The onset of Darcy–Brinkman convection in a porous layer using a thermal nonequilibrium model—part I: stress-free boundaries

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SUMMARY

The paper deals with the onset of convection in a porous layer heated from below, by considering the case when the fluid and solid phases are not in local thermal equilibrium and when form-drag and boundary effects are included in the analysis. Analytical progress is facilitated by taking stress-free boundaries conditions. Asymptotic solutions for both small and large values of the scaled inter-phase heat transfer coefficient, \( H \), are presented and comparisons with the numerical solutions are performed. Excellent agreement is obtained between the asymptotic and the numerical results. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: porous medium; local thermal nonequilibrium; convection; linear stability

1. INTRODUCTION

The research presented here focuses on the porous medium version of the Bénard problem which has been studied extensively since the pioneering works of Horton and Rogers (1945) and Lapwood (1948) first appeared. Particular emphasis is given to the effects of including a two-field model for heat transport through the medium where the volume averaged temperature of the solid and fluid phases are generally different from one another. This effect is also known as local thermal non-equilibrium since, from a macroscopic point of view, thermal equilibrium does not occur even in steady-state convection, although it clearly must do at a microscopic level.

The earliest analysis of such non-equilibrium effects were presented by Schumann (1929) who considered a one-dimensional semi-infinite bed subject to a step-change in the inlet fluid temperature. Most of the more recent studies of thermal non-equilibrium effects have considered forced convective flows, and many of these have been reviewed in the chapters by Vafai and Amiri (1998) and Kuznetsov (1993). However, we are interested in the onset of free convection and the literature associated with this aspect is very limited. Combarnous (1972) performed

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finite difference calculations of the strongly non-linear flow and heat transfer in a unit square at only one Rayleigh number. More recently Banu and Rees (2002) considered the more general onset problem where criteria were sought to determine at what value of the Rayleigh convection would first occur.

In the present paper we extend the work of Banu and Rees (2002) by the inclusion of boundary effects as modelled by the Brinkman terms. Form-drag is also included, but it is quickly shown that these terms have no effect on stability criteria since the basic state whose stability is being analysed is one of no flow. We consider how non-LTE effects affect the onset criterion for the case of stress-free boundaries since, for these boundary conditions, it is possible to proceed entirely analytically as in Banu and Rees (2002). This assumption is relaxed in our companion paper (Rees and Postelnicu, 2002). We find that in both the LTE ($h \to \infty$) and non-LTE limits ($h \to 0$), the critical wave number tends towards $\pi$, but our analysis shows that at intermediate values of $h$, the critical wave number is always above $\pi$.

2. ANALYSIS

We consider a layer of porous medium with depth $d$ which is heated from below and cooled from above, as depicted in Figure 1. The upper surface is held at a temperature $T_c$ while the lower one is at $T_h (> T_c)$. It is assumed that both form-drag and boundary effects are significant, that the porous medium is isotropic but that local thermal equilibrium does not apply. Thus the governing equations, i.e. the continuity equation, a suitably extended Darcy’s law and the energy equation, subject to the Boussinesq approximation, take the forms

$$\nabla \cdot \mathbf{V} = 0$$

(1)

$$\frac{\rho_l}{\varepsilon} \frac{\partial \mathbf{V}}{\partial t} + \frac{\rho_l}{\varepsilon^2} \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \mu_e \nabla^2 \mathbf{V} - \frac{\mu_e}{K} |\mathbf{V}| \mathbf{V} + \rho_l \beta (T - T_c) y - \frac{\rho_l b}{\sqrt{K}} |\mathbf{V}| |\mathbf{V}|$$

(2)

$$u = 0, \quad v = 0, \quad T = T_c$$

$$u = 0, \quad v = 0, \quad T = T_h$$

Figure 1. Definition sketch of the horizontal layer.
\[ \varepsilon (\rho c)_T \frac{\partial T_f}{\partial t} + (\rho c)_T \mathbf{V} \cdot \nabla T_f = \varepsilon k_f \nabla^2 T_f + h(T_s - T_f) \]  

(3)

\[ (1 - \varepsilon) (\rho c)_s \frac{\partial T_s}{\partial t} = (1 - \varepsilon) k_s \nabla^2 T_s - h(T_s - T_f) \]  

(4)

The constants and variables used in these equations are defined in the Nomenclature. The boundary conditions are

\[ u = 0, \ v_y = 0, \ T = T_h \text{ at } y = 0 \]  

(5a)

\[ u = 0, \ v_y = 0, \ T = T_c \text{ at } y = d \]  

(5b)

where the velocity conditions correspond to the stress-free case. Equation (1)–(4) are nondimensionalised using the transformations

\[ \tilde{x} = \frac{1}{d} x, \quad \tilde{t} = \frac{(\rho c)_T d^2}{k_f} t, \quad \tilde{V} = \frac{\varepsilon k_f}{(\rho c)_T d} V, \]

\[ \tilde{p} = \frac{\mu k_f}{(\rho c)_T K} p, \quad \tilde{\theta} = \frac{T_f - T_c}{T_h - T_c}, \quad \tilde{\varphi} = \frac{T_s - T_c}{T_h - T_c} \]  

(6)

and the governing equations become

\[ \nabla \mathbf{V} = 0 \]  

(7)

\[ \varepsilon F_1 \frac{\partial \mathbf{V}}{\partial \tilde{t}} + F_1 \mathbf{V} \cdot \nabla \mathbf{V} = -\varepsilon^2 F_1 \nabla p + D \nabla^2 \mathbf{V} - \mathbf{V} + R \tilde{y} \mathbf{y} - F_2 \mathbf{V}|\mathbf{V}| \]  

(8)

\[ \frac{\partial \tilde{\theta}}{\partial \tilde{t}} + \mathbf{V} \cdot \nabla \tilde{\theta} = \nabla^2 \tilde{\theta} + H (\tilde{\varphi} - \tilde{\theta}) \]  

(9)

\[ \frac{\partial \tilde{\varphi}}{\partial \tilde{t}} + \mathbf{V} \cdot \nabla \tilde{\varphi} = \nabla^2 \tilde{\varphi} + \gamma H (\tilde{\theta} - \tilde{\varphi}) \]  

(10)

where the tildes have been omitted, for convenience of presentation. In Equations (8)–(10), the following constants were introduced:

\[ F_1 = \frac{\rho_f \kappa K}{\varepsilon^2 d^2 \mu_f}, \quad F_2 = \frac{\rho_f \kappa K^{1/2}}{d \mu_f}, \quad D = \frac{\mu_s}{\mu_f} \cdot \frac{K}{d^2}, \]

\[ H = \frac{hd^2}{\varepsilon k_f}, \quad \gamma = \frac{\varepsilon k_f}{(1 - \varepsilon)k_s}, \quad \alpha = \frac{(\rho c)_s}{(\rho c)_f} \cdot \frac{k_f}{k_s} \]  

(11)

and \( R = \frac{\rho_u \beta (T_h - T_c) k_d}{\varepsilon \mu_k t} \) is the Darcy–Rayleigh number based on the fluid properties. We note that the usual Rayleigh number, which is based on the mean properties of the porous medium is given by \( R_{\gamma}/(1 + \gamma) \). The boundary conditions (5) become

\[ u = 0, \ v_y = 0, \ \theta = \varphi = 1 \text{ on } y = 0 \]  

(12a)

\[ u = 0, \ v_y = 0, \ \theta = \varphi = 0 \text{ on } y = 1 \]  

(12b)

The basic conduction profile, whose stability is the subject of this short paper, is given by

\[ \mathbf{V} = 0, \ \theta = \varphi = 1 - y \]  

(13)
We focus our attention to the 2D case and we introduce the stream-function $\psi$, according to

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$  \hfill (14)

Here, $u$ and $v$ are the components of the velocity in the Cartesian $x$ (horizontal) and $y$ (spanwise) directions. The basic conduction profile given by (13) are perturbed by setting:

$$\psi = \Psi, \quad \theta = 1 - y + \Theta, \quad \varphi = 1 - y + \Phi$$  \hfill (15)

and after linearization, we obtain

$$cF_1 \frac{\partial}{\partial t}\left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} - D\left( \frac{\partial^4 \Psi}{\partial x^4} + 2\frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} \right) = R \frac{\partial \Theta}{\partial x}$$  \hfill (16)

$$\frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial \Psi}{\partial x} + H (\Phi - \Theta)$$  \hfill (17)

$$z \frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \gamma H (\Theta - \Phi)$$  \hfill (18)

It is easy to show that system (16)–(18) obeys the principle of exchange of stabilities. Consequently, we may set the time derivatives to zero and the problem becomes

$$-D\left( \frac{\partial^4 \Psi}{\partial x^4} + 2\frac{\partial^4 \Psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} \right) + \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = R \frac{\partial \Theta}{\partial x}$$  \hfill (19)

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial \Psi}{\partial x} + H (\Phi - \Theta) = 0$$  \hfill (20)

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \gamma H (\Theta - \Phi) = 0$$  \hfill (21)

subject to

$$\Psi = \Theta = \Phi = 0, \quad \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad \text{on} \ y = 0 \ \text{and} \ y = 1$$  \hfill (22)

Equations (19)–(22) admit solutions in the form

$$\Psi = A_1 \cos kx \sin \pi y, \quad \Theta = A_2 \sin kx \sin \pi y, \quad \Phi = A_3 \sin kx \sin \pi y$$  \hfill (23)

where $k$ is the horizontal wave number and the $A$-coefficients are constants. By substituting (23) into Equations (19)–(21), the following set of equations is obtained:

$$\left[ (k^2 + \pi^2) + D(k^2 + \pi^2)^2 \right] A_1 + kRA_2 = 0$$  \hfill (24)

$$kA_1 + (k^2 + \pi^2 + H)A_2 - HA_3 = 0$$  \hfill (25)

$$-\gamma HA_2 + (k^2 + \pi^2 + \gamma H)A_3 = 0$$  \hfill (26)
The condition that this homogeneous system has non-trivial solutions leads to an eigenvalue equation for $R$ in terms of $k, D, H$ and $\gamma$. After some algebra, we obtain

$$R = (\pi^2 + k^2)^2[1 + D(\pi^2 + k^2)] \cdot \frac{\pi^2 + k^2 + H(\gamma + 1)}{k^2(\pi^2 + k^2 + \gamma H)}$$

(27)

In the remainder of the paper we determine not only the variation of $R$ with $k$ for selected values of $H$, $D$ and $g$ but we minimize $R$ with respect to $k$ in order to find the smallest Rayleigh number at which convection may be expected. In Section 3 we consider the extreme cases of small $H$ (i.e. for very poor inter-phase heat transport) and large $H$ (i.e. for the local thermal equilibrium limit). In Section 4 we present exact numerical values for $R$ and the minimizing value of $k$.

3. ASYMPTOTIC ANALYSIS FOR BOTH SMALL AND LARGE VALUES OF $H$

3.1. Small $H$ analysis

We first perform a small-$H$ series expansion of the expression given by (27) to obtain

$$R = 1 + \frac{D(\pi^2 + k^2)}{k^2} \left[ (\pi^2 + k^2)^2 + (\pi^2 + k^2)H - \gamma H^2 + \cdots \right]$$

(28)

which is correct to $O(H^2)$. The minimum value of $R$ is found by setting $\partial R/\partial k = 0$ in (28) and this leads to

$$(1 + D\pi^2)(k^4 - \pi^4) + (\pi^2 + k^2) \left[ 2Dk^4 - H(1 + D\pi^2) \right]$$

$$(1 + D\pi^2)H(k^2 + \gamma H) + Dk^4H + \cdots = 0$$

(29)

When $D = 0$, the Darcy flow limit, Equation (29) reduces to $(k^4 - \pi^4) - \pi^2H + \gamma H^2 + \cdots = 0$ which is in agreement with Eq. (13) from Banu and Rees (2002). By expanding $k$ using the expansion,

$$k = k_0 + k_1H + k_2H^2 + \cdots$$

(30)

and inserting (30) in (29), we obtain at $O(H^0)$

$$(1 + D\pi^2)(k_0^4 - \pi^4) + 2k_0^4D = 0$$

(31)

For $D = 0$, this equation gives $k_0 = \pi$ immediately, in agreement with Equation (14) from Banu and Rees (2002). Equation (31) yields the solution

$$k_0 = \pi \left[ 2(1 + D^2\pi^2) \right]^{1/2} \left[ 1 + D\pi^2 + (1 + 10\pi^2D + 9\pi^4D^2)^{1/2} \right]^{-1/2}$$

(32)

which is valid for all values of $D$.

At $O(H)$ we get

$$k_1 = \frac{\pi^2 + D(\pi^4 - k_0^4)}{4k_0^3(1 + 3D\pi^2 + 3k_0^2D)}$$

(33)
while at $O(H^2)$,
\[
k_2 = \left\{ 6k_0^2k_1^2[1 + \pi^2(D + 1) + k_0^2] + 2k_0^4k_1^2D + (1 + D\pi^2)(2k_0k_1 + \gamma) + 4Dk_0^3k_1 \right\}
\times (2k_0^3)^{-1} \left[ 2(1 + D\pi^2) + 2(\pi^2 + k_0^2) + Dk_0^2 \right]^{-1}
\]

(34)

In conclusion, the critical wave number $k_c$ is given by (30), while the critical Rayleigh number is given by (28).

3.2. Large $H$ analysis

For large values of $H$, $R$ takes the form
\[
R = \frac{(\pi^2 + k_0^2)^2[1 + D(\pi^2 + k_0^2)]}{k^2}, \gamma + 1 \cdot \left[ 1 - \frac{\pi^2 + k_0^2}{\gamma(\gamma + 1)} \frac{1}{1} \right] + \frac{(\pi^2 + k_0^2)^2}{\gamma^2H^2} \left[ 3k_0^2 - \pi^2 + D(3k_0^4 + 2\pi^2k_0^2 - \pi^4) \right]
\]

On minimizing with respect to $k$, we obtain
\[
\frac{\gamma + 1}{\gamma} \left[ k^2 - \pi^2 + D(2k^4 + \pi^2k^2 - \pi^4) \right] - \frac{\pi^2 + k_0^2}{\gamma^2H^2} \left[ 2k_0^2 - \pi^2 + D(3k_0^4 + 2\pi^2k_0^2 - \pi^4) \right]
\]
\[
+ \frac{(\pi^2 + k_0^2)^2}{\gamma^3H^2} \left[ 3k_0^2 - \pi^2 + D(4k_0^4 + 3\pi^2k_0^2 - \pi^4) \right] = 0
\]

(36)

We notice that, when $D = 0$, Equation (36) reduces to (18) from Banu and Rees (2002). By expanding $k$ in terms of inverse powers of $H$ we also obtain,
\[
k = k_0 + \frac{k_1}{H} + \frac{k_2}{H^2} + \cdots
\]

(37)

After inserting (37) into (36) the $O(1)$ terms yield the expression.
\[
k_c^2 - \pi^2 + D(2k_0^4 + \pi^2k_0^2 - \pi^4) = 0
\]

For $D = 0$, this equation gives also $k_0^2 = \pi$. In the general case we again have to apply (32). At $O(1/H)$ we get
\[
k_1 = (2k_0)^{-1} \left[ 1 + D(4k_0^2 + \pi^2) \right]^{-1} (\pi^2 + k_0^2) \cdot \left[ 3Dk_0^4 + 2(1 + \pi^2D)k_0^2 - \pi^2(1 + \pi^2D) \right]
\]

(38)

Further, at $O(1/H^2)$, we obtain
\[
k_2 = \frac{A}{2k_0(4Dk_0^2 + 1 + D\pi^2)}
\]

(39)

where
\[
A = -k_0^2(12Dk_0^2 + 1 + D\pi^2) + \frac{2k_0k_1}{\gamma(\gamma + 1)} \left[ 2(\pi^2 + k_0^2) \cdot (3Dk_0^2 + 1 + D\pi^2) \right]
\]
\[
+ k_0^2(Dk_0^2 + 1 + D\pi^2) \right] - \frac{2k_0^2}{\gamma^2(\gamma + 1)} (\pi^2 + k_0^2)^2(Dk_0^2 + 1 + D\pi^2)
\]

In conclusion, the critical wave number $k_c$ is given by (37), whilst the critical Rayleigh number is given by (35).
4. NUMERICAL RESULTS

With the aim to find the critical wavenumber, which corresponds to the minimum Rayleigh number, we calculate $\partial R / \partial k = 0$, and on using (27), we get

$$
\left[ k^2 - \pi^2 + D(2k^4 - \pi^4 + k^2\pi^2) \right] \left[ \pi^2 + k^2(1 + \gamma)H \right] 
\times (\pi^2 + k^2 + \gamma H) - k^2 H(\pi^2 + k^2) \left[ 1 + D(\pi^2 + k^2) \right] = 0
$$

(40)

Eq. (40) reduces, in the case when $D = 0$, to Eq. (21) from Banu and Rees (2002). Eq. (40) gives the critical value of $k$ (for various values of $H, \gamma$ and $D$), which, when inserted into (27), provides the critical value of the Rayleigh number.

Figure 2 gives a selection of neutral curves ($R$ against $k/\pi$) for various values of $\gamma$ and $D$ with $H = 100$. These all follow the familiar shape for Benard-like problems with a single well-defined minimum value. There is a general trend towards the values of $R$ becoming smaller as $D$ decreases and as $\gamma$ increases. The size of $D$ is related to the importance of viscous effects at the

Figure 2. Neutral curves for $H = 100, \gamma = 0.001, 0.01, 1, 5, 10, 50$ and $100$: (a) $D = 10^{-2}$; (b) $D = 10^{-3}$; (c) $D = 10^{-4}$; (d) $D = 0$.
Table I. Comparison of the exact and asymptotic values of the wave number and the critical Rayleigh number for large $H$ and $\gamma = 1$. “E” denotes the exact solution and “A” the asymptotic solution.

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boundaries, and reductions in $D$ decrease this effect, which allows the fluid to move more easily, thereby decreasing the critical Rayleigh number. Large values of $\gamma$ mean that heat is transported through both the solid and fluid phases, whereas small values correspond to transport primarily through the fluid phase. Thus convection is established more readily for larger values of $\gamma$ when

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all other parameters are held fixed. A similar argument may be put forward for the variation in $R$ when $H$ varies, since $H$ measures the ease with which heat is transferred between the phases.

The LTE case is recovered in the large $H$-limit. In Table I are provided comparisons, in the LTE limit, of the exact and asymptotic values of the critical wavenumber and the critical Rayleigh number for large $H$ with $g = 1$. Agreement between the asymptotic results and the exact solutions are very good, especially for the larger values of $H$. However, as the value of $D$ increases, agreement reduces slightly for any fixed value of $H$.

In Table II are presented the exact and asymptotic values of the wavenumber and the critical Rayleigh number for small values of $H$ (the non LTE case) with $g = 1$. We have increasingly good agreement as $H$ decreases, but, for fixed values of $H$ the agreement becomes poorer as $D$ increases.

In Figures 3 and 4, respectively, we summarize the behaviour of the critical Rayleigh number and wavenumber, respectively, as functions of $H$ and $\gamma$ for $D = 0$, $D = 10^{-3}$ and $D = 1$. The variation of $R_c$ with $\log_{10} H$ is depicted in Figure 3. We again see the fact that $R_c$ increases as $H$ increases and $\gamma$ decreases. When $H$ is small all the curves asymptote to a single value which is independent of $\gamma$ and given by (32). At the opposite extreme we see, from (35), that the asymptotic values of $R$ are such that $R\gamma/(1 + \gamma)$ attains a value which is independent of $\gamma$.
this should be so is not surprising since the constant, \( R_g = \frac{\gamma}{1 + \gamma} \) precisely the Rayleigh number which is based on the mean properties of the porous medium, rather than on the fluid properties as given in (11).

In Figure 4 we see that \( k_c \) attains the value \( p \) when \( H \) is small. That \( k_c \) should attain the value \( p \) when \( H \) is small may be understood from the fact that Equations (19) and (20) form the standard Darcy–Brinkman stability equations (subject to a rescaling of the Rayleigh number) and that the equation for the solid phase temperature decouples from (19) and (20). Therefore we would expect the wave number to take that value when there is no microscopic conduction between the phases.

5. CONCLUSIONS

In this paper we have analysed in detail the combined effects of boundary (Brinkman) and non LTE on the onset of convection in a porous layer of infinite extent. Although we have also
included inertia effects in the formulation, their presence does not affect the stability criterion, since the basic state is motionless. With the aim of undertaking an analytical study, we have considered the case of stress free boundaries. This restriction has been relaxed in Rees and Postelnicu (2002).

The present study shows that variations in the values of $H$, $\gamma$ and $D$ have a significant effect on the criterion for the onset of convection. Detailed numerical and asymptotic solutions have been presented for $R_c$ and $k_c$ as functions of $D$, $H$ and $\gamma$. As in Banu and Rees (2002), it has been shown that LTE is recovered in the large $H$ limit.

On the other hand, at fixed values of $\gamma$ and $H$, the critical wave number $k_c$ reduces and the critical Rayleigh number $R_c$ increases as $D$ increases. Conversely, at fixed $D$, the convection onset occurs at lower values of $R$ as $H$ decreases or $\gamma$ increases.

**NOMENCLATURE**

- $b$ = form drag coefficient
- $c$ = specific heat
- $d$ = depth of the convection layer
- $D$ = Darcy number
- $F_1, F_2$ = dimensionless coefficients
- $g$ = gravity
- $h$ = inter-phase heat transfer coefficient
- $H$ = scaled inter-phase heat transfer coefficient
- $k$ = wave number
- $K$ = permeability
- LTE = Local thermal equilibrium
- $P$ = Pressure
- $R$ = Darcy–Rayleigh number
- $t$ = Time
- $u, v$ = fluid flux velocities
- $\mathbf{V}$ = dimensional velocity vector
- $x, y$ = Cartesian co-ordinates

**Greek letters**

- $\alpha$ = diffusivity ratio
- $\beta$ = coefficient of expansion
- $\rho$ = density
- $\kappa$ = diffusivity
- $\varepsilon$ = porosity
- $\mu$ = viscosity
- $\psi$ = streamfunction
- $\theta, \Theta$ = scaled temperature of the fluid phase
- $\phi, \Phi$ = scaled temperature of the solid phase
- $\gamma$ = porosity-scaled conductivity ratio

REFERENCES


**Superscripts and subscripts**

\[ c = \text{cold or critical} \]
\[ e = \text{effective (viscosity)} \]
\[ f = \text{fluid} \]
\[ h = \text{hot} \]
\[ s = \text{solid} \]
\[ ' = \text{η-derivative} \]
\[ 0,1,2 = \text{successive terms in series expansions} \]