CONVECTIVE FLOW OF A BINGHAM FLUID IN INTERNALLY-HEATED POROUS CAVITIES

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We study the onset of convection within a rectangular porous cavity which is saturated with a Bingham fluid and subjected to a uniform internal heat generation. When such a cavity is saturated by a Newtonian fluid then convection takes place at all nonzero values of the Darcy-Rayleigh number, \( \text{Ra} \). In such cases convection takes the form of two contra-rotating cells with flow down the cold sidewalls when \( \text{Ra} \) first increase from zero. However, when the enclosure is saturated by a Bingham fluid, then we find that the cavity remains stagnant until the Darcy-Rayleigh number is sufficiently large that buoyancy overcomes the yield threshold.

Numerical solutions are obtained using a second order accurate finite difference methodology where convergence is accelerated using line-relaxation. The presence of the yield surfaces, which mark the boundaries of stagnant regions, is modelled by means of a regularisation of the yield threshold. It is found that the critical value of \( \text{Ra} \) above which convection arises depends roughly linearly on the value of \( \text{Rb} \), which may be described as a convective porous Bingham number. When the cavity has a sufficiently large aspect ratio the fluid admits more than one stable steady state solution.

KEY WORDS: Convection; Porous Medium; Bingham fluid; Onset; Internal Heat Generation

1. INTRODUCTION

The flow of Bingham fluids in porous media is a well-established topic with a variety of important industrial applications, particularly some in the oil industry. A Bingham fluid is characterized by having a yield stress, by which it is meant that the fluid exhibits a zero rate of strain unless the local stresses exceed a critical value called the yield stress. When such a fluid occupies a pipe and is subject to pressure gradient along that pipe, then no flow arises until the pressure gradient exceeds a critical value which is dependent on the yield stress of the fluid and the radius of the pipe. Thereafter the flow consists of two regions, an outer annular region which exhibits a shearing motion and a central region which is of the form of a moving but unyielded plug. In the context of porous media, for which this pipe flow is a simplified model, there is also a critical pressure gradient (or, equivalently, buoyancy forces when the medium is subjected to heating) below which no flow takes place. At sufficiently large pressure gradients the rate of flow is a linear function of the pressure gradient. Near the threshold gradient the variation in the rate of flow depends strongly on the microstructure of the porous medium; see Nash and Rees (2017).

The present work is part of a project which is examining the effect of the presence of a yield threshold on convective flows of a Bingham fluid in a porous medium. Previous works include some free convective boundary layer flows (Rees 2015, Rees and Bassom 2015, 2016, 2019a, 2019b). Rees (2015) shows that classical free convection boundary layer flows cannot arise within a semi-infinite domain because of the impossibility of entrainment into those boundary layers, although this result doesn’t preclude the existence of boundary layers with entrainment in finite domains. The other four papers are concerned with steady and unsteady one-dimensional boundary layer flows where entrainment does not occur. Some examples of the computation of nonlinear convection of a Bingham may be found in Rees (2016) and Rees (2020). The former is concerned with flow within a sidewall-heated rectangular cavity. For such cavities a Newtonian fluid would begin to convect as soon as the Darcy-Rayleigh number is nonzero, but when the fluid has a yield stress then a critical value of the Darcy-Rayleigh number needs to be exceeded before convection ensues. The latter paper forms the Bingham fluid analogue of the Darcy-Bénard problem. In this case the

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convection of a Newtonian fluid arises when the Darcy-Rayleigh number exceeds \(4\pi^2\), but when a Bingham fluid saturates the medium then the conduction state remains linearly stable. However, it remains possible for strongly nonlinear convection to arise if a thermal disturbance has a sufficiently large amplitude, and the nonlinear onset of convection is then via a fold bifurcation.

The present paper is concerned with how the presence of a yield threshold alters the stability characteristics of convection in a rectangular porous cavity which is subject to a uniform internal heating and where all four boundaries are maintained at the same temperature. This particular configuration has been the subject of quite a large number of studies when the fluid is Newtonian. When the cavity has an infinite aspect ratio the basic conducting state, which is realisable when the Darcy-Rayleigh number is sufficiently small, is independent of the horizontal coordinate, \(x\). Gasser and Kazimi (1976) then used a linearised stability theory to show that convection ensues once \(Ra\) exceeds a certain critical value; Nouri-Borujerdi et al. (2007) subsequently gave this value to be 471.3787 to seven significant figures. However, when the aspect ratio of the cavity is finite but large, the basic state consists of two weak circulations in the end-zones with the maximum temperature being obtained just above the centre of the cavity. The transition to a steady cellular convective state is now smooth in the sense that there is no bifurcation to a strongly convecting regime, although Choi et al. (1998) found a subcritical bifurcation to an alternative convection pattern where new cells appear near the middle of the upper surface of the cavity. Eventually, the flow becomes unsteady (Banu et al. 1998), and Banu (2000) indicates that, for example, for a cavity of aspect ratio equal to 2, an unsteady convecting state may be obtained when \(Ra\) is as low as 2200. Both Nield and Bejan (2017) and Storesletten and Rees (2019) list a very substantial number of authors who have varied the work of Gasser and Kazimi (1976) by the inclusion of other effects.

In this work we shall confine our attention to rectangular cavities with the aspect ratios of 1, 2 and 4. Two further nondimensional parameters govern the flow, namely the familiar Darcy-Rayleigh number, \(Ra\), and the Rees-Bingham number, \(Rb\), which may be regarded as porous convective Bingham number or, equivalently, a balance between the yield threshold and the applied buoyancy force. The general aims here are two-fold: (i) to determine how the critical value of \(Ra\) varies with \(Rb\) and the aspect ratio, and (ii) to obtain an understanding of how the fluid begins to flow once the critical value of \(Ra\) is exceeded. Our chosen range of values of \(Ra\) is below when might expect persistently unsteady convection of a Newtonian fluid. Given the extra resistance to flow which is caused by the Bingham fluid, it is an \textit{a priori} and perfectly reasonable expectation that flows will be steady under otherwise identical conditions.

2. GOVERNING EQUATIONS

We consider the convection of a Bingham fluid within a rectangular porous cavity where \(0 \leq x \leq L\) and \(0 \leq z \leq H\). The four boundaries are impermeable and are held at the temperature, \(\theta = \theta_c\).
The simplest unidirectional Darcy-Bingham law was given by Pascal (1981) and it may be written in the form,

\[
\begin{cases}
  -\frac{K}{\mu} \left(1 - \frac{G}{|p_z|}\right) p_z & \text{if } |p_z| > G, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( G \) is a threshold pressure gradient whose presence is due to the Bingham fluid having a yield stress. The values \( K \) and \( \mu \) denote permeability and dynamic viscosity, respectively, while \( p \) is the pressure. This may be extended to a frame-independent form in two dimensions as follows,

\[
\begin{cases}
  -\frac{K}{\mu} \left[1 - \frac{G (p_x^2 + p_z^2)^{1/2}}{p_z^2}\right] \begin{pmatrix} p_x \\ p_z \end{pmatrix} & \text{when } p_x^2 + p_z^2 > G^2, \\
  0 & \text{otherwise},
\end{cases}
\]

When buoyancy occurs in the vertical \((z)\) direction then Eq. (2) may be extended to the form,

\[
\begin{cases}
  -\frac{K}{\mu} \left[1 - \frac{G B}{B^2}\right] \begin{pmatrix} p_x \\ p_z \end{pmatrix} - \frac{\rho g \beta}{k_m} (\theta - \theta_c) & \text{when } B > G, \\
  0 & \text{otherwise},
\end{cases}
\]

where \( B = \sqrt{p_x^2 + (p_z - \rho g \beta (\theta - \theta_c))^2} \) and where the Boussinesq approximation has been assumed. The equation of mass conservation takes the familiar form,

\[
u_x + w_z = 0,
\]

while the steady form of the heat transport equation is,

\[
u \theta_x + w \theta_z = \alpha \left( \theta_{xx} + \theta_{zz} + \frac{Q}{k_m} \right),
\]

where \( \alpha \) is the thermal diffusivity, \( Q \) the rate of internal heat generation and \( k_m \) the thermal conductivity of the porous medium.

The governing equations given above may be recast in dimensionless form using the transformations,

\[
(x, z) \rightarrow H(x, z), \quad (u, w) \rightarrow \frac{\alpha \mu}{K} (u, w),
\]

\[
\theta \rightarrow \theta_c + \frac{qH^2}{k_m} \theta, \quad p \rightarrow \frac{\alpha \mu}{K} p.
\]

Hence Eqs. (4), (3) and (5) become,

\[
u_x + w_z = 0,
\]

\[
\begin{cases}
  -\frac{R b}{B} \begin{pmatrix} p_x \\ p_z - \frac{\alpha}{R} \theta \end{pmatrix} & \text{when } B > G, \\
  0 & \text{otherwise},
\end{cases}
\]

and

\[
\nabla^2 \theta + \nu \theta_x + w \theta_z + 1 = 0,
\]
respectively, where $B$ is now given by $B = \sqrt{p_z^2 + (p_z - Ra \theta)^2}$. We note that the final term in Eq. (10) represents the uniform generation of heat, but some other authors, including Gasser and Kazimi (1976), adopt a slightly different nondimensionalisation and therefore this constant may sometimes take different values. In these equations the values,

$$Ra = \frac{\rho g \beta (Q H^2/k_m) K H}{\mu \alpha} \quad \text{and} \quad Rb = \frac{G K H}{\mu \alpha},$$

are the Darcy-Rayleigh and Rees-Bingham numbers, respectively (see Rees 2016). The third nondimensional parameter is $A = L/H$, the aspect ratio of the cavity.

### 2.1 Regularisation

The simulation of the flow of Bingham fluids is complicated greatly by the need to determine where the yield surface is. In porous media this locus divides regions of flow from regions which are stagnant. We may motivate the required regularisation by comparing the following dimensionless form of Eq. (1),

$$w = \begin{cases} 
- \left(1 - \frac{Rb}{|p_z|}\right) p_z & |p_z| > Rb, \\
0 & \text{otherwise},
\end{cases}$$

with its regularized form,

$$w + Rb \tanh(cw/Rb) = -p_z.$$  

Here $c$ is an adjustable parameter and its role is to soften/smoothen the effect of the piecewise linear model of Pascal (1981). Indeed Pascal’s threshold model is recovered when $c \to \infty$ (see Rees 2016). When $w$ is sufficiently small, the left hand side of Eq. (13) takes the form, $(1 + c)w$, and therefore stagnant regions are modelled by a fluid with a viscosity that is $(1 + c)$ larger than the post-yield viscosity. When $w$ is sufficiently large and positive, both Eqs. (12) and (13) reduce to $w = -p_z - Rb$. This well-tested regularisation plays the same sort of role here as that of the regularisation of Papanastasiou (1987) for pure Bingham fluids.

The analogous frame-invariant system which governs isotropic two-dimensional convection is, therefore,

$$\left[1 + Rb \frac{\tanh(cq)}{q}\right] u = -p_z,$$

$$\left[1 + Rb \frac{\tanh(cq)}{q}\right] w = -p_z + Ra \theta,$$

where the fluid speed, $q$, is given by,

$$q^2 = u^2 + w^2.$$  

The streamfunction may be introduced in the usual way using $u = \psi_z$ and $w = -\psi_x$, and we obtain the following momentum equation,

$$\nabla^2 \psi + \frac{Rb \tanh(cq/Rb)}{q^3} \left[ \psi_z^2 \psi_{xx} - 2\psi_x \psi_z \psi_{xz} + \psi_z^2 \psi_{zz} \right] + \frac{c \sech^2(cq/Rb)}{q^2} \left[ \psi_z^2 \psi_{xx} + 2\psi_x \psi_z \psi_{xz} + \psi_z^2 \psi_{zz} \right] = -Ra \theta_x,$$

where $q^2 = \psi_z^2 + \psi_x^2$. Any potential for apparently singular behaviour of the coefficients when $q$ is small is avoided by replacing the left hand side of Eq. (17) by a five-term Maclaurin series in $q$ when $q < 10^{-3}$; this gives a seamless transition (to Fortran double precision accuracy) as $q$ crosses the $10^{-3}$ threshold. Finally, the heat transport equation is given by,

$$\theta_t = \nabla^2 \theta - \psi_z \psi_x + \psi_x \psi_z + 1.$$
To recap, we confine our attention to rectangular cavities of aspect ratio, $A$, where $0 \leq x \leq A$, and to values of $Ra$ that depend on the aspect ratio and the chosen value of $Rb$. Within these ranges of parameters the streamfunction field is antisymmetric about $x = \frac{1}{2} A$ and the temperature is symmetric. Given these symmetries we may adopt the following boundary conditions instead:

$$\psi = \theta = 0 \text{ on } x = 0, z = 0 \text{ and } z = 1, \quad \psi = \theta = 0 \text{ on } x = \frac{1}{2} A,$$

in order speed up the computations.

### 2.2 Numerical scheme

Two numerical codes were written. Each employed a standard second-order accurate finite difference discretisation on a uniform grid. The first code used a pseudo-transient method (Mallinson and de Vahl Davis 1973) whereby an additional single time derivative of $\psi$ was added to the right hand side of Eq. (17). A simple Euler time-stepping scheme was then used to march the solutions forward toward a steady-state. Steady state was deemed to be when the maximum change in $\theta$ between neighbouring timesteps was less than $10^{-8}$ in magnitude for 100 successive timesteps. The second code employed the line-relaxation Gauss-Seidel method to solve the steady state forms of Eqs. (17) and (10). We note that neither the use of Successive over-Relaxation nor multigrid acceleration could be used with universal success. The convergence criterion for this code was that the residual for the temperature field needed to be less than $10^{-6}$.

As has been discussed in detail in Rees (2016), there is a trade-off between the requirements of solving the governing equations accurately on a fine grid and the need to use as large a value of $c$ as is possible in order to model the yield surface accurately. Rees (2016) demonstrated that, for each chosen grid, there is a optimum value of $c$ which allows for the most accurate modelling of the yield threshold whilst still providing a robust convergence to a solution. In almost all cases here we use a uniform grid with the meshwidth, $1/128$, in both directions and with $c = 50$, although contour plots of the streamlines and isotherms use $c = 100$. The value of $c$ corresponds, in effect, to a fluid with a viscosity which is $(c + 1)$ times that of the equivalent Newtonian flow when body forces are sufficiently small, therefore the regularisation represents a pseudo-plastic fluid.

The two codes gave precisely the same results when the convergence criteria were made stronger and the codes were run until the difference equations had been solved to machine accuracy. However, the steady-state code was considerably faster and was therefore chosen. Regions which are deemed to be stagnant are defined in the same way as described in Rees (2016, 2020) and are shaded in orange in the streamline plots shown later.

### 3. NUMERICAL SOLUTIONS

Solutions will be presented for cavities of aspect ratio, $A = 2, 1$ and $4$, in that order, where $A = 2$ forms the datum case against which teh others are compared.

#### 3.1 Solutions when $A = 2$

Streamlines and isotherms for the cavity with aspect ratio, $A = 2$, are presented in Fig. 1 for a selection of values of $Rb$ and $Ra/Rb$. Here we were able to use $c = 100$ for regularisation — this is also true for streamlines and isotherms shown later. In all figures of this type we display the computational region which was used, namely the left hand half of the full cavity; the right hand half will appear as the mirror image although the streamfunction values will have the opposite sign. Regions of stagnant fluid are displayed using the orange shading. The respective rows in Fig. 1 correspond to $Rb = 5, 10$ and $20$, while the respective columns correspond to $Ra/Rb = 44, 60, 100$ and $200$.

The sidewall-heated cavity which was considered in Rees (2016) was shown to have the property that flow arises only when the Darcy-Rayleigh number exceeds a value which is a function of $Rb$. The same is true here, for when $Ra$ is too small, then buoyancy forces are too weak to overcome the yield threshold and thus the temperature field is given by the solution of $\nabla^2 \theta = -1$. The first column in Fig. 1 corresponds to $Rb = Rb = 44$ which is very close to the yield threshold. This may be inferred because the shape of the isotherms are almost identical to the conduction solution.
and thus the induced flow is very weak. Given that we have adopted a regularised form of Pascal’s law, this case represents the transition region between a highly viscous pseudo-plastic and a viscous fluid and therefore the precise details of the flow are at their least accurate. This is a reason why some streamlines appear to lie within that region which is deemed to be stagnant. This aspect may be improved by increasing the value of $c$ but, as pointed out by Rees (2016), this may only be achieved by increasing still further the resolution which is already composed of a $128 \times 128$ uniform mesh. Nevertheless, these flow patterns show clearly that there is a relatively strong downward-moving jet of fluid close to the left-hand sidewall, a characteristic which is shared by all the subfigures in Fig. 1.

For all three of the chosen values of $Rb$, each of the rows in Fig. 1 shows, not surprisingly, that the strength of the flow increases as $Ra$ increases, and therefore buoyancy forces become increasingly strong when compared with the yield threshold. Thus the region occupied by stagnant fluid decreases and, when $Ra$ is sufficiently large, the temperature field becomes increasingly distorted and the location at which $\theta$ achieves its maximum begins to rise as the overall anticlockwise circulation becomes stronger.

We note that, for a given value of $Ra/Rb$, flow patterns are quite similar, particularly for relatively small values of $Ra/Rb$. This is seen more clearly in Table 1 which lists some numerical data drawn from the computations plotted in Fig. 1. Thus when $Ra/Rb = 44$ the values of $\psi_{\text{max}}/Ra$, $\theta_{\text{max}}$ and $\phi$ are independent of the value of $Rb$, but this
independence is degraded as $Ra/Rb$ increases. The primary reason for this degradation is that the flow eventually becomes sufficiently strong that the nonlinear terms in the heat transport equation cease to be negligible. It may be noticed, for example, that the percentage of the cavity which is not stagnant is smaller when $Rb$ is large for fixed values of $Ra/Rb$; this is seen both in the numerical data in Table 1 and in the extent of the orange regions in Fig. 1.

Table 1. Values of the maximum values of $\psi$ and $\theta$ for chosen values of $Ra$ and $Rb$. Also shown is $\phi$, which is the fraction of the cavity in which the fluid is moving.

| Rb | Ra | $Ra/Rb$ | $|\psi|_{max}$ | $|\psi|_{max}/Rb$ | $\theta_{max}$ | $\phi$ |
|----|----|---------|----------------|-----------------|--------------|-------|
| 5  | 220| 44      | 0.0664         | 0.0133          | 0.1139       | 0.380 |
| 5  | 300| 60      | 0.364          | 0.0728          | 0.1135       | 0.714 |
| 5  | 500| 100     | 1.404          | 0.2808          | 0.1090       | 0.917 |
| 5  | 1000| 200     | 3.463          | 0.6926          | 0.0944       | 0.959 |
| 10 | 440| 44      | 0.133          | 0.0133          | 0.1138       | 0.379 |
| 10 | 600| 60      | 0.714          | 0.0714          | 0.1125       | 0.709 |
| 10 | 1000| 100    | 2.384          | 0.2384          | 0.1020       | 0.890 |
| 10 | 2000| 200    | 5.060          | 0.5060          | 0.0831       | 0.932 |
| 20 | 880| 44      | 0.264          | 0.0132          | 0.1137       | 0.378 |
| 20 | 1200| 60     | 1.343          | 0.0672          | 0.1094       | 0.700 |
| 20 | 2000| 100    | 3.584          | 0.1792          | 0.0932       | 0.859 |
| 20 | 4000| 200    | 6.897          | 0.3449          | 0.0726       | 0.913 |

The above Figure and Table have summarised the general behaviour of the flow and temperature fields for a $2 \times 1$ cavity when well within the regime where the flow is steady, and therefore no further Figures of this type need to be presented. However, a comprehensive summary of these flows is given in Figures 2 and 3 which respectively show how $\psi_{max}$ and $\theta_{max}$ vary with $Ra$ for integer values of $Rb$ which are in the range 0 to 10. The numerical solutions were obtained at increments of 10 in the Darcy-Rayleigh number and the regularisation constant was set at $\epsilon = 50$, a value which presented no convergence difficulties.

Figure 2. Variation with $Ra$ of $\psi_{max}$ for $Rb$ taking integer values between 0 and 10 and for $A = 2$. 

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Figure 3. Variation with $Ra$ of $\theta_{\text{max}}$ for $Rb$ taking integer values between 0 and 10 and for $A = 2$.

For Figure 2 we see that the strength of the flow, as measured by $\psi_{\text{max}}$, increases as $Ra$ increases due to the increasing strength of buoyancy forces, but it decreases as $Rb$ increases because the yield threshold of the Bingham fluid becomes stronger. Thus the overall strength of the flow at $Ra = 1000$ when $Rb = 10$ is roughly half that for a Newtonian fluid, for which $Rb = 0$. The effect of the regularised form of Pascal’s threshold model may also be seen when $\psi_{\text{max}}$ is small, i.e. when $Ra$ is below its critical value for flow. Each curve joins an envelope which has a slope that is roughly $1/51$ (i.e. $1/(c + 1)$) that of the $Rb = 0$ curve at $Ra = 0$. For example, when $Ra = 10$ then $\psi_{\text{max}} = 0.07052$ when $Rb = 0$, and $\psi_{\text{max}} = 0.00138$ for all the $Rb \neq 0$ curves. The trend which seen in Figure 2 for increasing values of $Rb$ continues when $Rb > 10$; ideally a larger value of the regularisation parameter, $c$, will be required in order to retain good accuracy when $\psi_{\text{max}}$ is small.

A similar behaviour may be found in Figure 3 which displays how $\theta_{\text{max}}$ varies with $Ra$ for the same values of $Rb$. Here, the natural tendency is for the maximum temperature in the cavity to decrease as $Ra$ increases. This counter-intuitive fact may be explained by the presence of a strong upflow in the middle of the cavity which brings cold fluid up from below. But here we see a straightforward monotonic variation in $\theta_{\text{max}}$ as $Ra$ and $Rb$ vary. In this case the upper envelope of the curves has the value 0.1139, which corresponds to pure conduction.

### 3.2 Solutions when $A = 1$

Figure 4. Streamlines (continuous) and isotherms (dashed) for $c = 100$ and $Rb = 10$ for a $1 \times 1$ cavity.
Now we shall consider the unit cavity and therefore the computations take place in the range, $0 \leq x \leq \frac{1}{2}$. We have chosen to use a $96 \times 128$ grid in order to ensure good resolution in the horizontal direction. A representative set of streamlines and isotherms is shown in Figure 4 and here we have chosen $Rb = 10$. Once again the subfigures correspond to the left hand half of the cavity.

When $Ra = 600$ we obtain a near-critical flow which is very much like the leftmost column of cases in Figure 1 where the flow is weak. The closer proximity of the sidewalls compared with those shown in Figure 1 means that more energy has to be expended in order to turn the downward flow near the left hand wall into the upward flow at the right-hand computational boundary. Therefore it is to be expected that this near-critical state arises at a larger value of $Ra/Rb$ when compared with $A = 2$.

From Figure 4 we see once again that, as the Darcy-Rayleigh number rises, the fraction of the cavity which is
Figure 7. Streamlines (continuous) and isotherms (dashed) for $c = 100$ and $Rb = 10$ for a $4 \times 1$ cavity, noting that the $Ra = 440$ case used $c = 400$.

stagnant decreases, and height at which the maximum temperature occurs rises from the centreline, $z = \frac{1}{2}$.

More detailed information may be found in Figures 5 and 6 which, like Figures 2 and 3, show respectively how $\psi_{\text{max}}$ and $\theta_{\text{max}}$ vary with $Ra$. These figures display many of the same tendencies as do Figures 2 and 3, namely that $\psi_{\text{max}}$ increases with $Ra$ but decreases with $Rb$, that the $\theta_{\text{max}}$ decreases with $Ra$ but increases with $Rb$. However, it is also clear that, for a given pair of values of $Ra$ and $Rb$ the strength of the flow is quite weak when compared with when $A = 2$ because of the greater lateral confinement.

3.3 Solutions when $A = 4$

This forms the final aspect ratio which will be discussed in detail and it represents a fairly shallow cavity. Yet again we depict the left-hand half of the cavity which occupies $0 \leq x \leq 2$.

The $Ra = 440$ case in Figure 7, for which $A = 4$, may be compared directly with its $Rb = 10$ and $Ra = 440$ counterpart in Figure 1, for which $A = 2$. There is a great deal of similarity between the two but, despite the fact that the respective flows are quite weak, the temperature fields in the range, $0 \leq x \leq 1$, differ slightly because of the quite different aspect ratios. Nevertheless, the fact that the isotherms at $x = 2$ in Figure 7 are horizontal suggests that the streamline and isotherm patterns in the range, $0 \leq x \leq 3$ will be independent of $A$ when $A > 4$. As before, we note that this case represents onset conditions where the regularisation has a very significant effect, and it is this which causes the presence of streamlines in the deemed stagnant region. For this particular case $c = 400$ was used in order to attempt to compute the nonstagnant region as carefully as possible; smaller values cause the streamlines to stray further into the shaded region.

Once $Ra$ exceeds 600, corresponding cases in Figures 1 and 7 no longer match because the nonstagnant region now spreads beyond $x = 1$ when $A = 4$ whereas it is forced to turn upwards at $x = 1$ when $A = 2$. As $Ra$ continues to increase, the region of moving fluid continues to expand until it occupies the full cavity in the $x$-direction. Given
that the flow is relatively weak at $x = 2$, the point at which $\theta$ is maximised is still very close to $z = \frac{1}{2}$, unlike when $A = 2$ or $A = 1$.

![Figure 8](image8.png)

**Figure 8.** Variation with Ra of $\psi_{\text{max}}$ and $\psi_{\text{min}}$ for Rb taking integer values between 0 and 10 and for $A = 4$. The red curves correspond to the second convection pattern. The bullet symbol denotes a fold bifurcation when Rb = 0.

![Figure 9](image9.png)

**Figure 9.** Variation with Ra of $\theta_{\text{max}}$ for Rb taking integer values between 0 and 10 and for $A = 4$. The red curves correspond to the second convection pattern.

Figures 8 and 9 show how the extremum values of $\psi$ and the maximum value of $\theta$ vary with Ra for the same range of values of Rb. Unlike when $A = 1$ and $A = 2$, the relatively large aspect ratio of the cavity admits two possible solutions for each value of Rb. Each of the black curves connects back to the zero-flow state at low values of Ra, and it is these which are computed naturally as Ra rises from zero. In all the cases depicted, including Rb = 0, the black curve terminates at a point where the iteration scheme is unable to continue — this is characterised by the scheme needing an increasing number of iterations to converge as Ra increases. For the type of iteration scheme which we
have used, such terminations have one of two causes: either (i) the flow is too strong or (ii) there is a bifurcation to a different solution branch. On the other hand, the red curves correspond to flows with a different number of cells, and both types of flow may be seen in Figure 10. For these curves the number of iterations also increase greatly as $Ra$ decreases towards their terminations. The shapes of the red $\psi_{\text{min}}$ curves in Figure 8 suggest very strongly that the lower limits form fold bifurcations, the solution curves then being presumed to turn back towards increasing values of $Ra$ and eventually joining the black curves at a subcritical bifurcation. While this sounds like a conjecture, the paper by Choi et al. (1998), used a spectral method with a curve-tracking methodology to demonstrate just that for Darcy-Brinkman flow in an internally heated unit-aspect-ratio cavity. Given that the present solution curves shown in Figures 8 and 9 were obtained using a Gauss-Seidel iteration scheme, all of them depict solutions which are stable to small perturbations. When saturated by a Newtonian fluid, we also obtain a fold bifurcation; The red $\psi_{\text{min}}$ curve for $Rb = 0$ was computed very carefully with very small increments in $Ra$ to approach the turning point as closely as possible. To the final three points a quadratic for $Ra$ in terms of $\psi_{\text{min}}$ was fitted, and the smallest value of $Ra$ estimated; this point forms the black disk in Figure 8, and therefore it must represent a fold bifurcation.

![Figure 10. Streamlines (continuous) and isotherms (dashed) for two different steady solutions for $Rb = 10$ and $Ra = 1400$ in a $4 \times 1$ cavity.](image)

### 3.4 Summary

Although the regularisation we have used softens the threshold for flow, we may, nevertheless, attempt to evaluate the critical value of $Ra$ at which convection begins. In the present case, we note that the $Rb = 0$ curve in Figure 3 rises almost perfectly linearly while $\psi_{\text{max}} < 1$, and this appears to be true also for all the other curves in the range $0.5 < \psi_{\text{max}} < 1$. Therefore we fitted a straight line to these separate sets of data and extrapolated back to give a rough indication of the value of $Ra$ for which $\psi_{\text{max}} = 0$. The result of this process is displayed in Figure 11, where we see an almost perfectly linear variation with $Rb$ for all three aspect ratios, although we think that this data becomes less reliable as $Rb$ approaches zero because of the nature of the regularisation.

We have already seen the similarity between the $(Ra, Rb) = (440, 10)$ cases depicted in Figures 1 and 7 because the flow and temperature fields are essentially unaffected by the width of the cavity, and therefore it is not surprising that the critical values of $Ra$ as a function of $Rb$ when $A = 2$ and $A = 4$ are almost identical. On the other hand, when $A = 1$, the width of the cavity restricts the flow, and therefore critical values of $Ra$ are larger. A straight line fit to the curves shown in Figure 11 suggests that critical Darcy-Rayleigh number for convection is given roughly by the values given in Table 2.
Figure 11. Variation with Rb of the critical value of Ra above which convection exists. Red denotes $A = 1$, green denotes $A = 1.25$, blue denotes $A = 1.5$, black denotes $A = 2$ and dotted red line denotes $A = 4$. The black disks represent the extrapolated critical values of Ra.

Table 2. Critical values of Ra/Rb as a function of the aspect ratio.

<table>
<thead>
<tr>
<th>$A$</th>
<th>Ra/Rb</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>55</td>
</tr>
<tr>
<td>1.25</td>
<td>48</td>
</tr>
<tr>
<td>1.5</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>44</td>
</tr>
<tr>
<td>4</td>
<td>44</td>
</tr>
</tbody>
</table>

4. CONCLUSIONS

Internally-heated porous cavities always admit flow when saturated by a Newtonian fluid, but when saturated by a Bingham fluid there is a critical value of Ra above which convection arises. Computationally, this critical value appears to depend linearly on the magnitude of the Rees-Bingham number, Rb. Given that the value of Rb expresses a balance between the magnitude of the buoyancy forces and the body force which is required to overcome the yield threshold of the fluid, then this suggests that Ra/Rb will be a constant for any given aspect ratio. The largest buoyancy force within the cavity occurs halfway down the vertical bounding surfaces, because this is where the basic conduction field finds its largest horizontal gradient. Therefore the convective pattern at onset consists of a narrow jet on the sidewalls with a wider recirculation region as the fluid has to ascend once more towards the upper surface. At larger values of Ra the fluid flow always remains the strongest near the sidewalls and there is a gradual reduction in the relative size of the stagnant regions as Ra increases.

We have found that the onset conditions are essentially independent of $A$ when $A \gtrapprox 2$ because the onset flow...
pattern and the temperature field for pure conduction are then almost independent of $A$. When the aspect ratio decreases from 2, the conduction temperature field becomes increasingly distorted, and the proximity of the sidewalls to one another also restricts fluid movement. Thus the critical value of $Ra$ increases as $A$ decreases.

We have also found that multiple steady states exist for moderate values of $Ra$ when the aspect ratio is sufficiently large. The computed steady states which we have presented were obtained using a Gauss-Seidel iteration scheme which means that they are linearly stable. Thus hysteresis could arise if one were to vary the strength of the internal heating effect as mediated by a slowly time-varying Darcy-Rayleigh number. While this hysteretical behaviour is not unknown for a Newtonian fluid (Choi et al. 1998), the range of values of $Ra$ over which it happens is much larger when the porous medium is saturated by a Bingham fluid.

For larger values of $A$ it is very likely that further complication will arise, and a careful study of the bifurcation structure will require the creation of a direct method of solution involving the Newton-Raphson scheme which would be capable of computing solutions which are linearly unstable.

REFERENCES

Banu, N., Free convection in fluid-saturated porous media, PhD thesis, Department of Mechanical Engineering, University of Bath, UK 2000.


