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The effects of boundary imperfections on convection in a saturated porous layer: non-resonant wavelength excitation

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The onset of Rayleigh–Bénard convection in a horizontally unbounded saturated porous layer is considered when the temperatures of both horizontal boundaries vary periodically in one direction about their respective mean values. Attention is focused on small-amplitude thermal modulations with a wavenumber not close to the critical value for the perfect layer. A stability analysis of weakly nonlinear convection is performed and the effects of different wavenumbers and symmetries of the thermal modulations are deduced systematically. It is shown that there are many special cases to be considered and that the convection patterns depend crucially on the particular configuration. Intuitively it might be expected that one-dimensional thermal modulation would always stimulate a two-dimensional motion. Surprisingly, however, for a wide range of modulation wavenumber a three-dimensional motion with a rectangular planform results from a resonant interaction between a pair of oblique rolls and the boundary forcing.

1. INTRODUCTION

The analysis of convection in fluid-saturated porous layers heated from below is of considerable interest in the fields of insulation technology and geothermal energy studies. It has been the subject of intensive research over the four decades or so since the pioneering studies of Horton & Rogers (1945) and Lapwood (1948). Analytical, numerical and experimental techniques have been used to investigate the onset of convection (see, for example, Lapwood 1948; Beck 1972), supercritical heat transfer (Elder 1967), the stability of finite-amplitude convection (Palm et al. 1972), the stability of large-amplitude convection (Straus 1974), oscillatory convection (Horne & O'Sullivan 1974; Schubert & Straus 1982), and routes to chaotic convection (Kimura et al. 1986). Other important aspects studied include non-Darcy effects (Borkowska-Pawlak & Kordylewski 1982), wavenumber selection (Geogiadis & Catton 1986) and non-Boussinesq effects (Straus & Schubert 1977). Further realism has been accommodated by, for example, relaxing the assumption of perfectly conducting horizontal boundaries (Riahi 1983) or of perfectly insulating sidewalls (Impey et al. 1987).

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In this paper we investigate the effects of small-amplitude thermal imperfections along the horizontal boundaries on the onset and stability of finite-amplitude convection. Such variations always occur to some extent in experiments. We have previously investigated the effects of wavy, isothermal boundaries on finite-amplitude convection (Rees & Riley 1986): cases were considered where the wavelength of the imperfection is equal to the critical wavelength of the 'perfect' problem. This has been extended recently (Rees & Riley 1988, 1989) to a study of the effects of near-resonant wavelength thermal modulations along planar boundaries. It should be noted that the effects on finite-amplitude convection of isothermal wavy boundaries and thermal modulations along planar boundaries are qualitatively the same, at least for near-resonant wavelength imperfections. Here the study is further extended to include non-resonant wavelength thermal imperfections.

There are several papers dealing with the Rayleigh–Bénard analogue of the present problem. The first, to our knowledge, concerned the stability of finite-amplitude convection with a near-resonant thermal imperfection (Vozovoi & Nepomnyaschii 1974). Later, Kelly & Pal (1976, 1978) and Pal & Kelly (1978, 1979) studied finite-amplitude convection for various choices of modulation wavenumber. In common with the present problem they found that certain wavenumbers, or ranges of wavenumbers, give rise to different resonant effects. In some cases the supercritical bifurcation typical of the 'perfect' problem is unfolded by the imperfection and a smooth transition from subcritical to supercritical flow results. In other cases the bifurcation remains supercritical but occurs at a different value of the Rayleigh number. Moreover, in certain circumstances the most unstable mode has the form of a rectangular cellular pattern. Some of this work has been extended recently by Wynne (1987).

The present work is concerned with analysing the effects of different thermal modulation wavenumbers and symmetries on the onset and stability of finite-amplitude convection in an infinite layer. The modulations are factorized into sinuous (in-phase) variations, and varicose (out-of-phase) variations, along the horizontal boundaries. We assume that the modulations are one-dimensional, with wavenumber $k_w$, and have amplitude $O(\delta)$, where $\delta \ll 1$.

At subcritical values of the Darcy–Rayleigh number, $Ra$, the flow is unique, two-dimensional, and driven solely by the thermal modulations via baroclinic effects: this is termed the quasi-conduction régime and is studied in detail in §3. As $Ra$ increases to near $Ra_c$, the critical value for the unmodulated problem, we have, as mentioned above, either a smooth transition to the convective régime (Rees & Riley 1986, 1988), or a supercritical bifurcation to convective motion that may be either two- or three-dimensional. We perform a perturbation expansion valid near $Ra_c$ in powers of $\delta$ and $\epsilon$, where $\epsilon \ll 1$ is the order of magnitude of the finite-amplitude motion. Details of this are contained in §4 and the Appendix. By considering a single roll of arbitrary orientation relative to the thermal modulations we are able to deduce systematically the various resonant cases that exist for this problem.

In the absence of imperfections ($\delta = 0$) rolls are the only stable planform and Landau–Ginzburg equations governing the amplitudes of these rolls may be
obtained by applying orthogonality conditions to the $O(\epsilon^3)$ equations in an $\epsilon$-expansion. In effect, this removes resonant forcing terms from the equations.

In the presence of imperfections further resonant terms appear. By using a suitable scaling for $\delta$ in terms of $\epsilon$ it is always possible to cause the lowest-order resonant terms to appear in the $O(\epsilon^3)$ equations, and the application of orthogonality conditions results in modified forms of the Landau–Ginzburg equations. In this study we choose to take $\delta$ to be the reference small quantity, and thus different imperfections have different asymptotic regions of importance about $Ra_c$. To take two examples, (i) when the imperfection is sinuous with $k_w = k_c$, we require $\delta = \epsilon^2$ and therefore $Ra - Ra_c = O(\epsilon^4)$ (i.e. $O(\delta^2)$), (ii) when it is varicose with $k_w < 2k_c$, then $\delta = \epsilon^2$ and $Ra - Ra_c = O(\delta)$. Thus the region of influence of the latter imperfection is vanishingly small compared with that of the former.

The remainder of the paper is devoted to the various special cases arising from the analysis of §4. In §5 we show that varicose imperfections with $0 < k_w < 2k_c$ result in a decrease in the critical Rayleigh number. In contrast to the perfect case, the corresponding stable convective pattern consists of rectangular cells rather than rolls. These cells, whose aspect ratio depends on $k_w$, are composed of two rolls of equal amplitude. At slightly higher values of $Ra$ this pattern loses stability to a rectangular pattern consisting of a pair of rolls of unequal amplitudes. Rolls of other orientations are also possible, but all are initially unstable. Varicose imperfections with $k_w \sim 2k_c$ are discussed next in §6. The results of the stability analysis are involved and are difficult to summarize succinctly. However, it is worth noting here that the first mode to appear depends on $k_w - 2k_c$: when $(k_w - 2k_c)/\delta < 0.1725$ rectangular cells of large aspect ratio are preferred, otherwise it is transverse rolls (defined here to have axes oriented perpendicularly to those of the quasi-conductive motion). The varicose layer with $k_w > 2k_c$ is studied in §7 where it is shown that resonant effects give critical values of $Ra$ that are $k_w$ and orientation dependent. In all cases transverse rolls are the first to appear.

In §8 we study cases where the imperfections have a sinuous form and where $k_w \neq k_c$. When $k_w < k_c$ rectangular cellular patterns arise via a similar resonance mechanism to that studied in §5. Unlike §5, however, the phase of the rectangles relative to the phase of the imperfection depends on $k_w$. Furthermore, single rolls of arbitrary orientation have critical values of $Ra$ that depend on that orientation. The first mode to appear as $Ra$ increases is found to depend on $k_w$: it may be either a transverse roll or a rectangular cell. When $k_w > k_c$ the results are generally qualitatively the same as those given in §7. However, when $k_c < k_w < 3k_c$ a resonance mechanism occurs that manifests itself by isolating two $k_w$-dependent orientations, whereby a disturbance in the form of a roll of one orientation results in the growth of a roll of the other orientation, but not vice versa.

In §9 we study the two subharmonic resonances $k_w \sim \frac{1}{2}k_c$ and $k_w \sim \frac{1}{3}k_c$. The latter arises when the heating is sinuous, and the former when the heating contains both sinuous and varicose components. In both cases the resulting roll amplitude equations may be reduced to that for the sinuous $k_w \sim k_c$ case.

There are two important cases that are not dealt with here. A study of the onset
of finite-amplitude convection where the heating is varicose and \( k_w \sim 0 \) is given in Rees (1989). There it is shown that the most dangerous mode is, somewhat unexpectedly, a rectangular cell of large aspect ratio. A detailed study of the sinuous and varicose heating cases with \( k_w \sim k_c \) is given in Rees & Riley (1988, 1989), where it is shown that there exists stable solutions with spatially varying orientations or wavenumbers.

Finally, we compile the hierarchy of imperfections in §10, and discuss both the effects of having more than one imperfection present and the applicability of our results.

2. Formulation of the Problem

We consider a slot of vertical separation \( 2d \), of infinite horizontal extent and filled with a porous material. On invoking the Boussinesq approximation and assuming the Darcy–Prandtl number is large, the non-dimensional equations are

\[
\nabla^2 p - Ra \theta_z = 0, \tag{2.1}
\]

\[
\nabla^2 \theta + Ra \theta - p_z = Ra \theta_z - \nabla p \cdot \nabla \theta + \theta_t, \tag{2.2}
\]

where the pressure \( p \) and temperature \( \theta \) are measured relative to the equilibrium values, and \( Ra \) denotes the usual Darcy–Rayleigh number (see Rees & Riley 1988).

We assume that the boundary temperatures have small-amplitude sinusoidal variations about their mean values and consider variations that are either varicose (symmetric) or sinuous (antisymmetric), with common wavenumber \( k_w \). Accordingly, the boundary conditions along \( z = \pm 1 \) are

\[
\theta = \pm \frac{1}{2} \delta (e^{ik_w x} + e^{-ik_w x}), \quad \text{(varicose)} \tag{2.3}
\]

\[
\theta = \frac{1}{2} \delta (e^{ik_w x} + e^{-ik_w x}), \quad \text{(sinuous)} \tag{2.4}
\]

\[
p_z = Ra \theta, \tag{2.5}
\]

where \( \delta \ll 1 \). Solutions are assumed to be either periodic or quasi-periodic in the \( x \)- and \( y \)-directions; the latter possibility occurring when, for example, the boundary wavenumber and the roll wavenumber are incommensurate.

3. Quasi-conduction régime

At Rayleigh numbers well below \( Ra_c \) the fluid is driven solely by the thermal imperfections at the boundaries. Because the thermal variations are weak and one dimensional, the amplitude of the resulting two-dimensional motion is correspondingly weak. For this subcritical flow, steady solutions are easily found as series in \( \delta \):

\[
\psi = \delta \psi_1 + \delta^2 \psi_2 + \ldots, \tag{3.1}
\]

\[
\theta = \delta \theta_1 + \delta^2 \theta_2 + \ldots, \tag{3.2}
\]
where the streamfunction, \( \psi \), is defined as in Rees & Riley (1988, 1989) and

\[
\psi_1 = \frac{1}{2} \nu_0 (e^{ik_w x} - e^{-ik_w x}) \tag{3.3a}
\]
\[
\theta_1 = \frac{1}{2} \nu_0 (e^{ik_w x} + e^{-ik_w x}) \tag{3.3b}
\]

where, for varicose variations,

\[
h_0 = \frac{1}{2} Ra^{\frac{1}{2}} \left( \frac{\sinh(\chi z)}{\sinh(\chi)} - \frac{\sinh(\gamma z)}{\sinh(\gamma)} \right), \tag{3.4a}
\]
\[
g_0 = \frac{1}{2} \left( \frac{\sinh(\chi z)}{\sinh(\chi)} + \frac{\sinh(\gamma z)}{\sinh(\gamma)} \right), \tag{3.4b}
\]

and, for sinuous variations,

\[
h_0 = \frac{1}{2} Ra^{\frac{1}{2}} \left( \frac{\cosh(\chi z)}{\cosh(\chi)} - \frac{\cosh(\gamma z)}{\cosh(\gamma)} \right), \tag{3.5a}
\]
\[
g_0 = \frac{1}{2} \left( \frac{\cosh(\chi z)}{\cosh(\chi)} + \frac{\cosh(\gamma z)}{\cosh(\gamma)} \right), \tag{3.5b}
\]

where

\[
\gamma^2 = k_w^2 + k_w Ra^{\frac{1}{2}}, \tag{3.6a}
\]
\[
\chi^2 = k_w^2 - k_w Ra^{\frac{1}{2}}. \tag{3.6b}
\]

In the above it should be noted that \( \chi \) may take either real or imaginary values.

In figure 1 the \( O(\delta) \) streamlines given by (3.3a, 3.5a) are displayed for various values of the boundary wavenumber, \( k_w \), and \( Ra = 5 \approx \frac{1}{2} Ra_c \). As \( k_w \) increases the streamlines become concentrated near the walls with little flow within the central part of the layer. It should be noted, however, that although the strength of the flow measured by \( \psi_{1\text{max}} - \psi_{1\text{min}} \), say, decreases as \( k_w \) increases, the maximum vorticity \( \psi_{xx} \sim \frac{1}{2} k_w Ra \) for large \( k_w \), and therefore the flow becomes more vigorous within the boundary layers.

The mean heat transfer across the layer is a quantity of practical significance, and is defined by

\[
Nu = -\frac{k}{2\pi} \int_0^{2\pi/k} \theta_z |_{z=-1} dx, \tag{3.7}
\]
where $k = k_w$. After some algebra, we find that

\[ Nu = 1 + \delta^2 Nu_v + O(\delta^4), \quad \text{(varicose)} \]  
\[ Nu = 1 + \delta^2 Nu_s + O(\delta^4), \quad \text{(sinuous)} \]

where

\[ Nu_v = \frac{1}{16} k_w Ra^\frac{1}{2} \left[ \coth^2 \gamma - \coth^2 \chi + \chi^{-1} \coth \chi - \chi^{-1} \coth \gamma \right], \quad \text{(3.9a)} \]

\[ Nu_s = \frac{1}{16} k_w Ra^\frac{1}{2} \left[ \tanh^2 \gamma - \tanh^2 \chi + \chi^{-1} \tanh \chi - \chi^{-1} \tanh \gamma \right]. \quad \text{(3.9b)} \]

**Figure 2.** The second-order Nusselt numbers $Nu_v$ and $Nu_s$, given by (3.8) and (3.9), as functions of $Ra$ and $k_w$. 
\( N\nu \) and \( N_u \) as functions of \( Ra \) and \( k_w \) are displayed in figure 2, where it may be seen that both are positive for all non-zero values of \( Ra \) and \( k_w \). We conclude that the presence of the boundary imperfections serves to increase the heat transfer. It is pertinent to note that \( N_u \) becomes singular at \( Ra = Ra_c = \pi^2 \), \( k_w = k_c = \frac{1}{2} \pi \) reflecting the singular nature of the \( O(\delta) \) solution for the sinuous layer; this was also found by Rees & Riley (1986).

4. Finite-amplitude expansion and identification of resonances

The analysis presented above is valid only for subcritical Rayleigh numbers, because, when \( Ra \) is close to \( Ra_c \), the flows given by (3.1) become thermo-convectively unstable. In the absence of imperfections (\( \delta = 0 \)) a variety of flow patterns including rolls, rectangles, and hexagons are theoretically possible. Palm et al. (1972), using the method of Schlüter et al. (1965), showed that rolls are the only stable planform at slightly supercritical Rayleigh numbers. In this section and the remainder of the paper we concentrate on the convective motion that arises when \( Ra \sim Ra_c \) and investigate the effects of the boundary imperfections.

It is assumed that the post-critical convective motion has amplitude \( \epsilon \ll 1 \), and that the solutions to equations (2.6) and (2.7) can be expanded in powers of \( \delta \) and \( \epsilon \):

\[
(p, \theta, Ra) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \delta^m \epsilon^n (p^{mn}, \theta^{mn}, Ra^{mn}),
\]

where

\[
(p^{00}, \theta^{00}, Ra^{00}) = (0, 0, Ra_c).
\]

To facilitate the study of the effects of small detunings of the boundary wavenumber \( k_w \) away from particular values such as \( k_c \), \( 2k_c \) and \( \frac{1}{2} k_c \), we further write

\[
k_w = k + \epsilon K,
\]

and introduce the slow scales \( X = \epsilon x \), \( Y = \epsilon y \), \( Y^* = \epsilon^3 y \) and \( \tau = \frac{1}{2} \epsilon^2 t \).

We are now in a position to deduce the resonant cases that exist for this problem. The procedure we follow is to examine the governing equations at each order in the expansion and determine the cases where the forcing terms involve the natural eigenfunctions of the linear differential operator. At \( O(\epsilon) \) we have the roll eigensolution (or eigensolutions), and, on proceeding with the expansion in powers of \( \epsilon \), find that resonant terms appear in the \( O(\epsilon^3) \) equations. The application of simple solvability, or orthogonality, condition yields the familiar Landau–Ginzburg equation for the roll amplitude (see (A 20) and (A 21)). When thermal imperfections are present, further resonant forcing terms occur at \( O(\delta^a \epsilon^b) \) (\( a > 0 \)) and, to determine how these resonances modulate the amplitude, we define \( \delta \) in terms of \( \epsilon \) by setting \( \delta = \epsilon^{(a-b)/a} \). In this way the resonant terms arise in the equations governing the \( O(\epsilon^3) \) terms in the new expansion

\[
(p, \theta, Ra) = (0, 0, Ra_c) + \sum_{n=0}^{\infty} \epsilon^n (p_n, \theta_n, R_n),
\]
and manifest themselves as modifications to the Landau–Ginzburg equation. In this paper we restrict our attention to cases for which \( m + n \leq 3 \) in (4.1).

For the sinuous case the \( O(\delta) \) solutions \((p^{10}, \theta^{10})\) are unbounded in the limit \( k \to k_c \), which is a consequence of the fact that the thermal imperfections resonate with a longitudinal mode in that limit. The resonant forcing arises via the boundary conditions for this one configuration; all the other resonances we discuss arise as inhomogeneous terms in the governing equations. The resolution of this breakdown is by means of a singular perturbation analysis (cf. Tavantzis et al. 1978), which involves a local scaling \( \varepsilon = \delta^2 \). On using (4.4) and taking \( k = k_c \), the weakly nonlinear analysis goes through straightforwardly, and the resonance manifests itself mathematically as an additional space-dependent forcing term in the amplitude equation (A 20). For longitudinal rolls we take \( k_x = k_c \) and obtain

\[
A_\tau = R_2 A + 4A_{XX} - k_c^4 A^2 \bar{A} - 4ik_c e^{iKX},
\]

(4.5)

where \( A \) is the amplitude of the roll and the overbar denotes complex conjugation. A detailed analysis of (4.5) is outside the scope of this paper, and is reported in Rees & Riley (1988, 1989); where the zigzag, cross-roll and sideband instabilities are studied.

The \( O(\delta^2) \) equations (see (A 8)) have resonant forcing terms when \( k_w \sim \frac{1}{3} k_c \), because this wavenumber provides a forcing term with wavenumber \( 2k_w \sim k_c \) (cf. (A 9)). However, what is interesting, but not obvious \( \text{a priori} \), is that the boundary imperfections must be neither exactly sinuous not exactly varicose. This can be seen by considering the forcing terms in (A 8b): for imperfections that are either exactly sinuous or varicose the forcing is odd in \( z \), and is therefore non-resonant. When the imperfection has both sinuous and varicose components then \( p^{10} \) and \( \theta^{10} \) consist of even and odd terms in \( z \). Hence the products on the right-hand side of (A 8b) have components that are even in \( z \) and therefore resonant. For the Bénard problem this resonance was found by Pal & Kelly (1978) when they considered thermal variations on one wall only; their case is effectively a combination of the sinuous and varicose forms of imperfection. For the present case we set \( \varepsilon = \delta^2 \) and its analysis is presented in §9. As shall be seen below, however, this resonance is significant only when the flow is restricted to be two dimensional, because a three-dimensional resonance mechanism dominates this effect when the heating contains a varicose component (cf. §5 where the roll scaling is \( \varepsilon = \delta \)).

At \( O(\delta^3) \) a resonance occurs when \( k_w \sim \frac{1}{3} k_c \) and the heating is sinuous (or, more generally, has a sinuous component). This follows from (A 13b) on requiring \( J_1 \) to be even in \( z \) for resonance. In this case we set \( \varepsilon = \delta \) and the amplitude equation for the longitudinal roll (with \( k = \frac{1}{3} k_c \) and \( k_x = k_c \)) is

\[
A_\tau = (R_2 - I_4) A + 4A_{XX} - k_c^4 A^2 \bar{A} - iI_1 e^{iKX},
\]

(4.6)

where, in general

\[
I_n(k, k_x) = 2 \int_{-1}^{1} J_n(k, k_x, z) \cos(k_c z) \, dz,
\]

(4.7)

and here \( I_1 = I_1(\frac{1}{3} k_c) \) is independent of \( k_x \), and \( I_4 = I_4(\frac{1}{3} k_c, k_c) \). A brief analysis of this subharmonic resonance is given in §9. It should be noted again that any
varicose component with a wavenumber $\frac{1}{2}k_c$ gives rise to a more important resonance mechanism. Thus the flow must be restricted to be two-dimensional for this resonance to be significant. Note also that other resonant terms arise in (A 13b) where the thermal forcing contains a sinuous component with $k \sim k_c$. However, because the same configuration gives rise to a resonance at $O(\delta)$ (see above) we have no need to consider these equations, unless higher-order effects are to be considered.

The $O(\delta \varepsilon)$ equations have resonant terms when the layer is varicose and if $0 < k < 2k_c$. This may be seen by examining equation (A 16b) where the fourth term of the right-hand side has a component proportional to an eigensolution if $J_3(z)$ is non-zero and even, and $k = 2k_x$ (but note that $J_3 = 0$ when $k = 2k_c$ and $k_x = k_c$). A roll with wavevector $(k_x, k_y)$ interacts with the baroclinic motion and induces a forcing at $O(\delta \varepsilon)$ that reinforces a roll with wavevector $(k_x, -k_y)$. Thus we need to consider two rolls, one of orientation $\phi = \arccos(k/2k_c)$, the other $-\phi$. This type of resonant behaviour was previously noted by Pal & Kelly (1979) for the Bénard problem. By setting $\varepsilon = \delta^{\frac{1}{3}}$ the resonant terms arise in the $O(\varepsilon^3)$ equations and modify the equations for the respective roll amplitudes, $A$ and $B$. Thus we deduce that $A$ and $B$ satisfy

$$A_r = R_2 A + 4A_{XX} - k_c^4 A [|A|^2 + \Omega(2\phi)|B|^2] + I_3 B,$$

$$B_r = R_2 B + 4B_{XX} - k_c^4 B [|B|^2 + \Omega(2\phi)|A|^2] + I_3 A,$$

where $I_3 = I_3(k, \frac{1}{2}k)$, the coupling coefficient, $\Omega$, is given by (A 23), and $X_A, X_B$ are the slow horizontal scales perpendicular to the respective roll axes. This case is studied in §5 where we extend the work of Pal & Kelly (1979) by presenting a stability analysis of the solutions to (4.8) and (4.9).

At $O(\delta^2 \varepsilon)$ there are two different resonant terms. The first (the term in (A 18b) involving $J_4$, which is even in $z$) has wavevector $(k_x, k_y)$ and must be included in the solvability condition at $O(\varepsilon^3)$ for both sinuous and varicose cases when $\varepsilon = \delta$. For the two cases, $k > 2k_c$ (varicose imperfections) and $k > 3k_c$ (sinuous imperfections), it may be shown that no other resonant terms arise in the $\varepsilon$, $\delta$-expansion for $m + n \leq 3$, and therefore the equations governing the amplitudes, $A$ and $B$, of two rolls with orientations, $\phi_A$ and $\phi_B$, are

$$A_r = (R_2 - I_4^A) A + 4A_{XX} - k_c^4 A [|A|^2 + \Omega(\phi_A - \phi_B)|B|^2],$$

$$B_r = (R_2 - I_4^B) A + 4A_{XX} - k_c^4 B [|B|^2 + \Omega(\phi_A - \phi_B)|A|^2],$$

where $I_4^A = I_4(k, k_c \cos \phi_A)$, $I_4^B = I_4(k, k_c \cos \phi_B)$.

The second resonant term occurs when $k \leq k_c, k_x = k$, and the layer is varicose (see the $J_4$ term in (A 18b)). This resonance takes the form of a pair of mutually resonant rolls as is the case for the resonance at $O(\varepsilon \delta)$. However, the $O(\varepsilon \delta)$ resonance is a lower-order effect than the present one, and is therefore dominant. In only one configuration do we find the present resonance important, namely, for varicose layers with $k = k_c$. For this particular case the ‘pair’ have identical orientations $\phi = 0$, and therefore we consider only the longitudinal roll, and restrict the motion to be two dimensional. We find that the roll amplitude, $A$, satisfies

$$A_r = (R_2 - I_4) A + 4A_{XX} - k_c^2 A^2 A + I_6 e^{2ikx} A,$$
where $I_4 = I_4(k_c, k_c)$ and $I_5 = I_5(k_c, k_c)$. In Rees & Riley (1988) we show that solutions other than $A \propto e^{ikx}$ exist and are stable. This second resonant term is also important for sinuous layers when $k < k_c$, and is studied in §8. For this case a roll of orientation $\phi = \arccos (k/k_c)$ induces roll motion at an orientation $-\phi$. This is similar to the varicose layer ($k < 2k_c$) discussed above, except that the $I_4(k, k_x)$ term varies with the roll orientation and complicates the results.

Finally, at $O(\delta^2)$ there are two further resonances. The first occurs when the term multiplying $J_6$ in (A 19b) has wavenumber $k_c$ and the layer is sinuous. Hence we have

$$k_c < k < 3k_c \quad \text{and} \quad k_x = (k^2 + 3k_c^2)/4k.$$  \hspace{1cm} (4.13)

There are two rolls to consider whose orientations $\phi_A$ and $\phi_B$ are given by

$$\cos \phi_A = k_x/k_c, \quad \sin \phi_A = k_y/k_c,$$

$$\cos \phi_B = (2k_x - k)/k_c, \quad \sin \phi_B = 2k_y/k_c;$$  \hspace{1cm} (4.14)

where $k_y = \pm (k_c^2 - k_x^2)^{1/2}$, and the amplitude equations are given by

$$A_r = (R_2 - I_4^2) A + 4A_{X_A} A_{X_A} - k_c^4 A_{[A] + 4\Omega(\phi_A - \phi_B)|B|^2]},$$  \hspace{1cm} (4.15)

$$B_r = (R_2 - I_4^2) A + 4A_{X_B} A_{X_B} - k_c^4 B_{[B] + 4\Omega(\phi_A - \phi_B)|A|^2} - I_6 A^2,$$  \hspace{1cm} (4.16)

where $I_4^2 = I_4(k, k_c \cos \phi_A)$, $I_4^2 = I_4(k, k_c \cos \phi_B)$, $I_6 = I_6(k, k_c)$, and $k, k_x$ satisfy (4.13). This case is examined in §8.

The second resonance occurs for the varicose layers when $k_w \sim 2k_c$ and $k_x = k_c$, a superharmonic resonance (see the $J_7$ and $J_8$ terms in (A 19b)). The roll scaling is again $\epsilon = \delta$ and the amplitude, $A$, of the longitudinal roll satisfies

$$A_r = (R_2 - I_4) A + 4A_{XX} - k_c^4 A^2 + [KI_7 A + I_8 A_{X} + I_9 A Y_{*y*}] e^{ikx}$$  \hspace{1cm} (4.17)

for two-dimensional motion, where

$$I_4 = I_4(2k_c, k_c), \quad I_7 = I_7(2k_c, k_c),$$

and

$$I_8 = I_8(2k_c, k_c).$$

When weak three-dimensional effects, such as the zigzag instability, are considered, the term (A 19f) must be included in the $O(\epsilon^3)$ solvability condition; then $A$ satisfies

$$A_r = (R_2 - I_4) A + \left[ \frac{2 \partial}{\partial X} - \frac{i}{k_c} \left( \frac{\partial}{\partial \bar{Y}^*} \right)^2 \right] A - k_c^4 A^2 + [KI_7 A + I_8 A_{X} + I_9 A Y_{*y*}] e^{ikx},$$  \hspace{1cm} (4.18)

where $I_8 = I_8(2k_c, k_c)$. Both (4.17) and (4.18) are analysed in §6.

5. The varicose layer ($0 < k < 2k_c$)

(a) Rectangular cells

As noted above, for this configuration there is an interesting interaction of the modulation with sets of oblique rolls of orientations $\phi$ and $-\phi$ where $\phi = \arccos (k/2k_c)$, i.e. rolls with wavevectors $(k_x, k_y)$ and $(k_x, -k_y)$ such that $k_x^2 + k_y^2 = k_c^2$ and
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Figure 3. Schema illustrating the resonant interaction with $k_w = 2k_x$ (after Pal & Kelly 1979).

$k_w = 2k_x$ (see figure 3). The interaction of the modulation with one oblique roll stimulates the other roll. The surprising result, as we shall see, is that the Rayleigh number for the onset of this pair of oblique rolls (a three-dimensional motion) is lower than that for two-dimensional rolls.

The equations governing the amplitudes, $A$ and $B$, of rolls with orientations $\phi$ and $-\phi$, respectively, are given by

$$A_r = R_2 A + I_3 B - k_c^4 A |A|^2 + \Omega(2\phi) |B|^2, \quad \text{(5.1a)}$$

$$B_r = R_2 B + I_3 \tilde{A} - k_c^4 B |B|^2 + \Omega(2\phi)|A|^2, \quad \text{(5.1b)}$$

where

$$I_3 = \frac{2k_c^2(1-\cos^4\phi)}{(1+\cos^2\phi+\cos^4\phi)}. \quad \text{(5.2)}$$

To keep the analysis simple we have omitted the terms with spatial derivatives: this therefore restricts the analysis to rolls with critical wavenumber.

The simplest solution of (5.1) is the zero solutions and its stability is easily considered by linearising with respect to $A$ and $B$ and taking $A, B \propto e^{\lambda t}$. The growth rate of disturbances in the form of these rolls is given by

$$\lambda = R_2 \pm I_3. \quad \text{(5.3)}$$
Thus there are two unstable modes, the more unstable of which has a critical value of $R_2$ given by
\[ R_{2c}^+ = -I_3, \]  
which is shown in figure 4, and corresponds to a rectangular cellular motion with $A = \bar{B}$. From figure 4 it may be seen that $R_{2c}^+$ varies from $-2k_c^2$ at $k = 0$ to zero at $k = 2k_c$. It is important to note that neither of these limits are included in the analysis of this section because in both cases the rolls are parallel. When $k = 0$, they are transverse ($\phi = \frac{\pi}{2}$), and, when $k = 2k_c$, they are longitudinal ($\phi = 0$). To allow for the effects of parallel or nearly parallel rolls spatial derivatives have to be included when $k_w \sim 0$ and $k_w \sim 2k_c$; these are studied in Rees (1989) and §6, respectively. The second mode has a critical Rayleigh number given by $R_{2c}^- = I_3$ and has the rectangular cellular form $A = -\bar{B}$.

![Figure 4](image_url)

**Figure 4.** The $A = \bar{B}$ mode exists above $R_{2c}^+$ and is stable below $R_{2m}$. The $A = -\bar{B}$ mode exists above $R_{2c}$ and is unstable. The $\alpha A = \bar{B}$ mode exists and is stable above $R_{2m}$. $R_{2c}$ is shown for different values of $\Phi$ and corresponds to values of $R_2$ above which rolls of orientation $\Phi$ are stable.

We consider now the stability of the finite-amplitude flows that evolve from the above two modes of instability. In both cases they evolve without change of planform. The finite-amplitude solution corresponding to the second disturbance is found by setting $A = -\bar{B}$ in (5.1), hence we obtain
\[ A = -\bar{B} = \frac{c k_c^2}{[R_2 - I_3]} \left[ \frac{1 + \Omega(2\phi)}{1 + \Omega(2\phi)} \right] = A_0 \]  
where $|c| = 1$; we set $c = 1$ in the subsequent analysis since taking $c = e^{i\xi}$ results simply in a phase shift in the $y$-direction. This solution may be shown to be unstable.
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by looking at the growth rate of disturbances in the form of its constituent rolls. Let

$$A = A_0 + \alpha A, \quad B = -A_0 + \alpha B$$

(5.6)
in (5.1) and linearize with respect to $\alpha_A$ and $\alpha_B$. The exponential growth rate of the disturbances is given by

$$\lambda = R_2 - (1 + \Omega(2\phi)) k_c^4 A_0^2 \pm I_3$$

(5.7a)

$$= 0, 2I_3,$$

(5.7b)

and therefore this solution is unstable, because $I_3 > 0$. Similarly we may determine the stability of the other finite-amplitude solution: $A = B = A_0$ where

$$A_0 = \frac{c}{k_c^2} \left[ \frac{R_2 + I_3}{1 + \Omega(2\phi)} \right]^{\frac{1}{2}},$$

(5.8)

and $|c| = 1$. Again, because $c$ represents a phase shift in the $y$-direction, it may be set to unity. The perturbations may be considered separately as purely real or purely imaginary, and the growth rate of the real part of the linearized disturbances is given by

$$\lambda = -2(R_2 + I_3) \quad or \quad \frac{2[R_2 - 2I_3/(\Omega(2\phi) - 1)]}{\Omega(2\phi) + 1},$$

(5.9)

and therefore the flow is stable unless

$$R_2 > R_{2m} = \frac{2I_3}{\Omega(2\phi) - 1}.$$ 

(5.10)

This stability bound is also shown on figure 4. The growth rates for the imaginary parts of the linearized disturbances are given by

$$\lambda = -2I_3, 0.$$ 

(5.11)

The first value implies stability, and the second, neutral stability that corresponds to disturbances translating the cell pattern in the $y$-direction.

The instability of the flow given by $A = A_0$ when $A_0$ is given by (5.8) and when (5.10) is satisfied implies that there must be yet another solution of (5.1). This is given by

$$A = A_0 \alpha^{-\frac{1}{2}}, \quad B = A_0 \alpha^{\frac{1}{2}},$$

(5.12a)

where

$$A_0 = [I_3/(\Omega(2\phi) - 1)]^{\frac{1}{2}} k_c^{-2}$$

(5.12b)

$$\alpha = [R_2(\Omega - 1) \pm (R_2^2(\Omega - 1)^2 - 4I_3^2)^{\frac{1}{2}}]/2I_3.$$ 

(5.12c)

Thus we have the following scenario as $R_2$ increases. When $R_2 < -I_3$ the only solution is the zero solution. At $R_2 = -I_3$ a stable solution of the form $A = \bar{B}$ bifurcates from the zero solution and remains stable until $R_2 = R_{2m}$ after which point the resulting flow consists of rolls of unequal amplitude. This progression is summarized in figure 5 together with the bifurcation of the mode $A = \bar{B}$ from the zero solution. From (5.12) we note that as $R_2$ becomes large one of $A$ or $B$ tends to zero, thereby recovering the stable single-roll situation typical of the Lapwood problem.
In the absence of experimental work with which to compare these results, it is of interest to consider the planform of these stable (in the sense considered above) solutions. On using (A 14) and \( A = B = A_0 \), as given by (5.8), the convective cell pressure term is found to be

\[
p^{01} = -4k_c A_0 \sin (k_x x) \cos (k_y y) \sin (k_z z)
\]

and the corresponding horizontal velocities, \((u^{01}, v^{01})\) at the top of the layer are given by

\[
\begin{align*}
(u^{01}) &= 4k_c A_0 \left( -k_x \cos (k_x x) \cos (k_y y) \right) \\
v^{01} &= k_y \sin (k_x x) \sin (k_y y)
\end{align*}
\]

Typical velocity vectors for this flow, and for the flow when \( R_2 > R_{2m} \) are shown in figure 6. Note that the maximum and minimum temperature differences across the layer occur at \( x = n\pi/k_x \), for integer \( n \), which corresponds to stations of maximum velocity in the \( x \)-direction for the flow given by (5.14).

\((b)\) Stability of rectangular cells with respect to roll disturbances

Consider now the stability of the above ‘stable’ solutions with respect to disturbances in the form of rolls of arbitrary orientation. We need only consider one roll disturbance because linearization of the disturbance equations decouples them from one another. On taking a roll of orientation \( \Phi \) and amplitude \( C \), the required amplitude equations are given by

\[
A_\tau = R_2 A + I_3 \bar{B} - k_c A [|A|^2 + \Omega (2\phi)|B|^2 + \Omega (\phi - \Phi)|C|^2],
\]

\[
B_\tau = R_2 B + I_3 \bar{A} - k_c B [|B|^2 + \Omega (2\phi)|A|^2 + \Omega (\phi + \Phi)|C|^2],
\]

\[
C_\tau = R_2 C - k_c C [|C|^2 + \Omega (\phi - \Phi)|A|^2 + \Omega (\phi + \Phi)|B|^2].
\]
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Figure 6. Velocity vectors at the top of the layer for the four flows (a) \( A = B \), (b) \( \alpha A = B \), \( \alpha \leq 1 \), (c) \( \alpha A = B \), \( \alpha \) small, (d) \( \alpha A = B \), \( \alpha \to 0 \). The cells appear square in the plots because of the scaling; generally they are rectangular.

We now linearize these equations about the solution \( A = B \), and find that the equation for the \( C \)-mode disturbance decouples from the \( A \)- and \( B \)-disturbance equations. Thus the growth rate of a disturbance in the form of a roll of orientation \( \phi \) is given by

\[
[1 + \Omega(2\phi)] \lambda = R_2[1 + \Omega(2\phi) - \Omega(\phi + \Phi) - \Omega(\phi - \Phi)] - I_3(\phi) [\Omega(\phi + \Phi) + \Omega(\phi - \Phi)]^T.
\]

(5.16)

Numerically we find that \( \lambda < 0 \) for all \((R_2, \phi)\) such that \( R_{2c}(\phi) < R_2 < R_{2m}(\phi) \), and for all choices of \( \Phi \). Hence this solution is linearly stable.
The growth rate for $C$-mode disturbances to the solution given by (5.12) is given by

$$\lambda = [\alpha^2(1 - \Omega(\phi + \Phi)) + (1 - \Omega(\phi - \Phi))] I_3/\alpha[\Omega(2\phi) - 1], \quad (5.17)$$

which is easily seen to be negative because $\Omega \geq \frac{10}{7}, I_3 > 0$, and $\alpha > 0$. Therefore this solution is also linearly stable.

Hence the conclusions of §5a regarding the stability of these rectangular cells remain valid when considering roll disturbances of arbitrary orientation.

(c) Stability of single rolls of arbitrary orientation

The above analysis demonstrates that the presence of boundary imperfections stimulates a pair of rolls with orientation $\pm \phi$, where $\phi = \arccos(k/2k_c)$. It may be expected, however, that single rolls of arbitrary orientation $\Phi \neq \phi$ could exist given appropriate initial conditions. If we were to concentrate on this roll and analyse its stability in the manner of Newell & Whitehead (1969) then we would obtain identical conclusions to those found without thermal imperfections (namely, that there is a finite continuous band of stable wavenumbers for rolls), but only if we disregarded disturbances in the form of rolls of orientation $\pm \phi$. It is natural, therefore, to extend the study of the stability of the single roll to the case where the boundary imperfections are present, and concentrate on the stability with respect to this mutually resonant pair of rolls. On linearizing (5.15) with respect to $A$ and $B$ where $C = R_1^2/k_c^2$, we obtain

$$\lambda = R_2^2[2 - \Omega^+ - \Omega^-] \pm [R_2^2(\Omega^+ - \Omega^-) + 4I_3^2]^{1/2} \quad (5.18)$$

for the growth rate where $\Omega^\pm = \Omega(\phi \pm \Phi)$. Hence we obtain two conditions for stability:

$$R_2 \geq 0, \quad (5.19a)$$
$$R_2 \geq I_3(\Omega^+ - 1)^{-\frac{1}{2}}(\Omega^- - 1)^{-\frac{1}{2}} \equiv R_{2s}. \quad (5.19b)$$

The former is satisfied trivially whereas the latter bound is drawn in figure 4 for various values of $\Phi$. A roll of orientation $\Phi$ is therefore unstable when $0 < R_2 < R_{2s}$, and stable if $R_2 > R_{2s}$.

To summarize this section, we have found that the most unstable mode takes the form of a pair of rolls of orientations $\pm \arccos(k/2k_c)$, and results in a rectangular finite-amplitude convection cell consisting of these rolls with equal amplitudes. This is stable to small perturbations in the form of rolls of other orientations. Above a second critical value of $R_2 (R_{2m})$ this motion becomes unstable with respect to amplitude disturbances in either roll. The resulting motion is also composed of the two rolls due to a resonant effect of the thermal imperfections, but they now have unequal amplitudes. This motion is also stable with respect to oblique roll disturbances. Convective motion in the form of single rolls of arbitrary orientation may also occur, but this is unstable unless $R_2 \geq R_{2s}$.

This sequence of events is illustrated in figures 4 and 5.
6. The varicose layer \((k_w \sim 2k_c)\)

(a) Two-dimensional analysis

The equation governing the amplitude of the longitudinal roll when \(k_w \sim 2k_c\) is given by (4.17); on integration we find that

\[ I_7 = -k_c \quad \text{and} \quad I_8 = 2k_c, \]  \hspace{1cm} (6.1)

and therefore the amplitude of the longitudinal roll satisfies

\[ A_r = (R_2 - I_4) A + 4A_{XX} + k_c (2iA_X - K A) e^{iKX} - k_c^2 A^2 A, \]  \hspace{1cm} (6.2)

where \(I_4 = I_4(2k_c, k_c) = -1.5967\).

We begin the analysis of this section by considering the stability of the zero solution. On substituting \(A = \alpha e^{iKX}\) in (6.2), where \(|\alpha| \ll 1\), and linearizing in \(\alpha\), we obtain

\[ \alpha_r = (R_2 - I_4 - K^2) \alpha. \]  \hspace{1cm} (6.3)

Therefore this mode, for which there is exactly one roll per imperfection wavelength, grows if

\[ R_2 > R_{c1} = I_4 + K^2. \]  \hspace{1cm} (6.4)

Eventually this growing disturbance evolves into the finite-amplitude solution

\[ A = c k_c^{-2}(R_2 - R_{c1})^{1/2} e^{iKX}, \quad |c| = 1. \]  \hspace{1cm} (6.5)

Note that to \(O(\epsilon^3)\) the roll phase is undetermined, that is, \(c\) is undetermined. Although a higher-order analysis would yield the phase we have found that \(c = \pm 1\) by employing the numerical scheme of Rees & Riley (1986) to calculate large-amplitude convection. We find that rolls with other values of the phase migrate to one or the other of these equilibrium positions. Thus the roll axes lie at the positions of maximum temperature difference across the layer, as illustrated in figure 7.

We know from Newell & Whitehead (1969) that the linearized form of amplitude equations such as (6.2), which contain space-dependent terms explicitly, admits pairs of solutions in the form

\[ A = \alpha_1 e^{i((K+L)X)} + \alpha_2 e^{i((K-L)X)} \]  \hspace{1cm} (6.6)

\[ \bar{\theta} = 1 \quad \text{or} \quad \bar{\theta} = 1 - \delta \cos(k_x x) \]

**Figure 7.** A sketch of the longitudinal mode for the superharmonic case \(k_w \sim 2k_c\) showing the phase of the mode relative to the thermal modulation on the lower boundary.
On substituting this into (6.2) and linearizing we obtain

\[
\begin{pmatrix}
\alpha_1r \\
\bar{\alpha}_2r
\end{pmatrix} = \begin{bmatrix}
R_2 - I_4 - (K + L)^2 & -k_c L \\
-\frac{k_c L}{R_2 - I_4 - (K - L)^2}
\end{bmatrix} \begin{pmatrix}
\alpha_1 \\
\bar{\alpha}_2
\end{pmatrix}.
\] (6.7a)

\[
R_s - I_4 - K^2 - L^2 + L(4K^2 - k_c^2)^h,
\] (6.7b)

The growth rate is given by

\[
\lambda = R_2 - I_4 - K^2 - L^2 + L(4K^2 - k_c^2)^h,
\] (6.8)

which attains a maximum when

\[
L = \begin{cases}
(K^2 - \frac{1}{4}k_c^2)^h & |K| \geq \frac{1}{2}k_c, \\
0 & \text{otherwise,}
\end{cases}
\] (6.9)

so that

\[
\lambda_{\text{max}} = \begin{cases}
R_s - I_4 - \frac{1}{4}k_c^2 & |K| \geq \frac{1}{2}k_c, \\
R_s - I_4 - K^2 & \text{otherwise.}
\end{cases}
\] (6.10)

Therefore the critical value of \( R_s \), \( R_{c2} \), for this pair of modes is given by

\[
R_{c2} = I_4 + \frac{1}{4}k_c^2 \quad \text{when} \quad |K| \geq \frac{1}{2}k_c,
\] (6.11)

which is depicted in figure 8a together with \( R_{c1} \). Note that when \( |K| < \frac{1}{2}k_c \) we have \( L = 0 \), and the above single-mode analysis is recovered. It is interesting to consider the form of (6.6) as \( K \rightarrow +\infty \). From (6.9), \( L \) also becomes infinite, but such that \( K - L \rightarrow 0 \), and it is simple to show that \( |\alpha_1| / |\alpha_2| \rightarrow 0 \). Hence in the limit \( K \rightarrow +\infty \) a single roll is recovered with wavenumber, \( k_c + \frac{1}{2}(K - L) = k_c \). Therefore the limiting value as \( k \rightarrow 2k_c^\pm \) of the critical value of \( R_2 \), \( R_{2\text{crit}} \), for single longitudinal rolls when \( k > 2k_c \) (which is considered in §7) should match with \( R_{c2} \) as \( K \rightarrow +\infty \). This is indeed the case as

\[
R_{2\text{crit}} = I_4(k, k_c) \rightarrow I_4(2k_c, k_x = k_c) + \frac{1}{4}k_c^2 = R_{c2} \quad \text{as} \quad k \rightarrow 2k_c^\pm.
\] (6.12)

For a graphical confirmation of this limiting behaviour we refer the reader to figure 9. An explanation is required at this point of why the term \( \frac{1}{4}k_c^2 \) occurs in (6.12). We note in the Appendix that the \( O(\varepsilon^3) \) equations contain resonant terms: the corresponding solutions, \((F_1, G_1)\), become infinite as \( k \rightarrow 2k_c \) for longitudinal rolls \((k_x = k_c)\). The contribution, however, of those singular solutions to \( R_{2\text{crit}} \) remains finite and approaches the value \( \frac{1}{4}k_c^2 \) in that limit.

We may investigate the stability of the single-mode finite-amplitude solution given by (6.5) to sideband disturbances by setting

\[
A = e^{iKX}[A_0 + \alpha_1 e^{iLX} + \alpha_2 e^{-iLX}]
\] (6.13)

in (6.2) where \( A_0 = k_c^{-\frac{1}{2}}(R_2 - R_{c1})^\frac{1}{2} \), and linearizing with respect to \( \alpha_1 \) and \( \alpha_2 \). The growth rate for these disturbances is given by

\[
\lambda = R_2 - I_4 - K^2 - L^2 - 2k_c^4 A_0^2 + [(4K^2 - k_c^2)L^2 + k_c^8 A_0^4]^\frac{1}{2},
\] (6.14)

which achieves a maximum when

\[
L^2 = [(4K^2 - k_c^2)^2 - k_c^8 A_0^4] / 4(4K^2 - k_c^2).
\] (6.15)
Because $L^2$ must be positive

$$R_2 \leq I_4 + 3K^2 - \frac{1}{2}k_c^2, \quad |K| > \frac{1}{2}k_c$$

(6.16)

has to be satisfied for this analysis to be valid. The marginal stability curve, $R_{c4}$, for this disturbance may be found easily, and therefore when

$$R_{c1} < R_2 < R_{c3} = I_4 + 3K^2 - \frac{1}{2}k_c^2$$

(6.17)

the solution (6.5) is unstable.

(b) Three-dimensional analysis

In this subsection attention will be focused mainly on the stability of the above two-dimensional flows to zigzag and cross-roll disturbances, and partly on the stability of transverse rolls. The equation governing the amplitude of the longitudinal roll when weak three-dimensional effects are included is given by (4.18): on integration $I_9 = -\frac{1}{3}$. Thus the equation is

$$A_r = (R_2 - I_4)A + \left[2 \frac{\partial}{\partial X} - \frac{i}{k_c} \left(\frac{\partial}{\partial Y^*}\right)^2 A - k_c^4 A^2 A + (2ik_c A_x - k_c KA - \frac{1}{2}A_{Y^*Y^*})e^{iKX}\right].$$

(6.18)

The stability of the zero solution to disturbances in the form of a pair of rolls aligned at a small angle about the longitudinal roll direction (the zigzag instability) may be deduced by setting

$$A = \alpha_1 e^{i[(K+L)X+MY^*]} + \alpha_2 e^{i[(K-L)X-MY^*]}$$

(6.19)

in (6.18) and linearizing. The growth rate of disturbances may be shown to be

$$\lambda = R_2 - I_4 - L^2 - K^2 - \frac{KM^2}{2k_c} - \frac{M^4}{16k_c^2} + \left[L^2 \left(2K + \frac{M^2}{2k_c}\right)^2 + \left(\frac{1}{2}M^4 - L^2k_c^2\right)\right]^{\frac{1}{2}}$$

(6.20)

which is to be maximized with respect to both $L$ and $M$. One maximum arises when $M = 0$, but this reproduces the analysis of §6a. The other corresponds to $L = 0$, and

$$M^2 = 4k_c^2 \left(\frac{3}{8} - K/k_c\right) \quad (K < \frac{3}{8}k_c).$$

(6.21)

Therefore the critical value of $R_2, R_{c4}$, is found by substituting (6.21) and $\lambda = 0$ into (6.20) to give

$$R_{c4} = I_4 - \frac{\frac{1}{2}k_c^2}{\frac{3}{8}} + \frac{1}{2}k_cK,$$

(6.22)

which is shown in figure 8. Hence a pair of slightly oblique rolls of the form $\alpha_1 = \alpha_2$ grow when $R_2 > R_{c4}$, $M$ is given by (6.21) and $K < \frac{3}{8}k_c$. Note that the critical Rayleigh numbers given by (6.22) for the zigzag instability and (5.4) for rectangular cells match asymptotically: setting $k_w = 2k_c - \beta \delta$, i.e., $\cos \phi = 1 - \beta \delta^2/2k_c$ in (5.4) and letting $\delta \to 0$, we obtain the critical Rayleigh number

$$Ra_{crit} = Ra_c - \frac{3}{8}k_c \beta \delta + o(\delta^3)$$

(6.23)

for the rectangular solution $A = B$ of §5. The critical Rayleigh number for the zigzag instability is $Ra_c + \delta^2 R_{c4}$: if we take $K = -\beta \delta^{-1}$ and let $\delta \to 0$ we recover
(6.23). The effect of the nonlinear term in (6.18) is to cause complicated mutual and self interactions of the constituent rolls, and a numerical computation is generally required to determine the structure of the resulting finite-amplitude solutions. However, these nonlinear effects decrease in magnitude as $-K$ becomes large, thereby producing the pure two-mode result in §5. It is possible, therefore, to assess the stability of the fully evolved finite-amplitude solution for large negative $K$; we find that it is unstable if $R_2 > R_{c6} \sim -\frac{2}{3}k_cK$ as $K \to -\infty$. This again matches with (5.10) as $\delta \to 0$ in the same way as above. It is also worth commenting that a second eigenmode grows when $R_2 > R_{c6}$ where

$$R_{c6} = I_4 + \frac{4}{3}k_c(k_c - K) \quad (K < \frac{2}{3}k_c)$$

(6.24)

and $M$ is given by (6.21). In this case $\alpha_1 = -\bar{\alpha}_2$, which is identical in form to the unstable mode given by (5.5). Once more the critical Rayleigh numbers of these two modes match in an intermediate asymptotic region.

Having determined the stability of the zero solution to zigzag disturbances aligned about the longitudinal roll direction we now turn to the stability of the single-mode solution given by (6.5) to similar disturbances. On following a similar analysis to that above we find that the growth rate of these disturbances is maximized when (6.21) is satisfied. Both the $\alpha_1 = \bar{\alpha}_2$ and $\alpha_1 = -\bar{\alpha}_2$ modes of instability are important, and the corresponding growth rates are

$$\lambda = -2(R_2 - I_4) + (K - \frac{2}{3}k_c)^2 \quad (\alpha_1 = \bar{\alpha}_2),$$

(6.25a)

$$\lambda = (K - \frac{2}{3}k_c)(K + 2k_c) \quad (\alpha_1 = -\bar{\alpha}_2).$$

(6.25b)

Hence the single-mode solution is unstable either when $R_2 < R_{c7}$, where

$$R_{c7} = I_4 + \frac{1}{2}(K - \frac{2}{3}k_c)^2,$$

(6.26a)

or when

$$K < -2k_c.$$  

(6.26b)

Both these bounds are shown in figure 8.

To complete the analysis of the stability of the single-mode longitudinal roll it is necessary to consider disturbances in the form of transverse rolls. The growth rate of such disturbances is maximised if the transverse roll, whose critical value of $R_2$ is given by $R_{c8} = I_4(2k_c, 0)$, has wavenumber $k_c$. Omitting the details we find that the single longitudinal roll is unstable when $R_{c8} < R_2 < R_{c9}$ where

$$R_{c9} = \frac{1}{3}(10R_{c1} - 7R_{c8}).$$

(6.27)

It is easy to show that the steady finite-amplitude transverse roll with wavenumber $k_c$ is stable to small perturbations in the form of longitudinal rolls. Therefore we consider its stability to zigzag perturbations aligned about the longitudinal roll direction. Once more the growth rate is maximized when (6.21) is satisfied, and transverse rolls are found to be stable when $R_2 > R_{c10}$ where

$$R_{c10} = \frac{1}{3}[10R_{c8} - 7R_{c2}] + \frac{28}{3}k_c[\frac{1}{3}k_c - K]$$

(6.28a)

and

$$K \leq \frac{3}{4k_c} \left[ \frac{4k_c^2}{9} + R_{c8} - R_{c2} \right].$$

(6.28b)
Figure 8. The stability curves $R_{c1}$ to $R_{c10}$ for the superharmonic configuration $k_w \sim 2k_1$, as defined in the text. (a) The curves $R_{c1}$, $R_{c2}$, $R_{c4}$, and $R_{c6}$ corresponding to points above which the zero solution is unstable to perturbations of the form of the single longitudinal roll, a pair of longitudinal rolls, a pair of ‘zigzag’ rolls, and the transverse roll, respectively. (b) Regions of instability of the single longitudinal roll: $R_{c3}$, $R_{c7}$, and $R_{c9}$ are the curves below which it is unstable to perturbations of the form of a pair of longitudinal rolls (sideband instability), an $\alpha_1 = \bar{\alpha}_2$ zigzag pair, and the transverse roll, respectively. The line denoted * refers to instability with respect to the $\alpha_1 = -\bar{\alpha}_2$ zigzag pair, points to the left being unstable. (c) Regions of stability for three-dimensional ‘zigzag’ solutions. $R_{c4}$ is the line above which the $\alpha_1 = \bar{\alpha}_2$ zigzag pair exists, $R_{c5}$ is the line above which it is unstable (asymptotically for large, negative $K$), and $R_{c8}$ the line above which the $\alpha_1 = -\bar{\alpha}_2$ pair exists and is unstable. (d) Region of stability for the transverse roll. $R_{c8}$ is the line above which the roll exists and $R_{c10}$ the line above which it is stable.
Conditions (6.28b) simply ensures that $R_{e10} \geq R_{e8}$, or, that the stability boundary lies within the region of existence of the transverse roll.

We have determined the stability of various flows to different types of small disturbance and determined the critical values of $R_2$ for the onset and marginal stability of these flows. Our calculations are illustrated in figure 8(a–d). Figure 8a shows the marginal stability curves for the longitudinal roll, transverse roll, and those flows generated by a pair of ‘sideband’ roll disturbances and a pair of ‘zigzag’ roll disturbances. When $K > 3(R_{e8} - I_4 + \frac{\pi}{2}k_c^2)/4k_c = 0.1725$ the flow with the smallest critical value of $R_2$ is the transverse roll. For smaller values of $K$ the most dangerous disturbance is in the form of a pair of almost parallel rolls equally oriented about the longitudinal roll direction. The finite-amplitude solution corresponding to this disturbance must be calculated numerically since the cubic terms in the governing amplitude equation generate modes of other orientations. However, for large, negative $K$, the original rolls dominate the other components and the rectangular cells of §5 are recovered.

In figure 8b the stability curves for the single-mode longitudinal roll are displayed. For positive values of $K$ the most dangerous disturbance is the cross-roll instability. However, for larger values of $R_2$ than displayed, the zigzag instability becomes the most dangerous instability mechanism for negative $K$.

Results for the ‘zigzag’ solutions are displayed in figure 8c. The critical values of $R_2$ above which the two convection patterns described in §6a exist are denoted $R_{e4}$ and $R_{e6}$. Also shown is the line $R_{e5}$ which marks the boundary between cellular motion with $\alpha_1 = \bar{\alpha}_2$(below the line) and motion with $\alpha_1 \neq \bar{\alpha}_2$ (cf. (6.19)); this is akin to the situation described in §5. It should be noted that $R_{e5}$ is the result of an asymptotic analysis for large negative $K$, and could therefore be inaccurate for small values.

The regions of stability and instability of the transverse roll are shown in figure 8d where it may be seen that when $K > 0.1725$ the roll is linearly stable.

7. The Varicose Layer ($k > 2k_c$)

For this configuration the amplitude equations are simpler than those considered in §§5 and 6, because the only resonant terms that arise in the expansion (4.4) results in an orientation-dependent correction to the critical Rayleigh number (cf. the $J_4$ term in (A18)). To investigate the stability of a steady, finite-amplitude roll of any orientation to disturbances in the form of rolls of any other orientation it is necessary, therefore, only to consider the amplitude equations for those rolls. We have, from (4.10) and (4.11), the full nonlinear equations

\[
A_z = (R_2 - R_A)A + 4AX_{x_A}x_A - k_c^2A[|A|^2 + \Omega(\phi_A - \phi_B)]|B|^2, \tag{7.1a}
\]

\[
B_z = (R_2 - R_B)A + 4BX_{x_B}A_B - k_c^2B[|B|^2 + \Omega(\phi_A - \phi_B)]|A|^2, \tag{7.1b}
\]

where $R_A$, and $R_B$ are given by $I_4(k, k_c \cos \phi_A)$ and $I_4(k, k_c \cos \phi_B)$, respectively, $\phi_A$ and $\phi_B$ are the roll orientations, and $X_A, X_B$ are the slow horizontal scales.
perpendicular to the roll axes. The finite-amplitude roll which we analyse for stability is given by \( A = A_0 e^{i L X_A} \) where

\[
|A_0| = (R_2 - R_A - 4L^2)^{\frac{1}{2}},
\]

and \( B = 0 \). On linearizing equations (7.1) about this solution, the growth rate of disturbances is given by

\[
\lambda = (R_2 - R_A - 4L^2) (1 - \Omega(\phi_A - \phi_B)) + R_A - R_B + 4L^2.
\]

Values of \( R_A \) as a function of \( k \) and \( \phi \) are shown in figure 9 for \( k > 2k_c \), and as a function of \( k \) for \( \phi_A = 0 \) and \( \phi_B = \frac{1}{2}\pi \) for \( k < 2k_c \). The latter case is shown because it is possible to restrict the layer to have a finite width in the \( y \)-direction to inhibit the more dangerous rectangular modes studied in \( \S \)5. From figure 9 it is worth noting that, for \( k > 2k_c \), \( R_A(\phi, k) \) is a monotonically decreasing function of \( \phi \) for \( 0 \leq \phi \leq \frac{1}{2}\pi \), and therefore transverse rolls constitute the most unstable mode.

Figure 9. Values of \( I_4 \), the critical value of \( R_2 \) for convection in the form of rolls in a varicose layer, as a function of \( k_\infty \) and \( \phi \).

Consider first the restricted situation where the convective motion consists solely of transverse and longitudinal modes. The stability of the longitudinal mode is found by setting \( \phi_A = 0 \), \( \phi_B = \frac{1}{2}\pi \), and \( \lambda = 0 \) in (7.3). Hence the roll is unstable when \( R_A < R_2 < R_{c1} \) where

\[
R_{c1} = \frac{1}{3}[10R_A - 7R_B + 40L^2].
\]
The corresponding result for the stability of the transverse roll is identical except that \( R_A \) and \( R_B \) are the critical values of \( R_2 \) for the transverse and longitudinal modes, respectively, and we denote the stability bound by \( R_{c2} \). Typical curves \( R_{c1} \) and \( R_{c2} \) are sketched in figure 10. Also shown is the result of analysing the sideband instability: transverse rolls are unstable when \( R_T < R_2 < R_{c3} = R_T + 12L^2 \), and longitudinal rolls when \( R_L < R_2 < R_{c4} = R_L + 12L^2 \), where \( R_T \) and \( R_L \) are the critical values of \( R_2 \) for transverse and longitudinal rolls, respectively. Both rolls are unstable to the zigzag instability when \( L < 0 \).

![Figure 10. Sketch of the stability curves for longitudinal and transverse rolls in a varicose layer with \( k_w > 2k_c \). \( R_L \) and \( R_T \) are the marginal curves for the longitudinal and transverse rolls, respectively. \( R_{c1} \) and \( R_{c2} \) are the cross-roll instability curves for longitudinal and transverse rolls, respectively. \( R_{c3} \) and \( R_{c4} \) denote the respective sideband instability curves. For a given value of \( L \) transverse rolls are unstable between \( R_2 \) and \( \max (R_{c2}, R_{c4}) \), whereas longitudinal rolls are unstable between \( R_1 \) and \( R_{c1} \).](image)

We turn briefly to the more complicated case of the unrestricted layer. The critical value of \( R_{c5} \), above which rolls of arbitrary orientation \( \phi_A \) are stable is easily shown to be

\[
R_{c5} = \max_{\phi_B} \left[ \frac{R_A - R_B + 4L^2}{\Omega(\phi_A - \phi_B) - 1} \right].
\]  

(7.5)

For longitudinal rolls \( R_{c5} = R_{c1} \) because the most unstable disturbance takes the form of transverse rolls. For other roll orientations the most unstable mode is not necessarily either the transverse roll or one which is perpendicular to the finite-amplitude roll, but depends on the precise values of \( \phi_A \), \( L \) and \( k \). In general, however, it is possible to make the following observations: (i) when \( L = 0 \) the transverse roll is linearly stable, (ii) all modes are unstable to the zigzag instability if \( L < 0 \), (iii) for large positive \( L \) the standard result of the perfect problem is recovered namely, that a mode of any orientation has a most dangerous disturbance in the form of a roll perpendicular to itself.
8. THE SINUOUS LAYER \((k \neq k_c)\)

(a) For \(k < k_c\)

The amplitude equations for a typical pair of rolls are given by (7.1 a, b) and are therefore identical in form to those considered in the last section. The values of \(R_\alpha\) are, of course, different from those in figure 9, and are shown in figure 11, where we have plotted \((20/\pi)\arctan(\frac{1}{2}R_\alpha)\) against \(k\), because of the presence of a singularity in \(R_\alpha\) at \(k = k_c\). Once more the transverse mode constitutes the most unstable roll orientation, with the value of \(R_\alpha\) increasing as the orientation decreases to zero. Thus the analysis of §7 also applies here.

![Figure 11. The sinuous-layer marginal stability curves \(I_\phi(k_w, \phi)\) above which rolls of orientation \(\phi = 0, \frac{1}{2}\pi, \frac{3}{2}\pi\), together with \(R_{c1}(k_w)\) and \(R_{c2}(k_w)\) for the two rectilinear solutions. Note that the ordinate is \((20/\pi)\arctan F/4\), where \(F = I_4, R_{c1}, R_{c2}\), and that \(0 < k_w < k_c\).](image)

Also important is a mutually resonant pair of rolls whose orientations are \(\phi\) and \(-\phi\), where \(\cos \phi = k/k_c\). The amplitudes \(A\) and \(B\) of rolls of orientation \(\pm \phi\) satisfy the equations

\[
A_r = (R_2 - I_4) A + I_3 \bar{B} - k_c^4 A[|A|^2 + \Omega(2\phi)|B|^2], \tag{8.1a}
\]

\[
B_r = (R_2 - I_4) B + I_3 \bar{A} - k_c^4 B[|B|^2 + \Omega(2\phi)|A|^2], \tag{8.1b}
\]

where \(I_3 = I_3(k, k_c \cos \phi)\) and \(I_4 = I_4(k, k_c \cos \phi)\). The analysis of the various equilibrium solutions and their stability with respect to disturbances in the form of their constituent rolls now follows in the same way as for the varicose layer with \(k < 2k_c\) (cf. § 5 a). The details are therefore omitted and the results summarized below. The critical values of \(R_2\) for the two solutions, \(A = \bar{B}\) and \(A = -\bar{B}\), are \(R_{c1}\) and \(R_{c2}\), respectively, where

\[
R_{c1} = I_4 - I_3, \quad R_{c2} = I_4 + I_3. \tag{8.2}
\]

For the varicose layer \((k < 2k_c)\) \(I_3\) is always positive and therefore the \(A = \bar{B}\) mode constitutes the more unstable disturbance to the zero solution. Here, however, \(I_3\) is negative for \(k \lesssim 0.632k_c\) and positive when \(0.632k_c < k < k_c\), so that \(A = -\bar{B}\)
becomes the more unstable mode for small values of \( k \). Plots of \( R_{c1} \) and \( R_{c2} \) are depicted in figure 11, but because it is difficult to see the behaviour of the curves for \( k > \frac{1}{3} k_c \), we have rescaled the ordinate of figure 11 and re-presented the plot in figure 12. Here, the changeover of the rectangular modes’ critical Rayleigh numbers can be seen clearly. As \( k \to k_c \) all the values in figures 11 and 12 become infinite, and we note that \( R_{c1} \sim \frac{1}{3} k_c^2 (k_c - k)^{-4} \) for which there is an asymptotic match with the behaviour of the zigzag instability curve for \( k \sim k_c \) given in Rees & Riley (1988), and which has the same planform, \( A = B \).

![Figure 12](image)

**Figure 12.** As figure 11, but with the ordinate \((20/\pi) \arctan (F/200)\).

The stability of the various solutions has been studied, and these are summarised in figure 13(a–c). There we show sketches of the curves depicted in figure 12 together with the stability boundaries for each equilibrium solution assuming that each roll component has wavenumber \( k_c \). In figure 13a the regions of stability of the rectangular modes are shown, in figure 13b, the transverse roll and, in figure 13c, the longitudinal roll. The first mode to appear as \( R_4 \) is increased may be seen to depend on \( k \); for \( 0 < k < 0.572 k_c \) it is the \( A = -\hat{B} \) pair, for \( 0.572 k_c < k < 0.738 k_c \), the transverse roll, and, when \( 0.738 k_c < k < k_c \) it is the \( A = \hat{B} \) pair. The \( A = -\hat{B} \) solution is stable to disturbances in the form \( A = \hat{B} \) below curve I in figure 13, and \( A = \hat{B} \) is stable to disturbances in the form \( A = -\hat{B} \) below curve II. Above these curves the solution \( \alpha A = \hat{B} \) exists. Curve III denotes values of \( R_2 \) above which rectangular cells are stable to disturbances in the form of transverse rolls. Curve IV denotes values below which transverse rolls are stable to rectangular cell disturbances. Curves V and VI correspond to the marginal stability of longitudinal rolls with respect to disturbances in the form of rectangular cells and transverse rolls, respectively.

(b) **For \( k > k_c \)**

In general the analysis for this configuration is identical to that of §7 with the appropriate values of \( I_4 \) being as shown in figure 14. However, it was shown in §4 that when \( k_c < k < 3k_c \), a roll of orientation \( \phi_c = \arccos[(k^2 + 3k_c^2)/4kk_c] \) induces
Figure 13. Sketches of the stability regions for (a) rectangular cells, (b) transverse rolls, and (c) longitudinal rolls. The curves labelled I to VI are discussed in the text.
Figure 14. The sinuous-layer marginal curves $I_4(k_w, \phi)$ for $k_w > k_c$. The ordinate is scaled as in figure 11.

Figure 15. Sketch of the bifurcation diagram for the sinuous layer with $k_c < k_w < 3k_c$. (a) and (b) show the respective views in the negative and positive quadrants of $B$. Stable solutions are denoted $S$, and unstable $U$. 
motion in the form of a roll whose orientation \( \phi_B = \arccos [(3k_c^2 - k^2) / 2kk_c] \), and the equations governing this interaction are given by (4.15) and (4.16).

For all \( k \in (k_c, 3k_c) \) the \( B \)-mode has a lower value of \( I_6 \) than the \( A \)-mode because \( \cos^2 \phi_A > \cos^2 \phi_B \). From (4.16) it is seen that roll \( B \) can exist independently of \( A \) and its stability may be deduced by using the analysis of §7. However, roll \( A \) cannot have an independent existence because, from (4.16), any non-zero value of \( A \) induces a non-zero value for \( B \). Because it is possible to redefine \( A \) and \( B \) so that they take real values, we will assume that they are real in analysing possible mixed-mode solutions. Omitting the details, the result of such an analysis is sketched in figure 15 where \( A \) and \( B \) are shown as a function of \( R_2 \). From figure 15 it is seen that the main effect of the \( I_6 \) term in (4.16) is to replace the pure \( A \)-mode by a mixed mode where \( |B| \rightarrow |I_6| / (\Omega(\phi_A) - 1) \) as \( R_2 \rightarrow \infty \), and where \( |A|^2 \rightarrow (R_2 - I_4(\phi_A)) / k_c^4 \), its value if \( I_6 \) were zero. The point that divides the stable and unstable regions of the mixed-mode solution is always on the turning point of the mixed-mode branch.

9. The subharmonic resonances

In this section we discuss briefly the two subharmonic resonant cases, \( k_w \sim \frac{1}{2} k_c \) and \( k_w \sim \frac{3}{2} k_c \). These resonances were first noted for this type of problem by Pal & Kelly (1978) who also gave the appropriate scalings for \( \epsilon \) in terms of \( \delta \).

We consider first the configurations for which \( k_w \sim \frac{1}{2} k_c \). For the resonance to exist we find that the thermal imperfections must be a combination of sinuous and varicose modulations, because otherwise the \( O(\delta^2) \) solutions are bounded. As the required configuration has a varicose component the most dangerous disturbance will be in the form of a pair of rolls inclined at angles \( \pm \arccos (\frac{1}{3}) \) (cf. §5) unless the layer is sufficiently narrow in the \( y \)-direction not to accommodate them into the available space. Thus we will only consider two-dimensional flow.

Let the new boundary conditions replacing (2.3–5) on \( z = \pm 1 \) be

\[
\theta = \mp 1 \pm \delta(c_v e^{ik_w x} + \text{c.c.}) + \delta(c_s e^{ik_w x} + \text{c.c.}), \quad (9.1a)
\]

\[
p_z = Ra \theta, \quad (9.1b)
\]

where \( c_v \) and \( c_s \) are complex constants. On setting \( \epsilon = \delta^4 \) and proceeding with the expansion in \( \epsilon \), the following equation for the longitudinal roll is eventually obtained at \( O(\epsilon^3) \):

\[
A_y = R_2 A_x + 4 A_{xx} - k_c^4 A_x A - ic_v c_s I_{10} e^{2ikx}, \quad (9.2)
\]

where

\[
I_{10} = \frac{\sqrt{3} k_c^2}{18} \left( \tan \frac{\sqrt{3}}{2} k_c - \cot \frac{\sqrt{3}}{2} k_c \right) = 1.06078. \quad (9.3)
\]

Note that if either \( c_v \) or \( c_s \) is zero, then (9.2) reduces to the amplitude equation for the perfect problem, and the effects of the imperfections are felt at higher orders. In this eventuality we would set \( \epsilon = \delta \) to recover an amplitude equation with an \( I_4 \), or critical Rayleigh number correction, term. Also it may be shown that the stable phase of the steady roll will then be found at \( O(\epsilon^5) \). For the present problem it is an easy matter to show that the phase of the solution \( A = A_0 e^{2ikx} \) of (9.2)
is given by \( \text{phase}(A_0) = \text{phase}(ic_v s_0) \). Because it is possible to introduce substitutions to reduce (9.2) to a form identical to (4.5), the equation for the near-resonant configuration (sinuous layer, \( k \sim k_c \)), we shall omit further analysis. Thus the analysis of Rees & Riley (1988, 1989) for the cross-roll and sideband instabilities is appropriate here.

Finally we consider the sinuous layer with \( k_w \sim \frac{1}{3} k_c \). The amplitude equation for longitudinal rolls is given by (cf. (4.6))

\[
A_r = (R_2 - I_1) A + 4A_{XX} - k_c^4 A^2 A - i I_1 e^{3iKX},
\]

(9.4)

where \( I_1 \) is found to have the value \(-2.0368\). This equation admits solutions of the form \( A \propto e^{3iKX} \) and there is only one stable roll phase, namely \( \frac{1}{3} \pi \); this is illustrated in figure 16. Equation (9.4) may also be reduced to the form of (4.4) by suitable substitutions.

![Figure 16. Sketch of the lower boundary temperature and the streamlines of the stable longitudinal roll for a sinuous modulation with \( k_w \sim \frac{1}{3} k_c \).](image)

10. Discussion

We have studied the onset and stability of convection in a fluid-saturated porous layer heated from below and cooled from above, when small-amplitude thermal nonuniformities at the horizontal boundaries have an associated wavenumber, \( k_w \), not close to \( k_c \), the critical value for the uniformly heated layer. This has been accomplished by using a weakly nonlinear theory to generate equations governing the finite-amplitude convective motion which exists when the Darcy-Rayleigh number is close to \( Ra_c \). Our aim has been to assess the effects of thermal noise on the onset, pattern selection and stability of convection. Therefore we have systematically identified the resonances that arise for various configurations to evaluate the effects of each type of imperfection separately. It is now possible to rank the imperfections according to the severity of their effect on finite-amplitude Lapwood convection.

The most dangerous imperfection occurs in a layer with antisymmetric (or sinuous) modulation with \( k_w \sim k_c \), the near-resonant configuration. This case is studied in detail in Rees & Riley (1988, 1989). In the present paper, we have shown that this imperfection has a leading-order effect when \( Ra - Ra_c = O(\delta^3) \). In general when this particular imperfection is present, all other imperfections have higher-order effects.

The next most dangerous imperfection corresponds to symmetric (or varicose) modulation with \( k_w \in (0, 2k_c) \), which gives rise to a leading-order effect when \( Ra - Ra_c = O(\delta) \). In this case we showed that the weak baroclinic motion loses stability to a three-dimensional mode with a rectangular planform. It is worth
noting that this imperfection is important not only in the absence of the near-resonant imperfection, but also if the latter is sufficiently small (specifically if the typical magnitude of the near-resonant imperfection is \( \delta_{nr} \), then \( \delta_{nr} \ll \delta^1 \) ensures that the resonant terms corresponding to the rectangular cells appear in the \( \epsilon \)-expansion before those for the near-resonant imperfection). As the Darcy–Rayleigh number increases further, these rectangular cells, which are composed of two rolls of equal amplitude, in turn lose their stability at a supercritical bifurcation to a mode composed of two rolls of unequal amplitude. We have also shown that rolls of arbitrary orientation are linearly stable at Rayleigh numbers that are sufficiently supercritical.

The subharmonic modulation with \( k_w \sim \frac{1}{2} k_c \) is next in the hierarchy: when \( Ra - Ra_c = O(\delta^3) \) two-dimensional rolls result. The imperfection must possess both varicose and sinuous components for this case to be important (see §9). Therefore the instability corresponding to the varicose case, above, must be suppressed by having, for example, a layer with a finite width in the \( y \)-direction. It would also be necessary for the amplitude, \( \delta_{nr} \), of any near-resonant modulations that are present to satisfy \( \delta_{nr} \ll \delta^2 \) to relegate their effects to higher order.

All other imperfections have been shown to be important when

\[
Ra - Ra_c = O(\delta^2),
\]

but only when the amplitudes of the above three imperfections are sufficiently small. It is interesting to note that when the modulation is a superposition of modes with parallel wavevectors and wavenumbers \( k_1 \) and \( k_1 + k_c \), it may be shown that, provided one has a sinuous element and the other a varicose one, the \( O(\delta^2) \) solutions become singular at \( Ra_c \). The resulting roll scaling is then \( \epsilon = \delta^\frac{1}{3} \) (in contrast to the single-mode case where the scaling is \( \epsilon = \delta \)), and is therefore comparable in its effect with the subharmonic case above.

We have concentrated on one-dimensional imperfections, and it is natural to question what would be the effects of two or more imperfections with non-parallel wavevectors and different wavenumbers. Although this is outside the scope of the present paper, it is a simple matter to show that the number of different special cases to consider, even for two wavevectors, would be considerably greater than here. One interesting example is when the wavevectors are perpendicular, with corresponding wavenumbers equal, and close to \( k_c \). The flow occurring subcritically then consists of square cells, which persist until they eventually become unstable at \( Ra = Ra_c + O(\delta^\frac{1}{3}) \) to form a pair of rolls of unequal amplitude. As the Darcy–Rayleigh number becomes sufficiently supercritical one of the rolls decays to zero recovering the result of the perfect problem, namely that rolls are the only stable mode.

The analysis here has focused on the effects of finite, non-zero values of \( k_w \). It is of great interest, in view of the possible geothermal applications of this work, to consider what happens if the imperfections occur over a large lengthscale (i.e. \( k_w \sim 0 \)). A point to note is that the results of §5 suggest that the finite-amplitude motion which appears first consists of rectangular cells; this has been confirmed in Rees (1989).

Finally, there are two important questions regarding the range of validity of our
study. First, there is the question of whether we have used the correct scalings, for the bifurcation structure will only be accurately resolved if we have. In general we can only be sure that we have chosen correctly by cross validating with results from Lyapounov–Schmidt reduction and/or numerical computation. It is worth remarking, moreover, that modern continuation methods often make use of local (i.e. weakly nonlinear) results. Thus there seems to us no simple method of validating the scalings; one must be guided a priori, for example, by the form of the neutral stability curve and by the need to balance various effects in the amplitude equations. One then must make a posteriori assessments of the results.

The second question regards the ranges of the small parameters (Darcy–Rayleigh number and imperfection amplitude) for which these local results apply. This, in common with most asymptotic analyses is very difficult to answer, and is judged most easily (relatively speaking) against numerical results for the fully nonlinear cases. The local structure near a bifurcation point can, however, have global consequences; for example, the unfolding of a bifurcation results in a preferred direction of flow, and sometimes anomalous convection. These features persist for non-small values of the excess Darcy–Rayleigh number, and, moreover, for non-small values of the modulation amplitude for it is the symmetry of the modulation which is important rather than the size.

Generally it is the symmetries that determine the bifurcation structures in convection problems. Thus most of the results we have presented here should go over, at least qualitatively, to the pure-fluid problem.

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**Appendix. The finite amplitude expansions in \( \epsilon \) and \( \delta \)**

On substituting (4.1) into equations (2.1) and (2.2) the following equations for the \( O(\delta) \) terms are obtained:

\[
\mathcal{L}_1(p^{10}, \theta^{10}) = 0, \quad (A\ 1a)
\]

\[
\mathcal{L}_2(p^{10}, \theta^{10}) = 0, \quad (A\ 1b)
\]

which are to be solved subject to

\[
\theta^{10}(z = \pm 1) = \pm \frac{1}{2} (e^{i(kx+Kx)} + e^{-i(kx+Kx)}) \quad \text{(varicose)} \quad (A\ 1c)
\]

\[
\theta^{10}(z = \pm 1) = + \frac{1}{2} (e^{i(kx+Kx)} + e^{-i(kx+Kx)}) \quad \text{(sinuous)} \quad (A\ 1d)
\]

\[
p^{10}_z(z = \pm 1) = Ra_c \theta^{10}(z = \pm 1), \quad (A\ 1e)
\]

where the linear differential operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are given by

\[
\mathcal{L}_1(p, \theta) = \nabla^2 p - Ra_c \theta_z \quad (A\ 2a)
\]

\[
\mathcal{L}_2(p, \theta) = \nabla^2 \theta + Ra_c \theta - p_z. \quad (A\ 2b)
\]

These equations are easily solved by substituting

\[
(p^{10}, \theta^{10}) = \frac{1}{2} (f_0(z), g_0(z)) [e^{i(kx+Kx)} + e^{-i(kx+Kx)}], \quad (A\ 3)
\]
to obtain

\( f_0'' - k^2 f_0 - Ra_c g_0' = 0, \)  
\( g_0'' + (Ra_c - k^2) g_0 - f_0' = 0, \)

subject to

\( f_0'(\pm 1) = \pm Ra_c, \ g_0(\pm 1) = \pm 1 \quad \text{(varicose),} \)
\( f_0'(\pm 1) = + Ra_c, \ g_0(\pm 1) = + 1 \quad \text{(sinuous).} \)

Equations (A 4) have the solutions

\( f_0 = \frac{\sqrt{Ra_c}}{2k} \left( \frac{\gamma \cosh (\gamma z)}{\sinh (\gamma)} - \frac{\chi \cosh (\chi z)}{\sinh (\chi)} \right) \)  
\( g_0 = \frac{1}{2} \left( \frac{\sinh (\gamma z)}{\sinh (\gamma)} + \frac{\sinh (\chi z)}{\sinh (\chi)} \right) \)

for varicose layers, and, for sinuous layers:

\( f_0 = \frac{\sqrt{Ra_c}}{2k} \left( \frac{\gamma \sinh (\gamma z)}{\cosh (\gamma)} - \frac{\chi \sinh (\chi z)}{\cosh (\chi)} \right) \)  
\( g_0 = \frac{1}{2} \left( \frac{\cosh (\gamma z)}{\cosh (\gamma)} + \frac{\cosh (\chi z)}{\cosh (\chi)} \right) \)

where

\( \gamma^2 = k^2 + kRa_c^{1/3} \)
\( \chi^2 = k^2 - kRa_c^{1/3} \)

It should be noted that \( \chi \) may be either real or imaginary.

At \( O(\delta^2) \) the following equations are obtained

\( \mathcal{L}_1(p^{20}, \theta^{20}) = 0, \)  
\( \mathcal{L}_2(p^{20}, \theta^{20}) = Ra_c \theta^{10}_x - p^{10}_x \theta^{10}_x - p^{10}_z \theta^{10}_z, \)

which are to be solved subject to \( \theta^{20}(z = \pm 1) = p^{20}(z = \pm 1) = 0. \) The solutions are found by substituting

\( (p^{20}, \theta^{20}) = (f_1(z), g_1(z)) + \frac{1}{2}(f_2(z) + g_2(z))[e^{21(kx+Kx)} + e^{-21(kx+Kx)}], \)

into (A 8) to give

\( f_1'' - Ra_c g_1' = 0, \)
\( g_1'' = \frac{1}{2}(Ra_c g_0 g_0' - k^2 f_0 g_0 - f_0' g_0'), \)

and

\( f_2'' - 4k^2 f_2 - Ra_c g_2' = 0, \)
\( g_2'' + (Ra_c - 4k^2) g_2 - f_2' = \frac{1}{2}(Ra_c g_0 g_0' + k^2 f_0 g_0 - f_0' g_0'), \)

subject to \( f_1' = g_1 = f_2' = g_2 = 0. \) The solutions to (A 11) are complicated and are omitted; the solutions to (A 10) are

\( \frac{f_1'}{Ra_c} = \frac{\sqrt{Ra_c} k}{32} \left( \frac{\sinh (2\chi z)}{\chi \sinh^2 (\chi)} - \frac{\sinh (2\gamma z)}{\gamma \sinh^2 (\gamma)} + \left( \frac{\sinh (2\gamma)}{\chi \sinh^2 (\chi)} \right) \right) \)
for varicose configurations, and,

\[
\frac{f'_1}{Ra_c} = g_1 = \frac{\sqrt{Ra_c} k}{32} \left( \frac{\sinh(2\chi z)}{\chi \cosh^2(\chi)} - \frac{\sinh(2\gamma z)}{\gamma \cosh^2(\gamma)} + \left( \frac{\sinh(2\gamma)}{\gamma \cosh^2(\gamma)} - \frac{\sinh(2\chi)}{\chi \cosh^2(\chi)} \right) z \right) \tag{A 12b}
\]

for sinuous configurations.

At \(O(\delta^0)\) we obtain

\[
\mathcal{L}_1(p^{30}, \theta^{30}) = 0, \tag{A 13a}
\]

\[
\mathcal{L}_2(p^{30}, \theta^{30}) = \frac{1}{2}J_1(z) \left[ e^{3i(kx+KX)} + e^{-3i(kx+KX)} \right] + \text{terms in } [e^{i(kx+KX)} + e^{-i(kx+KX)}], \tag{A 13b}
\]

where

\[
J_1 = 2k_c^2(g_0 g_2)' + k^2(f_0 g_2 + g_0 f_2) - \frac{1}{2}(g_0' f_2' + g_2' f_0'). \tag{A 13c}
\]

At \(O(\epsilon)\) we have \(\mathcal{L}_1(p^{01}, \theta^{01}) = \mathcal{L}_2(p^{01}, \theta^{01}) = 0\), which admits eigensolutions of the form

\[
\begin{pmatrix}
  p^{01} \\
  \theta^{01}
\end{pmatrix} = \frac{i}{2} [A(X, Y, \tau) e^{i(kx+ky)} - c.c.] \begin{pmatrix}
  2k_c \sin(k_c z) \\
  \cos(k_c z)
\end{pmatrix}. \tag{A 14}
\]

We assume, without loss of generality, that eigensolutions only occur in the \(O(\epsilon)\) solutions. Here, \(k_x^2 + k_y^2 = k_c^2\), \(\tau = \frac{1}{2} \epsilon^2 t\) is a slow timescale, and the orientation of this roll, \(\phi\), is defined by

\[
\cos \phi = k_x/k_c, \quad \sin \phi = k_y/k_c. \tag{A 15}
\]

In this paper the roll for which \(\phi = 0\) is termed a longitudinal roll, and the roll for which \(\phi = \frac{1}{2} \pi\) is termed a transverse roll because its orientation is perpendicular to the rolls constituting the quasi-conduction solution.

The first interaction between the roll and the quasi-conduction solution takes place at \(O(\delta \epsilon)\) where \((p^{11}, \theta^{11})\) satisfy

\[
\mathcal{L}_1(p^{11}, \theta^{11}) = Ra^{10} \theta^{01} + kK f_0 [e^{i(kx+KX)} + c.c.], \tag{A 16a}
\]

\[
\mathcal{L}_2(p^{11}, \theta^{11}) = -Ra^{10} \theta^{10} + kK g_0 [e^{i(kx+KX)} + c.c.]
+ \frac{1}{2} J_2(z) [A e^{i((k_x+k)y+KX)} - c.c.]
+ \frac{1}{2} J_3(z) [A e^{i((k_x-k)y-KX)} - c.c.] \tag{A 16b}
\]

where

\[
J_2 = \frac{1}{2} k_c f_0' \sin(k_c z) + \frac{1}{2} k_c k f_0 \cos(k_c z) + k_c (k_x k - 2k_c^2) g_0 \sin(k_c z) + k_c^2 g_0' \cos(k_c z), \tag{A 16c}
\]

\[
J_3 = \frac{1}{2} k_c f_0' \sin(k_c z) - \frac{1}{2} k_c k f_0 \cos(k_c z) - k_c (k_x k + 2k_c^2) g_0 \sin(k_c z) + k_c^2 g_0' \cos(k_c z). \tag{A 16d}
\]

Accordingly we set

\[
(p^{11}, \theta^{11}) = \frac{1}{2i} [A e^{i((k_x+k)y+KX)} - c.c.] (F_0(z), G_0(z))
+ \frac{1}{2i} [A e^{i((k_x-k)y-KX)} - c.c.] (F_1(z), G_1(z))
+ \frac{1}{2} K [e^{i(kx+KX)} + c.c.] (f_3(z), g_3(z)). \tag{A 17}
\]
In general, equations (A 17) do not possess solutions unless \( Ra^{10} = 0 \) because otherwise the right-hand sides contain terms proportional to the eigensolutions of the left hand sides.

At \( O(\delta^8 e) \) the equations for \((p^{21}, \theta^{21})\) are

\[
\mathcal{L}_1(p^{21}, \theta^{21}) = Ra^{20} \theta_z^{01} + \text{terms with wavevectors } (0,0) \text{ and } 2(k + \epsilon K,0), \quad (A 18a)
\]

\[
\mathcal{L}_2(p^{21}, \theta^{21}) = -Ra^{20} \theta_z^{01} + \frac{1}{2} i J_4(z) [A e^{i(k_x x + k_y y)} - \text{c.c.}] + \frac{1}{2} i J_6(z) [A e^{i((k_x - 2k)c x + k_y y - 2Kx)} - \text{c.c.}]
\]

\[+ \text{terms with wavevectors } (0,0), 2(k + \epsilon K,0), \text{ and } (k_x + 2k + 2\epsilon K, k_y), \quad (A 18b)\]

where

\[
J_4(z) = k_c f'_1 \sin (k_c z) + 2k_c^3 g_1' \cos (k_c z) - 4k_c^3 g_1 \sin (k_c z) + 2k_c^3 (g_0 G_0 + G_1)
\]

\[+ \frac{1}{2} k(k_x - k) (f_0 G_1 + g_0 F_1) - \frac{1}{2} k(k_x + k) (f_0 G_0 + g_0 F_0) - f'_0 (G_0' + G_1') - G_0' (F_0' + F_1'), \quad (A 18c)
\]

\[
J_6(z) = - k_c k f_2 \cos (k_c z) + \frac{1}{2} k_c f_2' \sin (k_c z) - 2k_c (k_c^2 + k) g_3 \sin (k_c z) + k_c^2 g_3' \cos (k_c z)
\]

\[+ 2k_c^2 (g_0 G_1)' - \frac{1}{2} k(k_x - k) (f_0 G_1 + g_0 F_1) - (f'_0 G_1' + g_0 F_1'). \quad (A 18d)
\]

These equations are insoluble unless \( Ra^{20} \) has a value determined by a solvability condition as the term multiplying \( J_4 \) is proportional to the roll eigensolution. The value of \( Ra^{20} \) thus obtained is the \( O(\delta^8) \) correction to the critical Rayleigh number, in general.

At \( O(\delta^2 e^2) \) the following equations are obtained:

\[
\mathcal{L}_1(p^{12}, \theta^{12}) = Ra^{11} \theta_z^{01} + Ra^{01} \theta_z^{11} - K^2 p_{xx}^{10} - 2p_{xx}^{11}, \quad (A 19a)
\]

\[
\mathcal{L}_2(p^{12}, \theta^{12}) = \frac{1}{2} i J_6(z) [A e^{i((k_x - k)c x + k_y y - Kx)} + \text{c.c.}]
\]

\[+ \frac{1}{2} i K J_7(z) [A e^{i((k_x - k)c x + k_y y - Kx)} + \text{c.c.}]
\]

\[+ \frac{1}{2} i J_8(z) [A e^{i((k_x - k)c x + k_y y - Kx)} + \text{c.c.}] - Ra_{11} \theta_z^{01} - Ra^{01} \theta_z^{11}
\]

\[+ \text{terms with wavevectors } (k,0), (2k_x + k, 2k_y), \text{ and } (k_x + k, k_y), \quad (A 19b)\]

where

\[
J_6 = \frac{1}{2} k(k_x - k_c^2) F_1 \cos (k_c z) + k_c (k_x k + k_c^2) G_1 \sin (k_c z) - k_c^2 G_1' \cos (k_c z), \quad (A 19c)
\]

\[
J_7 = k_c (k_c g_3' - k_c f_3 - f_0) \cos (k_c z) + k_c (f_3' - 4k_c g_3 - 2k_c g_0) \sin (k_c z), \quad (A 19d)
\]

\[
J_8 = k_c (g_0' - k_c f_0') \cos (k_c z). \quad (A 19e)
\]

Note that the terms \( J_7 \) and \( J_8 \) are important only when the layer is varicose and when \( k = 2k_c \) and \( k_x = k_c \). For this particular case, if we consider the zigzag instability (cf. Newell & Whitehead) by introducing the slow variable \( Y^* = e^{\delta y} \), then the expression

\[
\frac{1}{2} i J_6(z) [A Y^* Y^* e^{i((k_x - k)c x + k_y y - Kx)} - \text{c.c.}] \quad (A 19f)
\]

has to be added to the right-hand side of (A 19b), where

\[
J_9 = - \frac{1}{2} (kg_0 \sin (k_c z) + g_0' \cos (k_c z)). \quad (A 19g)
\]
At $O(\epsilon^3)$ we obtain the same equations as would arise from an analysis of the perfect problem; this results in the following equation for the amplitude, $A$, of a single roll:

$$A_r = (Ra_{92} A + 4A_{X_p} X_p - k_c^4 A^2 \bar{A}),$$  \hspace{1cm} (A 20)

where $X_p$ is the slow horizontal coordinate perpendicular to the roll axis, i.e. $X_p = X \cos \phi + Y \sin \phi$, and the overbar denotes complex conjugation. Following Newell & Whitehead (1969) the appropriate equation for considering the zigzag instability is

$$A_r = \left[ Ra_{92} A + \left( 2 \frac{\partial}{\partial X_p} - \frac{i}{k_c} \left( \frac{\partial}{\partial Y_p^*} \right)^2 \right)^2 A - k_c^4 A^2 \bar{A} \right],$$  \hspace{1cm} (A 21)

where $Y_p^* = e^{ik}(y \cos \phi - x \sin \phi)$, i.e. a slow coordinate parallel to the roll axis. The effect of considering a second roll of orientation $\phi_B$ is to replace (A 20) by

$$A_r = \left[ Ra_{92} A - k_c^4 A(|A|^2 + \Omega(\phi - \phi_B)|B|^2) \right],$$  \hspace{1cm} (A 22a)

$$B_r = \left[ Ra_{92} B - k_c^4 B(|B|^2 + \Omega(\phi - \phi_B)|A|^2) \right],$$  \hspace{1cm} (A 22b)

where the terms involving the spatial derivatives have been omitted, for simplicity, and where

$$\Omega(\alpha) = \frac{70 + 28 \cos^2 \alpha - 2 \cos^4 \alpha}{49 - 2 \cos^2 \alpha + \cos^4 \alpha}.$$  \hspace{1cm} (A 23)

This function attains a maximum of 2 at $\alpha = 0$, decreasing to a minimum of $\frac{10}{7}$ at $\alpha = \frac{\pi}{2}$.

References


The effects of imperfections on free convection


Rees, D. A. S. 1989 The effects of long wavelength boundary imperfections on the onset of convection in a porous layer. (In preparation.)


