an oscillating ring source of radius a to any point with $R > a$. If it is required to compute the field at points $R < a$, this can be done using the reciprocity relation which allows switching of r and a in the ring source integral of Eq. (5). The expansion is remarkably simple, containing only one special function, P^n_{n+2m}, given that the Bessel and Hankel functions can be evaluated as finite sums of elementary functions,[10]

$$J_{n+1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \times \sum_{q=0}^{n}\frac{1}{q!}\frac{(n+q)!}{(n-q)!}\frac{\cos[x-(n-q+1)\pi/2]}{(2x)^q}, \quad (9a)$$

$$H^{(1)}_{n-1/2}(x) = j^{-n}\left(\frac{2}{\pi x}\right)^{1/2} \times e^{jx}\sum_{q=0}^{n-1}\frac{j^q}{q!}\frac{(n+q-1)!}{(n-q-1)!}\frac{1}{(2x)^q}. \quad (9b)$$

We further note that this expansion has a form very similar to that of the expansion of the Helmholtz Green's function for a point source using the "summation theorem" for Bessel functions[10] or in terms of spherical harmonics.[12]

A. Higher order sources

An important feature of the derivation of Eq. (8) in physical variables is that it is easily differentiated to give the fields generated by higher order sources. Using standard relations for the special functions, given in mathematical tables,[10] the axial and radial dipole source expansions are

$$\frac{\partial R_n}{\partial z} = j^{2n+1}\frac{\pi}{4}\frac{1}{R}\frac{1}{(aR)^{1/2}}\sum_{m=0}^{\infty}(-1)^m \times \frac{(2n+4m-1)(2m-1)!!}{(2n+2m)!!}J_{n+2m+1/2}(ka) \times \Big[kR\cos\phi H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi) - (2m+1)H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m+1}(\cos\phi)\Big], \quad (10a)$$

$$\frac{\partial R_n}{\partial r} = j^{2n+1}\frac{\pi}{4}\frac{1}{R}\frac{1}{(aR)^{1/2}}\sum_{m=0}^{\infty}(-1)^m \times \frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!}J_{n+2m+1/2}(ka) \times \Big[n\frac{H^{(1)}_{n+2m+1/2}(kR)}{\sin\phi}P^n_{n+2m}(\cos\phi) + H^{(1)}_{n+2m+1/2}(kR)P^{n+1}_{n+2m+1}(\cos\phi) + kR\sin\phi H^{(1)}_{n+2m-1/2}(kR)P^n_{n+2m}(\cos\phi)\Big], \quad (10b)$$

$$\frac{\partial R_n}{\partial a} = j^{2n+1}\frac{\pi}{4}\frac{1}{a}\frac{1}{(aR)^{1/2}}\sum_{m=0}^{\infty}(-1)^m \times \frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!} \times H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi)[kaJ_{n+2m-1/2}(ka) - (n+2m+1)J_{n+2m+1/2}(ka)], \quad (10c)$$

with $\partial R_n/\partial a$ being required when the reciprocity relation for r and a is used. The tangential dipole field found by differentiation with respect to θ is given by jnR_n/r.

B. Finite disk source

A major application of a ring source evaluation method is in rotor acoustics where the radiated field is given by integrals of the form

$$p_n(k,a,r,z) = \int_0^a s(r_1)R_n(k,r_1,r,z)r_1\mathrm{d}r_1, \quad (11)$$

where $s(r_1)$ is a radial source function whose value depends on the rotor geometry and/or loading.

For points lying outside the sphere containing a rotor of radius a, substitution of Eq. (8) into Eq. (11) gives a series expansion for the acoustic field of the rotor,

$$p_n = j^{2n+1}\frac{\pi}{4}\frac{1}{R^{1/2}}\sum_{m=0}^{\infty}(-1)^m\frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!} \times H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi)s_{n+2m},$$

$$s_{n+2m} = \int_0^a s(r_1)J_{n+2m+1/2}(kr_1)r_1^{1/2}dr_1, \quad (12)$$

so that the coefficients s_{n+2m} are given by a Hankel transform of the radial source term. It has been known for many years that the far-field noise from a rotor is given by a Hankel transform of the radial source term[2,13] based on integer order Bessel functions. The expansion of Eq. (12) employs Hankel transforms of order integer plus one half, to give a prediction which is valid in both the near and far fields, with the coefficients s_{n+2m} being independent of observer position. A similar approach can be used to derive expansions for the dipole fields based on integrals over the source.

C. Far-field approximation

A far-field approximation to Eq. (8) can be derived on the assumption $kR \to \infty$ and using Eq. (9b) to write

$$H^{(1)}_{n+2m+1/2}(kR) \approx j^{-n-2m-1}\left(\frac{2}{\pi kR}\right)^{1/2}e^{jkR},$$

resulting in

$$R_n(k,a,r,z) \approx j^n \left(\frac{\pi}{2ka}\right)^{1/2} \frac{e^{jkR}}{R} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2n+2m)!!} \times (n+2m+1/2) J_{n+2m+1/2}(ka) \times P_{n+2m}^{n}(\cos\phi), \quad kR \to \infty. \tag{13}$$

Likewise, Eq. (12) can be approximated

$$p_n \approx \left(\frac{\pi}{2k}\right)^{1/2} \frac{e^{jkR}}{R} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2n+2m)!!} \times (n+2m+1/2) s_{n+2m} P_{n+2m}^{n}(\cos\phi), \quad kR \to \infty. \tag{14}$$

These series expansions can be truncated when $n + 2m + 1/2 > ka$ due to the exponential decay of the Bessel function for order larger than argument. In the case of a subsonic rotor, for which $ka < n$, only the first term of each series need be retained.

III. RESULTS

To assess the accuracy and efficiency of Eqs. (8) and (10), a number of test cases were calculated and compared to the results from numerical integration. The numerical evaluations were performed using a quadrature method implemented in the GNU Scientific Library.[14] This method is capable of dealing with the near singularities which occur when the field point approaches the ring source. The cases considered are $k = 14$, $n = 9$, and $a = 2^{-1/2}$ with $\phi = 45°$, $0 \le R \le 4$ and $\phi = 90°$, $1/2 \le R \le 1$. This second case was chosen to test the series behavior in the source plane to see how it handles singularities as the field point approaches the source. The convergence criterion for the series evaluation and for the numerical quadrature was an absolute error of 10^{-6} with the maximum number of terms evaluated in the series limited to 32. The series was evaluated for the monopole and dipole terms simultaneously and, for $R > a$, the calculation was performed for all points at once, taking advantage of the fact that the Bessel functions and associated Legendre functions need to be computed only once per value of ϕ. All calculations were performed using programs written in C, using the GNU C compiler and the GNU Scientific Library.[14]

Figures 2 and 3 show the results for $\phi = \pi/4$. In the first case, the monopole source, the series is seen to perform well in matching the numerically evaluated field. The computational time per point for the series evaluation was half of that required for the numerical integration. The dipole evaluation, Fig. 3, shows a similarly close match to the numerical result. In this case, the series was not evaluated near $R = a$, a point where, strictly, the series is not valid.

In the in-plane case, $\phi = \pi/2$, the range of R is restricted to allow examination of the behavior of the results around the source radius and, in particular, to see how the series deals with the singularity in $\partial R_n/\partial r$, the dipole term. The radial dipole has been chosen for testing since the axial dipole field is identically zero in the plane $z = 0$. Figure 4 shows a good match between the series and numerical approaches for the monopole integral with the singularity being well captured. The results for the radial dipole, Fig. 5, show similar behavior. The plot for the real part of $\partial R_n/\partial r$, Fig. 5(a), shows the singularity as the source radius is approached. In this case, the numerical integration method broke down around a, where a gap has been left in the curve. The evaluation time per point in this case was 1.2 times greater for the series than for the direct numerical integration. This is because a large proportion of the points lay in the region $R < a$ and the field had to be computed at each individually using the reciprocity relation, rather than taking advantage of the simultaneous evaluation method possible for $R > a$.

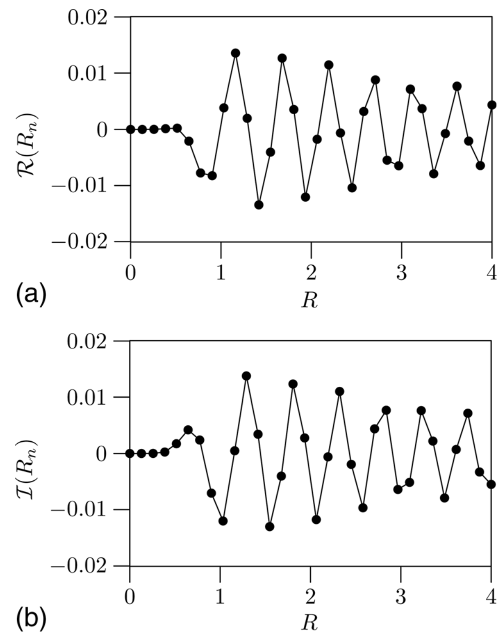

FIG. 2. Comparison of numerical (solid) and series (circle) computation of R_n, $\phi = \pi/4$: (a) Real part and (b) imaginary part.

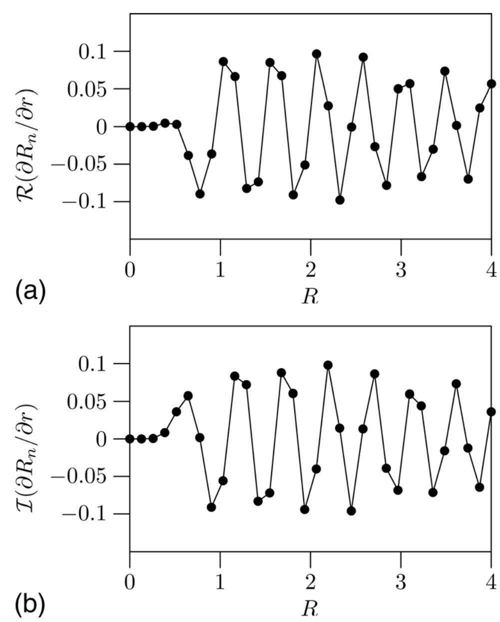

FIG. 3. Comparison of numerical (solid) and series (circle) computation of $\partial R_n/\partial r$, $\phi = \pi/4$: (a) Real part and (b) imaginary part.

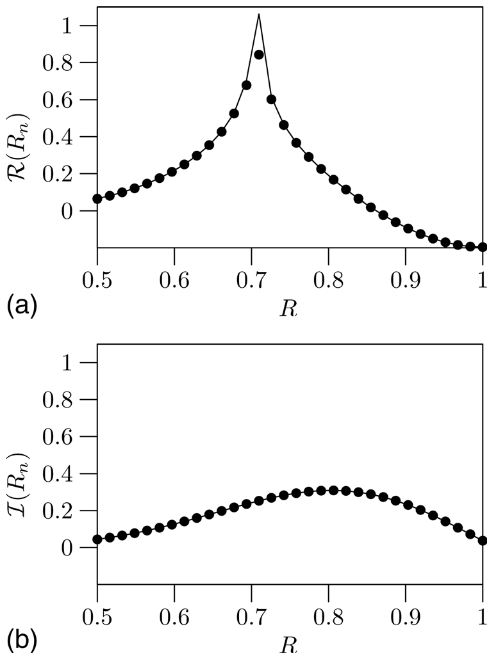

FIG. 4. Comparison of numerical (solid) and series (circle) computation of R_n, $\phi = \pi/2$: (a) Real part and (b) imaginary part.

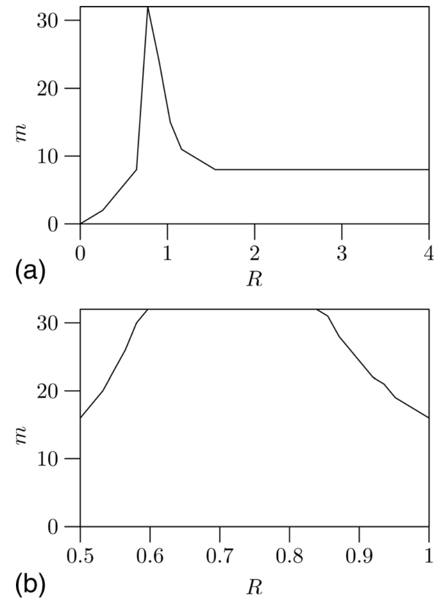

FIG. 6. Number of terms in series evaluation of field: (a) $\phi = \pi/4$ and (b) $\phi = \pi/2$.

Figure 6 shows the number of terms required to evaluate the series at the two polar angles considered. The number of terms was limited to 32, as can be seen around the source radius in both cases. Note that the number of terms is the maximum required for the evaluation of all terms, monopole and dipole, at each point so that the driving factor here is the number of terms required to evaluate the gradients near a. This is especially clear in Fig. 6(b) showing the behavior for a small region around a in the source plane.

Finally, Fig. 7 shows a comparison between the exact series and the single-term far-field approximation for a subsonic rotor of Eq. (13), for a subsonic ($ka < n$) source. The match is seen to be very good, especially as kR increases.

IV. CONCLUSIONS

A simple exact series expansion for the acoustic field radiated by a monopole ring source has been developed and derived from a previous result for a finite disk. The series has been tested numerically and compared to another recently published expansion for the Green's function for a Helmholtz problem in cylindrical coordinates. Since it is

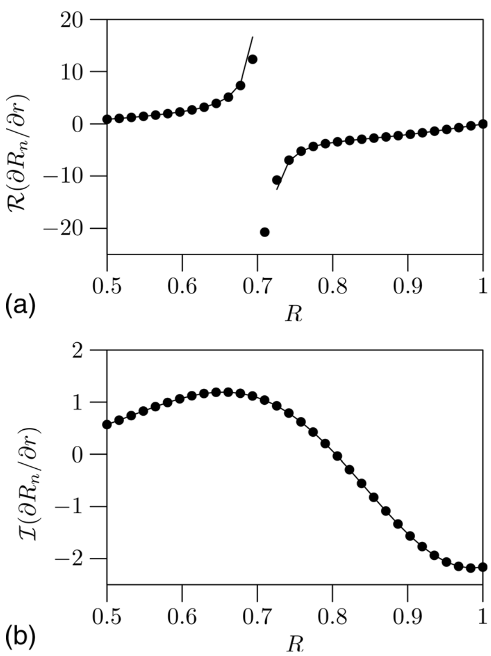

FIG. 5. Comparison of numerical (solid) and series (circle) computation of $\partial R_n/\partial r$, $\phi = \pi/2$: (a) Real part and (b) imaginary part.

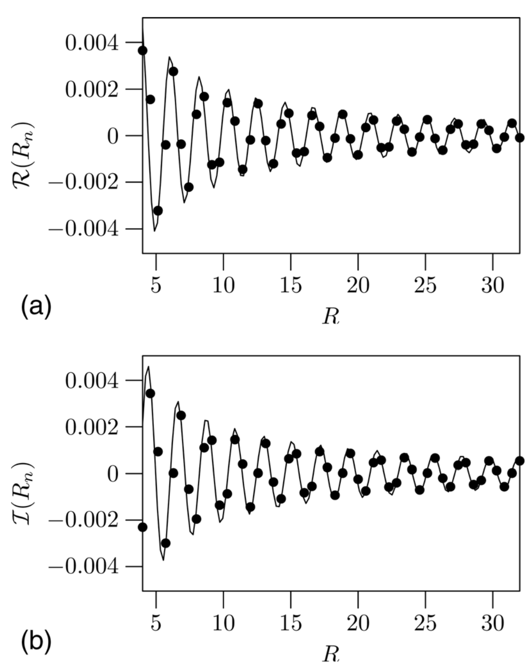

FIG. 7. Exact series (solid line) and single-term far-field approximation (circles) for $k = 3$, $n = 4$, $a = 1$, $\phi = \pi/4$: (a) Real part and (b) imaginary part.

based on physical variables, the series is easily integrated to give an expansion for finite sources, such as rotors, with arbitrary radial variation in source strength and is also easily differentiated to find the fields due to higher order sources such as dipoles and quadrupoles.

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