Natural convection in shallow porous cavities near the density maximum: the conduction and intermediate regimes

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We investigate two-dimensional natural convection in a shallow rectangular cavity filled with a porous medium which is saturated with a fluid which has a density maximum. One sidewall of the cavity is maintained at a temperature below the density maximum and the other sidewall is maintained at a temperature above the density maximum. The top and bottom boundaries are insulating. Attention is paid to the case when the aspect ratio, $A$, defined as the ratio of the height of the cavity to its width, is asymptotically small. The domain is divided into two end zone regions near the hot and cold walls, a density maximum turning zone region, and two central core regions in between the end-zone regions and the density maximum region. Asymptotic analysis is performed in the conduction regime when $Ra \sim O(1)$ and in the intermediate regime when $Ra \sim O(1/A)$, and heat transfer correlations are presented. Numerical solutions to the full governing solutions are obtained and these solutions are compared with the asymptotic results.

Keywords: natural convection; density maximum; porous media; asymptotic methods.

1. Introduction

The problem of convection in a shallow rectangular cavity filled with a fluid-saturated porous medium has been studied extensively (Walker & Homsy, 1978; Blythe et al., 1983; Daniels et al., 1989a,b) for the case where the density of the fluid is a linear function of temperature. This flow geometry is of great relevance to both geothermal energy extraction and to the design of solar energy collectors. Convection in shallow cavities filled with a porous medium shares many features with convection in shallow cavities filled with a Newtonian fluid, a problem which has also been extensively studied (Cormack et al., 1974a,b; Imberger, 1974). In this paper, we will examine convection in a shallow rectangular cavity filled with a fluid-saturated porous medium in the case where the fluid has a density maximum. Qualitatively, the key features of convection near the density maximum can be inferred from the previous studies of convection where density is a linear function of temperature. Quantitatively, however, there are fundamental differences due to the presence of a density maximum, and these differences will be highlighted in this paper.

Convection in a shallow rectangular cavity can be understood with reference to Fig. 1(a) for the case of a fluid with a linear dependence of temperature on density. The top and bottom walls are impermeable and insulating, and the vertical sidewalls are impermeable and maintained at constant temperatures of $T_c$. 

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Fig. 1. Natural convection in a shallow enclosure: (a) no density maximum between $T' = T_c$ and $T' = T_h$; (b) with density maximum between $T' = T_c$ and $T' = T_h$.

and $T_h = T_c + \Delta T$ (where $\Delta T > 0$), respectively. In the steady state a convection cell forms with fluid rising along the hot wall and then moving towards the cold wall in the top half of the cavity. Similarly, fluid sinks along the cold wall and moves towards the hot wall in the bottom half of the cavity. The exact nature of the convection cell, and the resultant temperature field, will depend on the imposed temperature difference, the geometry of the cavity, and properties of the fluid and of the porous media.

To identify the different flow regimes, we will compare the timescale for the diffusion of heat between the top and boundary walls with the time required for a fluid parcel to be advected between the hot and cold walls. The diffusive timescale is given by $t_{\text{diff}} = h^2/\kappa$, where $h$ is the height of the cavity and $\kappa$ is the effective thermal diffusivity of the fluid-saturated porous medium. The advective velocity scale $u_{\text{adv}}$ can be estimated by balancing the integrated effect of Darcy friction over the length $\ell$ of the cavity with the effect of buoyancy forcing over the height of the cavity to give $\nu u_{\text{adv}} \ell/K \sim g \beta \Delta T h$, where $\nu$ is the kinematic viscosity of the fluid, $K$ is the permeability of the porous media, $g$ is the acceleration of gravity and $\beta$ is the linear thermal expansion coefficient of the fluid. The advective timescale is given by $t_{\text{adv}} = \ell/u_{\text{adv}}$ and the ratio of the diffusive to the advective timescales is

$$\lambda = \frac{t_{\text{diff}}}{t_{\text{adv}}} = \frac{h^2/\kappa}{(\nu \ell^2/g \beta \Delta T k h)} = A^2 Ra_{\text{lin}},$$

where

$$A = \frac{h}{\ell}$$

is the aspect ratio of the cavity and

$$Ra_{\text{lin}} = \frac{g \beta \Delta T k h}{\nu \kappa}$$

is the Darcy–Rayleigh number for a fluid with a linear dependence of temperature on density. When $Ra_{\text{lin}} = O(1)$, the diffusive timescale will be considerably smaller than the advective timescale in the
limit as \( A \to 0 \). Thus, to leading order, thermal diffusion of heat smoothes out any vertical variations of temperature and advection only has an \( O(A^2) \) influence on the temperature field. The case when \( \text{Ra}_{\text{lin}} = O(1) \) is known as the conduction regime. When \( \text{Ra}_{\text{lin}} = O(1/A) \), the diffusive timescale will still be considerably smaller than the advection timescale and diffusion smoothes out any vertical variations in temperature to leading order. This regime is known as the intermediate regime and it is known (Daniels et al., 1989b) that advective effects will influence the temperature field at \( O(A) \). Finally, when \( \text{Ra}_{\text{lin}} = O(1/A^2) \), the ratio of diffusive and advective timescales is of \( O(1) \), and it is not possible to comment on the form of the leading order temperature field without performing more detailed calculations. The \( \text{Ra}_{\text{lin}} = O(1/A^2) \) limit is known as the boundary layer regime and it can be shown that the entire flow structure is controlled by thin vertical boundary layers along the vertical sidewalls of the cavity.

Convection due to sidewall heating in a cavity filled with a fluid-saturated porous medium is qualitatively very similar to convection in a cavity filled with a Newtonian fluid. Consequently, features of convection near the density maximum in a porous medium can be inferred from previous studies on convection in a Newtonian fluid (and vice versa). Ho & Tu (2001a) have examined convection near the density maximum experimentally in water in a tall cavity (\( A = 8 \)) and McDonough & Eaghri (1994) and Ivey & Hamblin (1989) have examined convection near the density maximum experimentally in shallow cavities (\( A = 0.75 \) and \( A = 0.1 \), respectively). Most of the experimental results by Ivey & Hamblin (1989) correspond to the boundary layer regime, however, they do provide some analytical results for convection in the conduction regime. Others have examined convection in a Newtonian fluid near the density maximum numerically for a range of aspect ratios (Tong & Koster, 1994; Tong, 1999; Ho & Tu, 2001b). From these studies, it can be inferred that in the case of convection in a shallow cavity near the density maximum a double cell structure should emerge (Fig. 1(b)) with fluid rising along the hot and cold walls, and travelling towards the location of the density maximum and then falling and returning to the end walls in the lower half of the cavity. While the previous studies of convection near the density maximum in a Newtonian fluid can be used to qualitatively describe features of fluid flow, there has been no systematic work examining the influence of increasing the value of the Rayleigh number in order to examine the transition between the different flow regimes for convection in a shallow cavity.

The purpose of this paper is to examine convection near the density maximum in a shallow cavity filled with a fluid-saturated porous medium. Results are presented in both the conduction regime (\( \text{Ra} = O(1) \)) and in the intermediate regime (\( \text{Ra} = O(1/A) \)) and the transition between these two regimes will be examined. As drawn in Fig. 1(b), there is a dividing streamline at the location of the density maximum which prevents fluid from the cell in the cold part of the cavity from entering the cell in the hot part of the cavity and vice versa. The location of this density maximum, therefore plays an important role in the dispersion of pollutants. Consider, for example, the scenario of nuclear waste buried in an Arctic region and suppose this waste generates heat and drives convection in a shallow cavity (due to large vertical variations in permeability associated with sedimentary layering, a shallow cavity is an appropriate geometry for many geophysical porous media flows). In this scenario, it is of considerable interest to be able to predict the size of the convection cell near the hot wall, in case there is a release of any of the nuclear waste. The results of this paper will show that the location of the density maximum can be predicted \textit{a priori} in the conduction regime, but that it depends strongly on the Darcy–Rayleigh number in the intermediate regime.

2. Mathematical model

We consider convection in a shallow rectangular cavity which is filled with a fluid-saturated porous medium (Fig. 2). The cavity has height \( h \) and width \( \ell \) and the aspect ratio of the cavity is defined as...
\( A = h/\ell \). The sidewall at \( x' = 0 \) is maintained at a temperature of \( T' = T_c \) and the sidewall at \( x' = \ell \) is maintained at a temperature \( T' = T_h \), while the top and bottom boundaries are insulating. The fluid has a density maximum \( \rho_m \) at temperature \( T' = T_m \) and it is assumed that \( T_c < T_m < T_h \). The porous medium has a permeability \( K \) and an effective thermal diffusivity \( \kappa \) and the fluid has a kinematic viscosity \( \nu \).

If we assume that the flow is steady and two-dimensional and that it obeys Darcy’s law, the governing equations are

\[
\begin{align*}
\rho \frac{\partial P'}{\partial x'} &= -K \frac{\partial}{\partial x'} \left( \rho_0 \nu \frac{\partial P'}{\partial x'} \right), \\
\rho \frac{\partial P'}{\partial y'} &= -K \frac{\partial}{\partial y'} \left( \rho_0 \nu \frac{\partial P'}{\partial y'} \right) - \frac{K}{\rho_0} \left( \rho - \rho_0 \right) g,
\end{align*}
\]

(2.1)

(2.2)

\[
\begin{align*}
u \frac{\partial T'}{\partial x'} + \rho \frac{\partial T'}{\partial y'} &= \kappa \left( \frac{\partial^2 T'}{\partial x'^2} + \frac{\partial^2 T'}{\partial y'^2} \right),
\end{align*}
\]

(2.3)

where \( x' \) and \( y' \) are the horizontal and vertical coordinates, and \( u' \) and \( v' \) are the horizontal and vertical Darcy velocities, respectively, \( P' = p' - \rho_0 g y' \) is the reduced pressure where \( p' \) is the thermodynamic pressure, \( g \) is the acceleration of gravity, \( \rho \) is density and \( \rho_0 \) is a reference density evaluated at temperature \( T' = T_c \). Water is the most common fluid which has a density maximum, and following Moore & Weiss (1973), we use a quadratic equation of state to give

\[
\rho - \rho_m = -\gamma \rho_m (T' - T_m)^2,
\]

(2.4)

where \( \gamma \) is a dimensional constant equal to \( 8.0 \times 10^{-6} \degree C^{-2} \) and \( \rho = \rho_m = 1.000 \text{ g cm}^{-3} \) when \( T' = T_m = 3.98 \degree C \). Moore & Weiss (1973) claim that Equation (2.4) is accurate to within 4% over a temperature range of 0–8\degree C. While there are more accurate parametrizations of the density maximum of water (see, for example, Chapter 9 of Gebhart et al., 1988), Equation (2.4) provides a convenient expression that has been used by many authors to examine the influence of the density maximum on convection.

We introduce a stream function \( \psi' \) defined by

\[
\begin{align*}
u &= -\frac{\partial \psi'}{\partial y'}, \quad \text{and} \quad \rho \frac{\partial \psi'}{\partial x'} = \kappa \left( \frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial y'^2} \right),
\end{align*}
\]

(2.5)

and non-dimensional variables defined by

\[
\begin{align*}
x &= \frac{x'}{h}, \quad y &= \frac{y'}{h}, \quad \psi &= \frac{\psi'}{A \kappa \text{Ra}}, \quad \theta &= \frac{T' - T_c}{T_h - T_c}
\end{align*}
\]

(2.6)
to give

\[ A \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = 2(\theta - \theta^*) \frac{\partial \theta}{\partial x}, \quad (2.7) \]

\[ \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = Ra A \left( \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} \right). \quad (2.8) \]

The relevant dimensionless parameters when examining convection in a porous medium near the density maximum are the aspect ratio \( A = h/\ell \) of the cavity, the density maximum Darcy–Rayleigh number

\[ Ra = \frac{g \gamma \rho_m (T_h - T_c)^2 K_h}{\rho_0 \kappa \nu} \quad (2.9) \]

and the dimensionless temperature of the density maximum

\[ \theta^* = \frac{T_m - T_c}{T_h - T_c}. \quad (2.10) \]

The appropriate boundary conditions are

\[ \psi = \frac{\partial \theta}{\partial y} = 0 \quad \text{when } y = 0, 1, \quad (2.11) \]

\[ \psi = \theta = 0 \quad \text{when } x = 0 \quad (2.12) \]

and

\[ \psi = 0, \theta = 1 \quad \text{when } x = 1/A. \quad (2.13) \]

In this paper, the solution of Equations (2.7) and (2.8) subject to boundary conditions (2.11–2.13) will be obtained in the asymptotic limit as \( A \to 0 \). Results will be presented in the conduction regime when \( Ra = O(1) \) and in the intermediate regime when \( Ra = O(1/A) \). The asymptotic solution procedure used in this paper will be described in Section 3, the results in the conduction regime will be presented in Section 4 and the results in the intermediate regime will be presented in Section 5. The asymptotic results in the conduction regime have previously been reported by Leppinen & Rees (2003). The full numerical solution of Equations (2.7) and (2.8) will be presented in Section 6 for a range of the governing parameters and these numerical solutions will be compared with the asymptotic results. Heat transfer correlations will be presented in Section 7 and conclusions in Section 8.

3. Asymptotic analysis

We seek asymptotic solutions to Equations (2.7) and (2.8) in the limit as \( A \to 0 \). As shown in Fig. 2, we assume that the flow can be divided into five regions, with approximately square end-zone regions near the hot and cold end walls, and a turning region near the density maximum. In between the end zones and the density maximum region there are two central core regions. In each of these regions we seek solutions of the form

\[ (\psi, \theta) = \sum_{i=0}^{N} A^i (\psi_i, \theta_i) \quad (3.1) \]

and we will perform asymptotic matching between the adjoining regions.
In the central core regions horizontal changes will occur over distances of $O(1/A)$ and we introduce the scaled variable $\hat{x} = Ax$. If $\psi$ and $\theta$ are denoted by $\hat{\psi}$ and $\hat{\theta}$ in the core regions, the governing equations become

\begin{equation}
A^2 \frac{\partial^2 \hat{\psi}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\psi}}{\partial \hat{y}^2} = 2(\hat{\theta} - \theta^*) \frac{\partial \hat{\theta}}{\partial \hat{x}},
\end{equation}

\begin{equation}
A^2 \frac{\partial^2 \hat{\theta}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}}{\partial \hat{y}^2} = A^2 \text{Ra} \left( \frac{\partial \hat{\psi}}{\partial \hat{x}} \frac{\partial \hat{\theta}}{\partial \hat{y}} - \frac{\partial \hat{\psi}}{\partial \hat{y}} \frac{\partial \hat{\theta}}{\partial \hat{x}} \right) .
\end{equation}

The only boundary conditions available in the core regions are those at $y = 0, 1$. Equations (3.2) and (3.3) are valid in the core regions on either side of the density maximum and when performing matching ($\hat{\psi}^c, \hat{\theta}^c$) will denote the core solutions on the cold side of the density maximum and ($\hat{\psi}^h, \hat{\theta}^h$) will denote the core solutions on the hot side of the density maximum. In common with convection in shallow cylindrical annuli (Pop et al., 1998; Leppinen, 2002; Leppinen et al., 2004), a key feature of Equation (3.3) is that the solution for $\hat{\theta}$ at $O(A^n)$ can only be completely specified by proceeding to $O(A^{n+2})$ in the asymptotic analysis. This feature will be commented upon further in Section 4.

In the cold end region it is expected that horizontal changes will take place over distances of $O(1)$ and so there is no need to rescale the horizontal coordinate. The governing equations are thus given by Equations (2.7) and (2.8) and in the cold end region we will denote the stream function and the temperature, respectively, as $\psi = \hat{\psi}$ and $\theta = \hat{\theta}$. The only boundary conditions available in the cold end region are those at $y = 0, 1$ and at $x = 0$.

In the hot end region we introduce $\xi = x - 1/A$ and we denote the stream function and the temperature using $\psi = \hat{\psi}$ and $\theta = \hat{\theta}$ so that the governing equations become

\begin{equation}
A \left( \frac{\partial^2 \hat{\psi}}{\partial \xi^2} + \frac{\partial^2 \hat{\psi}}{\partial y^2} \right) = -2(\hat{\theta} - \theta^*) \frac{\partial \hat{\theta}}{\partial \xi},
\end{equation}

and

\begin{equation}
\frac{\partial^2 \hat{\theta}}{\partial \xi^2} + \frac{\partial^2 \hat{\theta}}{\partial y^2} = A \text{Ra} \left( \frac{\partial \hat{\psi}}{\partial y} \frac{\partial \hat{\theta}}{\partial \xi} - \frac{\partial \hat{\psi}}{\partial \xi} \frac{\partial \hat{\theta}}{\partial y} \right),
\end{equation}

where it is noted that $\xi$ measures the distance away from the hot wall. The only boundary conditions available in the hot end region are those at $y = 0, 1$ and at $x = 1/A$ (i.e. $\xi = 0$).

In the density maximum region we introduce the new horizontal coordinate $\eta = x - x^*/A$ where $x = x^*/A$ (i.e. $\hat{x} = x^*$) is the location of the density maximum which must be determined as part of our analysis. Denoting the stream function and temperature using $\psi = \hat{\psi}$ and $\theta = \hat{\theta}$, the governing equations in the density maximum region are

\begin{equation}
A \left( \frac{\partial^2 \hat{\psi}}{\partial \eta^2} + \frac{\partial^2 \hat{\psi}}{\partial y^2} \right) = 2(\hat{\theta} - \theta^*) \frac{\partial \hat{\theta}}{\partial \eta},
\end{equation}

and

\begin{equation}
\frac{\partial^2 \hat{\theta}}{\partial \eta^2} + \frac{\partial^2 \hat{\theta}}{\partial y^2} = A \text{Ra} \left( \frac{\partial \hat{\psi}}{\partial \eta} \frac{\partial \hat{\theta}}{\partial y} - \frac{\partial \hat{\psi}}{\partial y} \frac{\partial \hat{\theta}}{\partial \eta} \right). \end{equation}

The only boundary conditions available in the density maximum region are those at $y = 0, 1$. 

Equations (3.2–3.7) are the appropriate equations for the conduction regime when \( Ra = O(1) \). The governing equations in the intermediate regime when \( Ra = O(1/A) \) are obtained directly from Equations (3.2–3.7) by introducing the rescaled Rayleigh number \( R = A \, Ra \). The solution procedure for both the conduction and the intermediate regimes is to first obtain asymptotic solutions in the central core regions. To be fully specified, these solutions will require numerical constants which can only be determined by matching with solutions in the cold end, hot end and density maximum regions.

4. The conduction regime: \( Ra = O(1) \)

In this section, we will compute the asymptotic solutions in the conduction regime when \( Ra = O(1) \). Ultimately, we will compare the solutions in the conduction regime with the solutions in the intermediate regime in the limit as \( R = A \, Ra \to 0 \). As it will turn out, we have to determine the solution to \( O(A^3) \) in the conduction regime in order to compare with the solution with \( O(A) \) in the intermediate regime.

4.1 Core region

When \( Ra = O(1) \), the governing equations in the core regions at \( O(1) \) are

\[
\frac{\partial^2 \hat{\theta}_0}{\partial y^2} = 0 \tag{4.1}
\]

and

\[
\frac{\partial^2 \hat{\psi}_0}{\partial y^2} = 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}}. \tag{4.2}
\]

Integrating Equation (4.1) from \( y = 0, 1 \) and applying the boundary conditions \( \frac{\partial \hat{\theta}}{\partial y} = 0 \) when \( y = 0, 1 \) gives

\[
\frac{\partial \hat{\theta}}{\partial y} \equiv 0 \tag{4.3}
\]

so that \( \hat{\theta}_0 = \hat{\theta}_0(\hat{x}) \) is a function of \( \hat{x} \) alone. The boundary conditions in the core region are such that \( \hat{\theta}_0 \) cannot be fully specified by examining the governing equations at \( O(1) \). As will be shown below, \( \hat{\theta}_0 \) is only determined by proceeding to \( O(A^2) \) in the asymptotic analysis. Noting that \( \hat{\theta}_0 = \hat{\theta}_0(\hat{x}) \), Equation (4.2) can be integrated subject to the appropriate boundary conditions to give

\[
\hat{\psi}_0 = (y^2 - y)(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}}. \tag{4.4}
\]

The governing equations in the core regions at \( O(A) \) are similar to those at \( O(1) \) and the solutions are \( \hat{\theta}_1 = \hat{\theta}_1(\hat{x}) \) and

\[
\hat{\psi}_1 = (y^2 - y) \left( (\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + \hat{\theta}_1 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right). \tag{4.5}
\]

The energy equation in the core at \( O(A^2) \) is

\[
\frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_2}{\partial y^2} = Ra \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial y} - \frac{\partial \hat{\psi}_0}{\partial y} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right), \tag{4.6}
\]
which, using known values, can be integrated from \(y = 0, 1\) to give

\[ \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} = 0 \]  

(4.7)

and hence

\[ \hat{\theta}_0 = c_0 + d_0 \hat{x}, \]  

(4.8)

where \(c_0\) and \(d_0\) are numerical constants which can only be determined by matching the core solutions with solutions in the neighbouring regions. Upon substitution of Equation (4.8), Equation (4.6) becomes

\[ \frac{\partial^2 \hat{\theta}_2}{\partial y^2} = -(d_0)^2 \text{Ra}(2y - 1)(\hat{\theta}_0 - \theta^*), \]  

(4.9)

which is integrated to give

\[ \hat{\psi}_2 = -2 \text{Ra}(\hat{x} - \theta^*)(d_0)^3 \left( \frac{y^5}{60} - \frac{y^4}{24} + \frac{y^2}{24} - \frac{y}{60} \right) + \left( \hat{\theta}_0 - \theta^* \right) \frac{\partial \tau_2}{\partial \hat{x}} + \hat{\theta}_1 \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + \tau_2 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) \right) \left( y^2 - y \right). \]  

(4.10)

where \(\tau_2 = \tau_2(\hat{x})\) is a yet to be determined function of \(\hat{x}\). The \(\frac{1}{12}\) in the polynomial on the right-hand side of Equation (4.10) has been included so that this term is symmetric about \(y = \frac{1}{2}\). This is allowed since \(\tau_2\) is an as yet undetermined function of \(\hat{x}\).

The stream function equation at \(O(A^2)\) is

\[ \frac{\partial^2 \hat{\psi}_0}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\psi}_2}{\partial y^2} = 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_2}{\partial \hat{x}} + 2\hat{\theta}_1 \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + 2\hat{\theta}_2 \frac{\partial \hat{\theta}_0}{\partial \hat{x}}, \]  

(4.11)

which can be integrated to give

\[ \hat{\psi}_2 = -2 \text{Ra}(\hat{x} - \theta^*)(d_0)^3 \left( \frac{y^5}{60} - \frac{y^4}{24} + \frac{y^2}{24} - \frac{y}{60} \right) \]

(4.12)

The energy equation in the core at \(O(A^3)\) is

\[ \frac{\partial^2 \hat{\theta}_1}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_3}{\partial y^2} = \text{Ra} \left( \frac{\partial \hat{\psi}_1 \partial \hat{\theta}_1}{\partial \hat{x} \partial y} - \frac{\partial \hat{\psi}_0 \partial \hat{\theta}_1}{\partial y \partial \hat{x}} + \frac{\partial \hat{\psi}_1 \partial \hat{\theta}_0}{\partial \hat{x} \partial y} - \frac{\partial \hat{\psi}_0 \partial \hat{\theta}_0}{\partial y \partial \hat{x}} \right), \]  

(4.13)

which can be integrated in a similar manner as the energy equation at \(O(A^2)\) to give

\[ \frac{\partial^2 \hat{\theta}_1}{\partial \hat{x}^2} = 0 \]  

(4.14)

and hence

\[ \hat{\theta}_1 = c_1 + d_1 \hat{x}, \]  

(4.15)
where again $c_1$ and $d_1$ are numerical constants which can only be determined by matching the core solutions with solutions in the neighbouring regions. In fact, matching will show (see Section 4.2) that $c_1 = d_1 = 0$ and we will use this result to simplify Equation (4.13) to give

$$\frac{\partial^2 \hat{\theta}_3}{\partial y^2} = 0, \quad (4.16)$$

which can be integrated to show that $\hat{\theta}_3 = \hat{\theta}_3(\hat{x})$.

Noting that $\hat{\theta}_1 = 0$ implies that $\hat{\psi}_1 = 0$, the stream function equation in the core at $O(A^3)$ reduces to

$$\frac{\partial^2 \hat{\psi}_3}{\partial y^2} = 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_3}{\partial \hat{x}} + 2\hat{\theta}_3 \frac{\partial \hat{\theta}_0}{\partial \hat{x}}, \quad (4.17)$$

which can be integrated directly to give

$$\hat{\psi}_3 = (y^2 - y) \left( (\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_3}{\partial \hat{x}} + \hat{\theta}_3 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right). \quad (4.18)$$

The energy equation at $O(A^4)$ is

$$\frac{\partial^2 \hat{\theta}_2}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_4}{\partial y^2} = Ra \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_2}{\partial y} - \frac{\partial \hat{\psi}_0}{\partial y} \frac{\partial \hat{\theta}_2}{\partial \hat{x}} + \frac{\partial \hat{\psi}_2}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial y} - \frac{\partial \hat{\psi}_2}{\partial y} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right), \quad (4.19)$$

where terms involving $\hat{\theta}_1 = \hat{\psi}_1 = 0$ have been omitted. After substituting known values, Equation (4.19) can be integrated from $y = 0, 1$ to yield

$$\frac{d^2 \tau_2}{d \hat{x}^2} = Ra \int_0^1 \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_2}{\partial y} - \frac{\partial \hat{\psi}_0}{\partial y} \frac{\partial \hat{\theta}_2}{\partial \hat{x}} + \frac{\partial \hat{\psi}_2}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial y} - \frac{\partial \hat{\psi}_2}{\partial y} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) \, dy \quad (4.20)$$

$$= -\frac{1}{15} (Ra)^2 (d_0)^4 (c_0 + d_0 \hat{x} - \theta^*), \quad (4.21)$$

so that

$$\tau_2 = -\frac{1}{90} (Ra)^2 (d_0)^2 (c_0 + d_0 \hat{x} - \theta^*)^3 + c_2 + d_2 \hat{x}, \quad (4.22)$$

where $c_2$ and $d_2$ are again numerical constants.

We complete the analysis in the core region by writing the energy equation at $O(A^5)$

$$\frac{\partial^2 \hat{\theta}_3}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_5}{\partial y^2} = Ra \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_3}{\partial y} - \frac{\partial \hat{\psi}_0}{\partial y} \frac{\partial \hat{\theta}_3}{\partial \hat{x}} + \frac{\partial \hat{\psi}_3}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial y} - \frac{\partial \hat{\psi}_3}{\partial y} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right), \quad (4.23)$$

where again use has been made of $\hat{\theta}_1 = \hat{\psi}_1 = 0$. Integrating Equation (4.23) from $y = 0, 1$ reveals that

$$\frac{\partial^2 \hat{\theta}_3}{\partial \hat{x}^2} = 0 \quad (4.24)$$

and hence

$$\hat{\theta}_3 = c_3 + d_3 \hat{x}. \quad (4.25)$$
4.2 Matching

The unknown constants \((c_n, d_n)\) for \(n = 0–3\) in the core region are determined by matching with solutions in the end-zone regions and in the density maximum region. In principle, we should have to determine matching constants \((c_n^c, d_n^c)\) on the cold side of the density maximum and \((c_n^h, d_n^h)\) on the hot side of the density maximum. As is shown in Appendix A, matching with solutions in the density maximum region will show that \((c_n^c, d_n^c) = (c_n^h, d_n^h)\). Thus, in this section we will only consider matching the core region solutions with solutions in the cold and hot end-zone regions. The only comment which will be made about the density maximum region in this section is its location. The leading order stream function in the core region is given by Equation (4.4). When \(\hat{\theta}_0 - \theta^* < 0\) (\(> 0\)), the leading order horizontal velocity in the core is positive (negative) when \(y > \frac{1}{2}\) and negative (positive) when \(y < \frac{1}{2}\). Intuitively, it follows that the density maximum \(\hat{x} = x^*\) should be located where \(\hat{\theta}(x^*) = \theta^*\). This assumption is in excellent agreement with the full numerical solutions discussed in Section 6 both when \(Ra = O(1)\) and \(Ra = O(1/A)\).

The matching conditions between the core region solutions and the end regions are

\[
\lim_{x \to \infty} (\hat{\psi}, \hat{\theta}) = (\hat{\psi}, \hat{\theta}) \tag{4.26}
\]

and

\[
\lim_{\xi \to \infty} (\bar{\psi}, \bar{\theta}) = (\hat{\psi}, \hat{\theta}) \tag{4.27}
\]

The matching constraints are applied by expressing the core region solutions in terms of the horizontal coordinate in the end regions and then taking the limit as \(A \to 0\) with \(\hat{x} = Ax = 1 - A \xi\).

The governing equations in the cold end region at \(O(1)\) are

\[
2(\bar{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial x} = 0 \tag{4.28}
\]

and

\[
\frac{\partial^2 \hat{\theta}_0}{\partial x^2} + \frac{\partial^2 \hat{\theta}_0}{\partial y^2} = 0. \tag{4.29}
\]

The solution to Equations (4.28) and (4.29) which satisfies the appropriate boundary conditions at \(x = 0\) and \(y = 0, 1\) is \(\hat{\theta}_0 = 0\). Similarly, it can be shown that the leading order temperature in the hot end region is given by \(\hat{\theta}_0 = 1\). The matching conditions given by Equations (4.26–4.27) are

\[
\hat{\theta}_0 = 0 \Leftrightarrow \hat{\theta}_0 = c_0 + d_0 Ax, \tag{4.30}
\]

\[
\hat{\theta}_0 = 1 \Leftrightarrow \hat{\theta}_0 = c_0 + d_0(1 - A \xi), \tag{4.31}
\]

where the symbol \(\Leftrightarrow\) is used to indicate that only terms which are \(O(1)\) in \(A\) are used when matching. For example Equation (4.30) implies that \(c_0 = 0\) and that the term \(d_0 Ax\) must be retained for matching the temperature field at \(O(A)\). Similarly, Equation (4.31) implies that \(c_0 + d_0 = 1\), which implies that \(d_0 = 1\) since \(c_0 = 0\). Thus the leading order temperature in the core is \(\hat{\theta}_0 = \hat{x}\), which is simply the conduction temperature profile between the cold and the hot walls.
Fig. 3. The leading order stream function in the cold end region with contours of $(\tilde{\psi}_0)' = 0.02, 0.04, \ldots, 0.22$.

The energy equation at $O(A)$ in the cold end region is

$$\frac{\partial^2 \tilde{\theta}_1}{\partial x^2} + \frac{\partial^2 \tilde{\theta}_1}{\partial y^2} = 0 \quad (4.32)$$

with a corresponding equation for $\tilde{\theta}_1$ in the hot end region. The matching conditions are

$$\tilde{\theta}_1 \leftrightarrow x + \hat{\theta}_1 = x + c_1 + d_1 A x \quad (4.33)$$

and

$$\tilde{\theta}_1 \leftrightarrow -\xi + (\hat{\theta}_1) = -\xi + c_1 + d_1 (1 - A \xi). \quad (4.34)$$

The only solution to Equation (4.32) which is compatible with the matching constraint (4.33) and all of the relevant boundary conditions is $\tilde{\theta}_1 = x$ with $c_1 = 0$. Similarly in the hot end region $\tilde{\theta}_1 = -\xi$ with $d_1 = 0$.

With $\tilde{\theta}_0$ and $\tilde{\theta}_1$ determined, the leading order stream function equation in the cold end region becomes

$$\frac{\partial^2 \tilde{\psi}_0}{\partial x^0} + \frac{\partial^2 \tilde{\psi}_0}{\partial y^0} = -2\theta^* \quad (4.35)$$

with a matching condition given by $\tilde{\psi}_0 \leftrightarrow -(y^2 - x^2)\theta^*$ as $x \to \infty$. The form of Equation (4.35) and the matching condition suggests the scaling $\tilde{\psi}_0 = \theta^*(\tilde{\psi}_0)'$. The rescaled version of Equation (4.35) can easily be solved numerically (or using Fourier series) and $(\tilde{\psi}_0)'$ is plotted in Fig. 3. The numerical solution to Equation (4.35) (and to the other numerical solutions discussed in Sections 4–5) was calculated using second-order central differencing on a uniform 1024 by 128 grid with the matching condition applied at $x = 8$ (although only the range $x \leq 4$ is plotted in Fig. 3). Solutions obtained using a uniform 256 by 64 with the matching condition applied at $x = 4$ are virtually identical to the solution plotted in Fig. 3. The leading order stream function in the hot end satisfies Equation (4.35) and its matching condition with $-\theta^*$ everywhere replaced by $(1 - \theta^*)$ and $x$ replaced by $\xi$. Hence the stream function in the hot end is
linearly related to that in the cold end region. The energy equation in the cold region at $O(A^2)$ is

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} + \frac{\partial^2 \tilde{\theta}}{\partial y^2} = -Ra \frac{\partial \tilde{\psi}_0}{\partial y},$$

(4.36)

which can be integrated from $y = 0, 1$ to give

$$\int_0^1 \frac{\partial^2 \tilde{\theta}}{\partial x^2} \, dy = d \left( \int_0^1 \tilde{\theta} \, dy \right) = 0.$$ 

(4.37)

Thus

$$\int_0^1 \tilde{\theta} \, dy = \tilde{a} + \tilde{b}x$$

(4.38)

for unknown constants $\tilde{a}$ and $\tilde{b}$. The boundary condition at $x = 0$ implies that $\tilde{a} = 0$ while the matching condition with the core region requires

$$\tilde{\theta} \leftrightarrow \dot{\theta} = c_2 + d_2 Ax - Ra(Ax - \theta^*) \left( y^3 - \frac{y^2}{2} + \frac{1}{12} \right) - \frac{1}{90} (Ra)^2 (Ax - \theta^*)^3.$$ 

(4.39)

Equation (4.38) is valid throughout the entire cold end region and, if we take the limit as $x \to \infty$, we can apply the matching constraint given by (4.39) to give

$$c_2 + \frac{1}{90} (Ra)^2 (\theta^*)^3 = 0$$

(4.40)

with $\tilde{b} = 0$. The corresponding result in the hot end region is

$$c_2 + d_2 - \frac{1}{90} (Ra)^2 (1 - \theta^*)^3 = 0,$$

(4.41)

which gives

$$c_2 = -\frac{1}{90} (Ra)^2 (\theta^*)^3, \quad d_2 = \frac{1}{90} (Ra)^2 ((\theta^*)^3 + (1 - \theta^*)^3).$$

(4.42)

With $c_2$ and $d_2$ determined, the matching condition for $\tilde{\theta}$ can be rewritten as

$$\tilde{\theta} \leftrightarrow \dot{\theta} = Ra \theta^* \left( \frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12} \right)$$

(4.43)

and it is clear from this condition and from Equation (4.36) that $\tilde{\theta}$ can be scaled as $\tilde{\theta} = Ra(\dot{\theta})'$. The numerical solution of $(\dot{\theta})'$ is plotted in Fig. 4 and it is seen that $(\dot{\theta})'$ has an odd symmetry about $y = \frac{1}{2}$ with positive perturbations in the lower half of the end region and negative perturbations in the upper half of the end region. A plot of $\tilde{\theta}$ in the hot end region would be a mirror image of Fig. 4 with the sign of the temperature perturbation reversed. After substituting known values, the equation for the $O(A)$ stream function perturbation is given by

$$\frac{\partial^2 \tilde{\psi}_1}{\partial x^2} + \frac{\partial^2 \tilde{\psi}_1}{\partial y^2} = -2\theta^* \frac{\partial \tilde{\psi}_2}{\partial x} + 2x$$

(4.44)
The $O(A^2)$ temperature perturbation in the cold end region with contours of $(\tilde{\theta}_2)' = 0, \pm 0.02, \pm 0.04, \ldots, \pm 0.08$ (negative contours are solid; positive contours are dashed).

with the matching constraint

$$\tilde{\psi}_1 \leftrightarrow x(y^2 - y) + \tilde{\psi}_1 = -x(y^2 - y). \quad (4.45)$$

The form of Equation (4.44) and the matching condition suggests the decomposition

$$\tilde{\psi}_1 = -2(\theta^*)^2\text{Ra}(\tilde{\psi}_1)' + (\tilde{\psi}_1)'', \quad (4.46)$$

where, from Equation (4.44), we write

$$\frac{\partial^2(\tilde{\psi}_1)'}{\partial x^2} + \frac{\partial^2(\tilde{\psi}_1)'}{\partial y^2} = \frac{\partial(\tilde{\psi}_2)'}{\partial x} \quad (4.47)$$

and

$$\frac{\partial^2(\tilde{\psi}_1)''}{\partial x^2} + \frac{\partial^2(\tilde{\psi}_1)''}{\partial y^2} = 2x \quad (4.48)$$

with

$$(\tilde{\psi}_1)' \leftrightarrow 0, \quad (4.49)$$

$$(\tilde{\psi}_1)'' \leftrightarrow x(y^2 - y). \quad (4.50)$$

Numerical solutions for $(\tilde{\psi}_1)'$ and $(\tilde{\psi}_1)''$ are shown in Fig. 5. The corresponding results for the $O(A)$ stream function perturbation in the hot end region can be inferred from Fig. 5.

Finally, the energy equation at $O(A^3)$ in the cold end region is

$$\frac{\partial^2\tilde{\psi}_3}{\partial x^2} + \frac{\partial^2\tilde{\psi}_3}{\partial y^2} = \text{Ra} \left( \frac{\partial\tilde{\psi}_0}{\partial x} \frac{\partial\tilde{\psi}_2}{\partial y} - \frac{\partial\tilde{\psi}_0}{\partial y} \frac{\partial\tilde{\psi}_2}{\partial x} - \frac{\partial\tilde{\psi}_1}{\partial y} \right) \quad (4.51)$$
Fig. 5. The $O(A)$ stream function perturbation in the cold end region with contours of: (a) $(\tilde{\psi}_1)' = 0, \pm 2 \times 10^{-4}, \pm 4 \times 10^{-4}, \ldots, \pm 16 \times 10^{-4}$ (negative contours are solid, positive contours are dashed); (b) $(\tilde{\psi}_1)'' = 0, -0.1, -0.2, \ldots, -0.9$.

with a matching condition given by

$$\tilde{\theta}_3 \Leftrightarrow -Ra \left( \frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12} \right) - \frac{(Ra)^2}{90} (3(\theta^*)^2 x) + c_3 + d_3 A x. \quad (4.52)$$

Equation (4.51) and the matching condition are such that we can write

$$\tilde{\theta}_3 = (Ra\theta^*)^2 \left( (\tilde{\theta}_3)' - \frac{x}{30} \right) + Ra(\tilde{\theta}_3)'', \quad (4.53)$$

where

$$\frac{\partial^2 (\tilde{\theta}_3)'}{\partial x^2} + \frac{\partial^2 (\tilde{\theta}_3)'}{\partial y^2} = \frac{\partial \tilde{\psi}_0}{\partial x} \frac{\partial (\tilde{\theta}_3)'}{\partial y} - \frac{\partial \tilde{\psi}_0}{\partial y} \frac{\partial (\tilde{\theta}_3)'}{\partial x} + 2 \frac{\partial (\tilde{\psi}_1)'}{\partial y}, \quad (4.54)$$
Fig. 6. The $O(A^3)$ temperature perturbation in the cold end region with contours of: (a) $(\tilde{\theta}_1)' = 0, 0.002, 0.004, \ldots, 0.018$; (b) $(\tilde{\theta}_3)'' = 0, \pm 0.005, \pm 0.01, \ldots, \pm 0.03$ (negative contours are solid; positive contours are dashed).

and

$$\frac{\partial^2 (\tilde{\theta}_3)''}{\partial x^2} + \frac{\partial^2 (\tilde{\theta}_3)''}{\partial y^2} = -\frac{\partial (\tilde{\psi}_1)''}{\partial y}, \quad (4.55)$$

and where

$$(\tilde{\theta}_3)' \leftrightarrow \frac{c_3}{(Ra\theta^*)^2} \quad (4.56)$$

and

$$(\tilde{\theta}_3)'' \leftrightarrow -x \left( \frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12} \right). \quad (4.57)$$

In the numerical solution of Equations (4.54) and (4.55), the matching conditions given by (4.56) and (4.57) are suitably replaced by conditions involving $\partial (\tilde{\theta}_3)'/\partial x$ and $\partial (\tilde{\theta}_3)'/\partial x$. Numerical solutions of $(\tilde{\theta}_3)'$ and $(\tilde{\theta}_3)''$ are plotted in Fig. 6. To emphasize the fact that $(\tilde{\theta}_3)'$ approaches a constant as $x \rightarrow \infty$, in Fig. 7, we plot $(\tilde{\theta}_3)'_{\min}(x) = \min_{0 \leq y \leq 1} (\tilde{\theta}_3)'(x, y)$ and $(\tilde{\theta}_3)'_{\max}(x) = \max_{0 \leq y \leq 1} (\tilde{\theta}_3)'(x, y)$. From Fig. 7, we
conclude that
\[ c_3 = 0.0199 (\text{Ra} \theta^*)^2. \] (4.58)

The corresponding result in the hot end region will give
\[ d_3 = -0.0199 (\text{Ra})^2 ((\theta^*)^2 + (1 + \theta^*)^2). \] (4.59)

5. The intermediate regime: \( \text{Ra} = O(1/A) \)

As noted in Section 3 the asymptotic analysis in the intermediate regime proceeds by introducing \( R = A \text{Ra} \) where \( R = O(1) \) when \( \text{Ra} = O(1/A) \). A distinguishing feature of the asymptotic results in the conduction regime is that the leading order temperature profile is the conduction profile \( \hat{\theta}_0 = \hat{x} \). In the intermediate regime it will be seen that the leading order temperature profile is dependent on \( R \) (and hence \( \text{Ra} \)) and is thus convectively determined. This is a significant result because it highlights a fundamental difference between convection near the density maximum and convection when density depends linearly on temperature: when density varies linearly with temperature, the leading order temperature in the intermediate regime is still the linear conduction profile, independent of the effects of convection.

5.1 Core region

In the intermediate regime the governing equations at \( O(1) \) are the same as in the conduction regime and the leading order temperature is again \( \hat{\theta}_0 = \hat{\theta}_0(\hat{x}) \) with \( \hat{\psi}_0 \) given by Equation (4.4). Once again, \( \hat{\theta}_0 \) will only be fully specified by examining the energy equation at \( O(A^2) \).

The energy equation at \( O(A) \) is
\[
\frac{\partial^2 \hat{\theta}_1}{\partial y^2} = R \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial y} - \frac{\partial \hat{\psi}_0}{\partial y} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) = -R(\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 (2y - 1), \tag{5.1}
\]
which can be integrated directly to give
\[
\hat{\theta}_1 = -R(\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \left( \frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12} \right) + \tau_1,
\] (5.2)

where again $\tau_1 = \tau_1(\hat{x})$ is a yet to be determined function of $\hat{x}$ and the factor of $\frac{1}{12}$ has been added to ensure that the first term on the right-hand side of Equation (5.2) is symmetric about $y = \frac{1}{2}$. Our objective is to determine the full solution to $O(A)$ since this is the order at which the temperature varies as a function of $y$. In order to determine $\tau_1$, we shall have to proceed to $O(A^3)$ in the asymptotic analysis.

The stream function equation at $O(A)$ is
\[
\frac{\partial^2 \hat{\psi}_1}{\partial y^2} = 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + 2\hat{\theta}_1 \frac{\partial \hat{\theta}_0}{\partial \hat{x}},
\] (5.3)

which, after substitution of known values, can be integrated to give
\[
\hat{\psi}_1 = \left( (\hat{\theta}_0 - \theta^*) \frac{\partial \tau_1}{\partial \hat{x}} + \tau_1 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) (y^2 - y) - 4R(\hat{\theta}_0 - \theta^*) \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^3 + (\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0 \partial^2 \hat{\theta}_0}{\partial \hat{x} \partial \hat{x}^2} \right) \left( \frac{y^5}{60} - \frac{y^4}{24} + \frac{y^2}{24} - \frac{y}{60} \right).
\] (5.4)

The energy equation at $O(A^2)$ is
\[
\frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_2}{\partial y^2} = Ra \left( \frac{\partial \hat{\psi}_0 \partial \hat{\theta}_1}{\partial \hat{x} \partial y} - \frac{\partial \hat{\psi}_0 \partial \hat{\theta}_1}{\partial y \partial \hat{x}} + \frac{\partial \hat{\psi}_1 \partial \hat{\theta}_0}{\partial \hat{x} \partial y} - \frac{\partial \hat{\psi}_1 \partial \hat{\theta}_0}{\partial y \partial \hat{x}} \right),
\] (5.5)

which, upon substitution of known values, can be integrated from $y = 0, 1$ to give
\[
\frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} = -\frac{R^2}{30} \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 (\hat{\theta}_0 - \theta^*) + 3(\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right)
\] (5.6)

so that
\[
\frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} = \frac{-(R^2/15) (\partial \hat{\theta}_0/\partial \hat{x})^4 (\hat{\theta}_0 - \theta^*)}{1 + (R^2/10)(\hat{\theta}_0 - \theta^*)^2 (\partial \hat{\theta}_0/\partial \hat{x})^2}.
\] (5.7)

This differential equation for $\hat{\theta}_0$ can be solved by first matching with solutions in the hot and cold end regions to provide the appropriate boundary conditions at $\hat{x} = 0$ and $\hat{x} = 1$. 
With \( \hat{\theta}_0 \) now known, at least implicitly in terms of Equation (5.7), we can now integrate Equation (5.5) to give

\[
\hat{\theta}_2 = \tau_2 - \frac{1}{2} \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} y^2 - R \left( + 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \frac{\partial \tau_1}{\partial \hat{x}} + \tau_1 \left( \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right)^2 \right) \\
+ 4R^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 \left( \hat{\theta}_0 - \theta^* \right) + (\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \left( \frac{y^6}{360} - \frac{y^5}{120} + \frac{y^3}{24} - \frac{y^2}{120} \right) \\
- R^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 \left( \hat{\theta}_0 - \theta^* \right) + 2(\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \left( \frac{y^6}{45} - \frac{y^5}{15} + \frac{y^3}{24} - \frac{y^2}{24} \right) \\
- R^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 \left( \hat{\theta}_0 - \theta^* \right) + (\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \left( \frac{y^6}{30} - \frac{y^5}{10} + \frac{y^4}{12} \right),
\]

(5.8)

where \( \tau_2 \) is a yet to be determined function of \( \hat{x} \).

At this stage we could solve the stream function equation at \( O(A^2) \) but since this is not necessary in order to fully specify the temperature field at \( O(A) \), we proceed to the energy equation at \( O(A^3) \) which requires

\[
\frac{\partial^2 \hat{\theta}_1}{\partial \hat{x}^2} + \frac{\partial^2 \hat{\theta}_3}{\partial y^2} = R \left( \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} + \frac{\partial \hat{\psi}_0}{\partial \hat{x}} \frac{\partial \hat{\theta}_0}{\partial \hat{x}} + \frac{\partial \hat{\psi}_1}{\partial \hat{x}} \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + \frac{\partial \hat{\psi}_1}{\partial \hat{x}} \frac{\partial \hat{\theta}_1}{\partial \hat{x}} + \frac{\partial \hat{\psi}_2}{\partial \hat{x}} \frac{\partial \hat{\theta}_2}{\partial \hat{x}} + \frac{\partial \hat{\psi}_2}{\partial \hat{x}} \frac{\partial \hat{\theta}_2}{\partial \hat{x}} \right).
\]

(5.9)

By substituting for known values, and applying the appropriate boundary conditions, we can integrate Equation (5.9) from \( y = 0, 1 \) to derive a differential equation for \( \tau_1 \) in the same manner we used to derive Equation (5.7) for \( \theta_0 \). The result is

\[
\frac{\partial^2 \tau_1}{\partial \hat{x}^2} = \left\{ \begin{array}{l}
- \frac{R^2}{30} \left( \left( \hat{\theta}_0 - \theta^* \right) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \frac{\partial \tau_1}{\partial \hat{x}} + \tau_1 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \right) \\
- \frac{R^2}{30} \left( \left( \hat{\theta}_0 - \theta^* \right) \frac{\partial \tau_1}{\partial \hat{x}} + \tau_1 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^3 + 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \right) \\
+ \frac{R}{12} \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^3 + 2(\hat{\theta}_0 - \theta^*) \frac{\partial \tau_1}{\partial \hat{x}} \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \\
+ \frac{R^3}{360} \left( 2(\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 + 3(\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 + (\hat{\theta}_0 - \theta^*) \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \right) \\
- \frac{R^2}{30} \left( 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \frac{\partial \tau_1}{\partial \hat{x}} + \tau_1 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \right) \left( \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 + (\hat{\theta}_0 - \theta^*) \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \right\}
\]

(5.10)
Thus in the cold end region \( \tilde{\theta} \) is observable in Fig. 8. First, when \( \theta^* = 0 \), the density maximum is fixed at \( \hat{x} = 0 \) (i.e. \( \bar{\theta}(0.5) = \theta^* \) when \( \theta^* = 0.5 \)). When \( \theta^* = 0.25 \), however, the density maximum moves when \( R \) increases: when \( R = 0 \), the density maximum occurs exactly at \( \hat{x}^* = \theta^* \) and as \( R \) increases the density maximum moves monotonically towards the cold end wall. Equations (2.7) and (2.8) have the property that solutions for \( \theta^* \) are mirror symmetric (in \( x \) and \( y \)) with solutions for \( 1 - \theta^* \). Thus, if \( \theta^* \) is >0.5, then the density maximum would move towards the hot wall as \( R \) increases. The second interesting feature of Fig. 8

\[
\begin{align*}
+ \frac{R^3}{360} (\theta_0 - \theta^*) \frac{\partial \theta_0}{\partial \hat{x}} \left( 2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^5 + 14 (\theta_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^3 \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \\
+ 6 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} (\theta_0 - \theta^*)^2 \left( \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right)^2 + 3 (\theta_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^3 \hat{\theta}_0}{\partial \hat{x}^3} \\
- \frac{R^2}{30} (\theta_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \left( 2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial \tau_1}{\partial \hat{x}} + 2 (\theta_0 - \theta^*) \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \frac{\partial \tau_1}{\partial \hat{x}} \right) \\
+ \frac{\partial \tau_1}{\partial \hat{x}} \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 + 2 \tau_1 \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right) \bigg/ \left( 1 + \frac{R^2}{10} (\theta_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \right).
\end{align*}
\] (5.10)

Matching will provide the appropriate boundary conditions for this equation and, by solving for \( \tau_1 \), we can fully specify the temperature field in the core to \( O(A) \).

5.2 Matching

The core region solutions when \( Ra = O(1/A) \) are considerably more complex than the core region solutions when \( Ra = O(1) \). In this section, we will perform matching between the core region and the end regions in order to determine the necessary boundary conditions to solve the differential equations for \( \bar{\theta}_0 \) and \( \tau_1 \). Once again, the matching between the core region and the density maximum region will be discussed in Appendix A.

The leading order energy equation in the cold end region when \( Ra = O(1/A) \) is the same as the leading order energy equation when \( Ra = O(1) \) and is given by Equation (4.28) with a corresponding equation in the hot end region. Thus in the cold end region \( \bar{\theta}_0 = 0 \) and in the hot end region \( \bar{\theta}_0 = 1 \). The matching conditions between the end regions and the core region are

\[
\bar{\theta}_0 = 0 \iff \hat{\theta}_0 = \hat{\theta}_0(0) + A \left. \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} x + \frac{A^2}{2} \left. \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right|_{\hat{x}=0} x^2 + \cdots \tag{5.11}
\]

and

\[
\bar{\theta}_0 = 1 \iff \hat{\theta}_0 = \hat{\theta}_0(1) - A \left. \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=1} \xi + \frac{A^2}{2} \left. \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right|_{\hat{x}=1} \xi^2 + \cdots \tag{5.12}
\]

which implies that the appropriate boundary conditions for \( \hat{\theta}_0 \) are \( \hat{\theta}_0(0) = 0 \) and \( \hat{\theta}_0(1) = 1 \). With these boundary conditions Equation (5.7) can now be solved using a shooting method and the results are presented in Fig. 8 for \( \theta^* = 0.5 \) and \( \theta^* = 0.25 \) for various values of \( R \). (The results in Fig. 8 were obtained by solving Equation (5.7) using NAG routine D02HAF.) There are several interesting features observable in Fig. 8. First, when \( \theta^* = 0.5 \), the density maximum is fixed at \( \hat{x} = 0.5 \) (i.e. \( \bar{\theta}_0(0.5) = \theta^* \) when \( \theta^* = 0.5 \).) When \( \theta^* = 0.25 \), however, the density maximum moves when \( R \) increases: when \( R = 0 \), the density maximum occurs exactly at \( \hat{x}^* = \theta^* \) and as \( R \) increases the density maximum moves monotonically towards the cold end wall.
Fig. 8. The leading order temperature \( \hat{\theta}_0 \) in the core regions for (a) \( \theta^* = 0.5 \); (b) \( \theta^* = 0.25 \). The different curves are for \( R = 1, 5, 10, 15, 20, 25 \) and 30 when looking from bottom to top on the right-hand side of each graph.

is that the derivative of temperature at the density maximum increases continuously as \( R \) increases. To leading order the heat transfer across the density maximum is solely via conduction, and thus, since we expect the heat transfer between the cold and hot wall to increase as \( R \) increases, it follows that the temperature gradient at the density maximum should increase as \( R \) increases.

The governing equations at \( O(A) \) in the cold end region are

\[
\frac{\partial^2 \tilde{\psi}_0}{\partial x^2} + \frac{\partial^2 \tilde{\psi}_0}{\partial y^2} = -2\theta^* \frac{\partial \hat{\theta}_1}{\partial \hat{x}}
\]  
(5.13)
and
\[
\frac{\partial^2 \tilde{\theta}_1}{\partial x^2} + \frac{\partial^2 \tilde{\theta}_1}{\partial y^2} = R \left( \frac{\partial \tilde{\psi}_0}{\partial x} \frac{\partial \tilde{\theta}_1}{\partial y} - \frac{\partial \tilde{\psi}_0}{\partial y} \frac{\partial \tilde{\theta}_1}{\partial x} \right)
\]  
(5.14)
and these equations must be solved subject to the matching constraints
\[
\tilde{\psi}_0 \Leftrightarrow -\theta^* \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} (y^2 - y)
\]  
(5.15)
and
\[
\tilde{\theta}_1 \Leftrightarrow \tau_1(0) + \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} x + R\theta^* \left( \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} \right)^2 \left( \frac{y^3}{3} - \frac{y^2}{3} + \frac{1}{12} \right),
\]  
(5.16)
where only the leading order terms in the matching have been written including the term carried forward from the matching at \(O(1)\). In order to solve Equations (5.13) and (5.14), we can use the scaling
\[
\tilde{\psi}_1 = \theta^* \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} (\tilde{\psi}_1)', \quad \text{and} \quad \tilde{\theta}_1 = \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} (x + (\tilde{\theta}_1)')
\]  
(5.17)
so that
\[
\frac{\partial^2 (\tilde{\psi}_0)'}{\partial x^2} + \frac{\partial^2 (\tilde{\psi}_0)'}{\partial y^2} = -2 \left( \left. \frac{\partial (\tilde{\theta}_1)}{\partial \hat{x}} \right|_{\hat{x}=0} + 1 \right)
\]  
(5.18)
and
\[
\frac{\partial^2 (\tilde{\theta}_1)'}{\partial x^2} + \frac{\partial^2 (\tilde{\theta}_1)'}{\partial y^2} = R^* \left( \frac{\partial (\tilde{\psi}_0)'}{\partial x} \frac{\partial (\tilde{\theta}_1)'}{\partial y} - \frac{\partial (\tilde{\psi}_0)'}{\partial y} \frac{\partial (\tilde{\theta}_1)'}{\partial x} - \frac{\partial (\tilde{\psi}_0)'}{\partial y} \frac{\partial (\tilde{\theta}_1)'}{\partial x} \right),
\]  
(5.19)
where
\[
R^* = \theta^* \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} R.
\]  
(5.20)
The appropriate matching conditions are
\[
(\tilde{\psi}_0)' \Leftrightarrow -(y^2 - y)
\]  
(5.21)
and
\[
(\tilde{\theta}_1)' \Leftrightarrow \tau_1(0) \left. \frac{\partial \tilde{\theta}_0}{\partial \hat{x}} \right|_{\hat{x}=0} + R^* \left( \frac{y^3}{3} - \frac{y^2}{3} + \frac{1}{12} \right),
\]  
(5.22)
where once again the matching condition for \( (\tilde{\theta}_1)' \) is suitably replaced by a matching condition for \( d(\tilde{\theta}_1)'/dx \) so that the unknown constant \( \tau_1(0) \) can be determined. Equations (5.18) and (5.19) are very similar to the coupled end region equations solved by Daniels et al. (1989a) at \(O(A)\) for convection when the fluid density is a linear function of temperature. The finite-differenced version of these equations can be solved iteratively and the results are presented in Fig. 9 for the specific value of \( R^* = 10 \). By examining Fig. 9(b), it is possible to determine the value of \( \tau_1(0) \), and Fig. 10 shows how \( \tau_1(0) \) varies as a function of \( R^* \). The value of \( \tau_1(1) \) can be determined by matching with the \(O(A)\) solution in the hot end region. The resulting equations are identical to those in the cold end regions except that \( \theta^* \) in
Equation (5.20) is replaced by $1 - \theta^*$ and the derivative in Equation (5.20) is evaluated at $\hat{x} = 1$. Thus, Fig. 10 can also be used to determine the value of $\tau_1(1)$. By performing a least squares fit to the data in Fig. 10, we obtain

$$\tau_1(0) = 0.0657R^* - 0.00015(R^*)^2$$

(5.23)

with a maximum error of $<0.05$ for $0 \leq R^* \leq 50$.

With the values of $\tau_1(0)$ and $\tau_1(1)$ determined, it is now possible to solve Equation (5.10) for $\tau_1(\hat{x})$. (As for Equation (5.7), Equation (5.10) is solved using NAG routine D02HAF.) Results are presented in Fig. 11 for $\theta^* = 0.5$ and 0.25 at different values of $R$. For $\theta^* = 0.5$ it is seen that $\tau_1$ has an odd symmetry about $\hat{x} = 0.5$ and that the slope of $\tau_1$ at $\hat{x} = 0.5$ becomes large as $R$ becomes large. For $\theta^* = 0.25$, the results are no longer symmetric. Indeed, as $R$ is increased from $R = 20$ to $R = 25$, it is seen that $\tau_1$ ceases to remain monotonic. For $\theta^* \neq 0.5$ an internal boundary layer forms with a peak located just on the cold (hot) wall side for $\theta^* < 0.5$ ($\theta^* > 0.5$) of the point where $\theta_0 = \theta^*$. While the peak in $\tau_1$ for $\theta^* \neq 0.5$ is unexpected, the excellent overall agreement between the asymptotic results presented in this section and the numerical solutions presented in Section 6 confirm that Equation (5.10) is indeed
being solved correctly (see also the discussion after Equation (7.5)). It should be noted, however, that the large gradients near $\hat{x} = 0.5$ when $\theta^* = 0.5$ and the peak in $\tau_1$ when $\theta^* = 0.5$ lead to numerical difficulties when solving Equation (5.10) and it has not been possible to obtain solutions for $R > 150$.

6. Numerical solutions

In this section, the full numerical solutions to Equations (2.7–2.8) subject to boundary conditions (2.11–2.13) are presented for a range of Ra and $\theta^*$ in order to examine the transition from the conduction regime to the intermediate regime and ultimately to the boundary layer regime. The numerical solutions will also be used to assess the accuracy and ranges of validity of the asymptotic solutions presented in Sections 4–5. The numerical solutions are obtained using a control volume technique on uniform grids as described in Leppinen (2003). The stream function equation is solved using the method of fast Fourier transforms and the energy equation is solved by adding a transient term to the right-hand side of Equation (2.8) and then time-stepping until a steady state is reached. (A convergence criterion of $\max_{i,j} |\theta_{i,j}^{n+1} - \theta_{i,j}^n| < 10^{-10}$ was used where $\theta_{i,j}^n$ is the value of $\theta$ at time-step $n$ and grid cell $i,j$.) Equation (2.8) was solved using (Roe’s, 1981) monotonic second-order upwind scheme as described by Tamamidis & Assanis (1993). Details of the numerical solutions discussed in this section are summarized in Table 1.

6.1 Results: $\theta^* = 0.5$

The influence of varying Ra for a fixed value of $\theta^* = 0.5$ and $A = \frac{1}{32}$ is shown in Fig. 12, which presents contour plots of $\theta$ and $\psi$, respectively (where it is noted that the plots in Fig. 12, and subsequently Fig. 14, have been compressed by a factor of four in the horizontal). Contours of temperature are plotted for $\theta = 0.0, 0.1, 0.15, \ldots, 1.0$ and contours of stream function are plotted for $\psi = 0, \pm 0.01, \pm 0.02, \ldots,$ up to the maximum/minimum values of $\psi$ which are noted in Table 1. It is noted that the magnitude of the dimensionless stream function maximum/minimum decreases as $R$ is increased, however, the
The magnitude of the dimensional stream function (cf. Equation (2.6)), and hence velocities, increases as $R$ increases. The results in Fig. 12 have been plotted for $R = A Ra = 0.1, 1, 10, 100$ and 1000. In all cases it is seen that the density maximum occurs at $x^* = 0.5$ with mirror symmetric temperature contours on either side of the density maximum. It is also seen that the flow is divided into mirror symmetric counter-rotating convection cells. For small values of $R$, the temperature field varies approximately linearly between the cold and hot walls, with little variation in the vertical. As $R$ increases, the vertical variation in temperature increases with relatively hotter fluid overlying relatively colder fluid on the cold side of the density maximum, and vice versa on the hot side of the density maximum. When $R$
Table 1 Summary of the numerical solutions plotted in Figs 12–15. In all cases $A = \frac{1}{32}$. $\text{Nu}_{\text{num}}$ refers to the Nusselt number determined from the numerical solutions, $\text{Nu}_{\text{cond}}$ refers to the Nusselt number calculated using Equation (7.2) and $\text{Nu}_{\text{int}}$ refers to the Nusselt number calculated using Equation (7.3).

<table>
<thead>
<tr>
<th>$\theta^*$</th>
<th>$R = A \text{Ra}$</th>
<th>Grid</th>
<th>$\psi_{\text{min}}$</th>
<th>$\psi_{\text{max}}$</th>
<th>$\text{Nu}_{\text{num}}$</th>
<th>$\text{Nu}_{\text{cond}}$</th>
<th>$\text{Nu}_{\text{int}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>1024 × 32</td>
<td>−0.1127</td>
<td>0.1127</td>
<td>1.000025</td>
<td>1.000025</td>
<td>1.000025</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>1024 × 32</td>
<td>−0.1123</td>
<td>0.1123</td>
<td>1.00247</td>
<td>1.00247</td>
<td>1.00247</td>
</tr>
<tr>
<td>0.5</td>
<td>10</td>
<td>1024 × 32</td>
<td>−0.0951</td>
<td>0.0951</td>
<td>1.2067</td>
<td>1.2467</td>
<td>1.2600</td>
</tr>
<tr>
<td>0.5</td>
<td>100</td>
<td>2048 × 64</td>
<td>−0.0748</td>
<td>0.0748</td>
<td>13.187</td>
<td>25.678</td>
<td>12.505</td>
</tr>
<tr>
<td>0.5</td>
<td>1000</td>
<td>8192 × 256</td>
<td>−0.0446</td>
<td>0.0446</td>
<td>269.14</td>
<td>2467.84</td>
<td>N/A</td>
</tr>
<tr>
<td>0.25</td>
<td>0.1</td>
<td>1024 × 32</td>
<td>−0.1741</td>
<td>0.0519</td>
<td>1.000045</td>
<td>1.000045</td>
<td>1.000045</td>
</tr>
<tr>
<td>0.25</td>
<td>1</td>
<td>1024 × 32</td>
<td>−0.1724</td>
<td>0.0520</td>
<td>1.00447</td>
<td>1.00447</td>
<td>1.00447</td>
</tr>
<tr>
<td>0.25</td>
<td>10</td>
<td>1024 × 32</td>
<td>−0.1280</td>
<td>0.0577</td>
<td>1.3160</td>
<td>1.4472</td>
<td>1.3151</td>
</tr>
<tr>
<td>0.25</td>
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<td>2048 × 64</td>
<td>−0.0997</td>
<td>0.0602</td>
<td>18.389</td>
<td>45.724</td>
<td>17.162</td>
</tr>
<tr>
<td>0.25</td>
<td>1000</td>
<td>8192 × 256</td>
<td>−0.0169</td>
<td>0.0239</td>
<td>459.60</td>
<td>4473.4</td>
<td>N/A</td>
</tr>
</tbody>
</table>

is increased to 100, it is seen that thin thermal boundary layers form near the cold and hot sidewalls and near the density maximum. As $R$ is increased further, these boundary layers become thinner and the vertical temperature gradients in the core regions begin to dominate the horizontal temperature gradients. Details of the boundary layer structure near the density maximum when $R = 1000$, $\theta^* = 0.5$ and $A = \frac{1}{32}$ are shown in Fig. 13, which shows contours of $\theta$ and $\psi$ for $15.5 < x < 16.5$. There are 256 by 256 grid cells over this range, with approximately 20 horizontal grid cells across the boundary layer. The plots in Fig. 13 have not been compressed in the horizontal, and it is seen that the isotherms are virtually vertical in the boundary layer and horizontal away from the boundary layer.

6.2 Results: $\theta^* = 0.25$

The influence of varying $R = A \text{Ra}$ for $\theta^* = 0.25$ and $A = \frac{1}{32}$ is shown in Fig. 14. In contrast to the results in Fig. 12 for $\theta^* = 0.5$, the results for $\theta^* = 0.25$ are no longer symmetric with a small convection cell on the cold side of the density maximum and a larger convection cell on the hot side. For small values of $R$ the temperature is seen to vary approximately linearly between the hot and cold walls and the location of the density maximum is approximately $\bar{x} = \theta^*$. As $R$ increases, the density maximum is seen to move towards the cold wall, as predicted in Fig. 8(b), and the vertical variation in temperature becomes more pronounced. A consequence of keeping $A$ fixed as $R$ increases is that eventually horizontal changes in the cold core region no longer occur over distances of $O(1/A)$, and the asymptotic solutions presented in this paper cease to be valid. When $R = 1000$, the hot convection cell fills virtually the entire domain and the dividing streamline between the hot and cold cells deviates strongly from the vertical. A detailed view of the temperature and stream function fields near the density maximum when $R = 1000$ is given in Fig. 15. It is seen that fluid rises along the cold wall, and even before it reaches the top it is deflected downwards, forming a triangular-shaped convection cell. As $R$ is increased further, it is expected that the cold convection cell will decrease further in size.

6.3 The effect of varying $A$

Numerical solutions were obtained for many different combinations of $A$, $\theta^*$ and $\text{Ra}$ and a general conclusion is that, for $\theta^*$ and $\text{Ra}$ fixed, the flow is similar to Fig. 14(a–d) with two counter-rotating
convection cells separated by an approximately vertical dividing streamline, provided that $A$ is sufficiently small with the necessary value of $A$ decreasing as $Ra$ is increased and as $\theta^* \to 0, 1$. For small values of $R = A Ra$ the location of the dividing streamline can be determined by assuming a linear temperature profile. As $R$ is increased, the density maximum will shift towards the cold wall if $\theta^* < 0.5$ or towards the hot wall if $\theta^* > 0.5$. For fixed, but large values of $R$, and fixed $A$, a situation similar to
Fig. 13. Temperature and stream function contours near the density maximum (15.5 < x < 16.5) for \( \theta^* = 0.5 \), \( A = \frac{1}{32} \) and \( R = A Ra = 1000 \): (a) \( \theta = 0, 0.05, 0.1, \ldots, 1; \psi = -0.03, -0.024, -0.018, \ldots, 0.03 \) (positive (negative) contours are to the right (left) of the density maximum).

Fig. 14(e) can occur with either the cold or the hot cell becoming very small or even vanishing as \( \theta^* \to 0 \) or 1. For example, numerical solutions were obtained for \( R = 100 \) and \( A = \frac{1}{16} \) and it was observed that the cold cell essentially vanishes for \( \theta^* < 0.15 \).

6.4 Comparison of the numerical solutions with the asymptotic solutions

The asymptotic solutions derived in Section 4–5 are valid in the limit as \( A \to 0 \). By comparing the numerical solutions discussed above with the asymptotic solutions, it is possible to determine the range of validity of the asymptotic results. Essentially we will be assuming that the numerical solutions represent the exact solution to Equations (2.7) and (2.8), and the validity of the asymptotic solutions will be assessed by measuring the deviation between the numerical solutions and the asymptotic solutions. In reality, the numerical solutions are not exact, but if the differences between the numerical solutions and the exact solutions are small compared with the differences between the asymptotic solution and the exact solution, then it is still possible to assess the validity of the asymptotic solutions. In this paper we will only compare the numerical solutions and the asymptotic solutions for the temperature field; however, it is expected that the same conclusions would be reached if the stream function fields were compared.

The numerical and asymptotic solutions are compared by vertically averaging the absolute difference between the two solutions. We define the error between the numerical and asymptotic solutions as

\[
\text{Error}(\hat{x}_i) = \frac{1}{jw} \sum_{j=1}^{jw} |\theta_{i,j} - \theta_{\text{asym}}(\hat{x}_i,y_j)|,
\]

where there are \( jw \) grid cells in the vertical, \( \theta_{i,j} \) is the value of the numerical solution at grid cell \((i,j)\), which is centred at \((\hat{x}_i,y_j)\), and \( \theta_{\text{asym}}(\hat{x}_i,y_j) \) is the appropriate asymptotic solution evaluated at \((\hat{x} = \hat{x}_i, y = y_j)\). In all cases, the core region asymptotic solutions will be used to evaluate (6.1) (i.e. we are extending the core region asymptotic solutions into the hot and cold end regions and into the density maximum region).
Fig. 14. Temperature and stream function contours for $\theta^* = 0.5$, $A = \frac{1}{32}$ and (a) $Ra = 0.1$; (b) 1; (c) 10; (d) 100; (e) 1000. The contours of temperature (above) are $\theta = 0, 0.05, 0.1, \ldots, 1$ and the contours of stream function (below) are $\psi = 0, \pm 0.01, \pm 0.02, \ldots$ (positive (negative) contours are to the right (left) of the density maximum).

The numerical and asymptotic solutions are compared in Fig. 16 for $\theta^* = 0.5$ and in Fig. 17 for $\theta^* = 0.25$. The numerical results in Figs 16–17 were obtained for $A = \frac{1}{32}$ and $Ra = 0.1, 1, 10$ and 100. The different curves in Figs 16–17 correspond to different asymptotic orders of solution in either the conduction (i.e. $Ra = O(1)$) or the intermediate regime (i.e. $R = O(1)$). The dashed and the dot-dashed curves in Figs 16–17 correspond to the conduction regime asymptotics and the solid curves correspond
Fig. 15. Temperature and stream function contours near the density maximum \(0 < x < 1\) for \(\theta^* = 0.5, A = \frac{1}{32}\) and \(R = A Ra = 1000\): (a) \(\theta = 0, 0.05, 0.1, \ldots, 1\); \(\psi = -0.03, -0.024, -0.018, \ldots, 0.024\) (positive (negative) contours are to the right (left) of the density maximum).

Fig. 16. Comparison between asymptotic theory and full numerical solutions for \(\theta^* = 0.5\) and \(R = Ra A\) (a) 0.1; (b) 1; (c) 10; (d) 100. See text for details.

to intermediate regime asymptotics. The dashed curves with symbols correspond to \(\theta_{\text{asymp}} = \hat{\theta}_0\), the dashed curves with no symbols correspond to \(\theta_{\text{asymp}} = \hat{\theta}_0 + A^2 \hat{\theta}_2\) and the dot-dashed curves correspond to \(\theta_{\text{asymp}} = \hat{\theta}_0 + A^2 \hat{\theta}_2 + A^3 \hat{\theta}_3\). The solid curves with symbols correspond to \(\theta_{\text{asymp}} = \hat{\theta}_0\) and the solid
Fig. 17. Comparison between asymptotic theory and full numerical solutions for $\theta^* = 0.25$ and $R = A Ra = (a) 0.1; (b) 1; (c) 10; (d) 100$. See text for details.

curve with no symbols correspond to $\theta_{\text{asym}} = \hat{\theta}_0 + A \hat{\theta}_1$. Note that some of the curves in Figs 16–17 have been scaled as indicated.

The errors plotted in Figs 16 for $\theta^* = 0.5$ are symmetric about $\hat{x} = 0.5$ as expected, and it is seen that the errors increase as $R$ is increased. When $R = 0.1$ (Fig. 16(a)), the leading order asymptotic solutions in the conduction and intermediate regimes are virtually identical with the dashed curve with symbols and the solid curve with symbols essentially overlying each other. The $O(A)$ correction to the intermediate regime asymptotic results is essentially exact with the difference between the numerical and asymptotic solutions vanishing away from the hot and cold walls. The $O(A^2)$ correction to the conduction regime asymptotic results is seen to be a substantial improvement upon the leading order solution, and the $O(A^3)$ correction is essentially exact with the dot-dashed curve overlying the curve for the $O(A)$ correction for the intermediate regime asymptotics. When $R = 1$ (Fig. 16(b)), the error curves have similar shapes to the corresponding curves when $R = 0.1$ except that the magnitudes of the errors have been scaled. The leading order errors in Fig. 16(b) are 10 times larger than the leading order errors in Fig. 16(a), while the errors corresponding to the high-order corrections are a factor of 100 times larger. This observation is consistent with the asymptotic results in the intermediate regime where it can be shown that, for small $R$, $(\hat{\theta}_0 - \hat{x}) \sim R$ and $\tau_1 \sim R^2$. For $R = 0.1$ and 1, the asymptotic solutions derived in the conduction regime are of comparable accuracy to the asymptotic solutions derived in the intermediate regime, where it is noted that in order to compare the $O(A)$ correction in the intermediate regime solution, it is necessary to proceed to $O(A^3)$ in the conduction regime solution. Figure 16(c) shows that when $R = 10$, the results from the conduction regime and the intermediate regime deviate considerably. The errors calculated from the conduction regime asymptotics have been divided by a
factor of 10 before being plotted in Fig. 16(c). It is seen that the leading order error associated with the conduction regime solution is $\sim 5$ times as large as the leading order error associated with the intermediate regime solution. Moreover, when $R = 10$, the conduction regime solutions are no longer asymptotic with the $O(A^2)$ correction to the leading order solution, resulting in larger errors, rather than smaller errors. In contrast, the $O(A)$ correction for the intermediate regime asymptotic solution provides a significant improvement upon the leading order solution. Finally, the results for $R = 100$ are plotted in Fig. 16(d) where it is noted that the $O(A^2)$ and $O(A^3)$ errors in the conduction regime have been divided by a factor of 50. It is seen that the results in the intermediate regime are still asymptotic with the $O(A)$ correction improving upon the leading order solution, although, for this large value of $R$, the improvement due to the $O(A)$ correction is relatively small. The spike near $\hat{x} = 0.5$ for the $O(A)$ correction in the intermediate regime is a consequence of extending the core region solutions into the density maximum region. For $R = 100$, the $O(A^2)$ correction to the conduction regime solution is clearly non-asymptotic. In fact, the $O(A^3)$ correction is non-physical for $R = 100$ since it predicts temperatures outside the range $0 \leq \theta \leq 1$.

The asymptotic results of Sections 4–5 are compared with full numerical solutions in Fig. 17 for $\theta^* = 0.25$, and the same general conclusions that were made for $\theta^* = 0.5$ again hold. When comparing the intermediate regime results between Figs 16 and 17, it is seen that the value of $\theta^*$ does not influence the accuracy of the intermediate regime results. The most significant difference when comparing the results for $\theta^* = 0.5$ and $\theta^* = 0.25$ is the value of $R$ at which the leading order solution in the conduction regime begins to deviate strongly from the leading order solution in the conduction regime. In Fig. 17(b), for $R = 1$ it is seen that there are significant differences when comparing the conduction regime results and the intermediate regime results when $\theta^* = 0.25$. In contrast, when $\theta^* = 0.5$, the results from the two asymptotic regimes are still in good agreement when $R = 1$. Thus, it is seen that the range of validity of the conduction regime asymptotic solutions is dependent both on $R$ and on $\theta^*$ with the conduction regime solutions remaining valid to higher values of $R$ when $\theta^*$ is nearer to 0.5, perhaps due to the inherent symmetry of the solutions when $\theta^* = 0.5$.

Comparisons between the asymptotic results and the numerical solutions were made for $A = \frac{1}{8}, \frac{1}{16}$, and $\frac{1}{32}$, and for various values of $\theta^*$ and $R = A Ra$, although results have only been presented in Figs 16–17 for $A = \frac{1}{32}$. These comparisons were made in order to (subjectively) assess the range of validity of the asymptotic results. A first criterion is that $A$ must be sufficiently small that the domain can be divided into two turning regions near the vertical walls, a density maximum turning region, and two shallow core regions. For $R \leq 50$, this is achieved if $\theta^*/A$ is $> 4$. For larger values of $R$ it is possible that either the hot or cold convection cell will no longer be shallow if $\theta^*$ is close to either 1 or 0, respectively. As noted above, the validity of the asymptotic results in the conduction regime depends both on $R$ and $\theta^*$. Nevertheless, a useful criterion that is valid for all $\theta^*$ is that the conduction regime solutions will give reasonable results for $R < 4$. It is noted that the intermediate regime solutions are very accurate in the limit of small $R$; however, the conduction regime solutions have the advantage of being fully analytic. The only comment that can be made on the transition from the intermediate regime to the boundary layer regime is that, for $A > \frac{1}{32}$, the flow is definitely within the boundary layer regime for all values of $\theta^*$ when $R > 200$. While no numerical solutions were obtained for $A < \frac{1}{32}$, it is assumed that transition will occur at larger values of $R$ as $A$ is reduced.

7. Heat transfer

Of great interest in convection problems of the type discussed in this paper is the overall rate of heat transfer. The results are usually expressed in terms of a Nusselt number, Nu, which is defined the ratio
of the total heat flux across any vertical plane between the hot and cold walls divided by the purely conductive heat flux. In terms of our non-dimensional variables,

$$\text{Nu} = \int_0^1 \left( \frac{\partial \theta}{\partial x} - A \text{Ra} u \theta \right) \, dy. \quad (7.1)$$

In the limit as $A \to 0$, the asymptotic solutions derived in Section 4–5 can be used to evaluate the Nusselt number in the conduction regime when $\text{Ra} = O(1)$ and in the intermediate regime when $\text{Ra} = O(1/A)$. In particular, after substituting the core region solutions when $\text{Ra} = O(1)$ into Equation (7.1), it is seen that

$$\text{Nu} = 1 + \frac{(\text{Ra})^2}{90} ( (\theta^*)^3 + (1 - \theta^*)^3 ) A^2 - 0.0199 (\text{Ra})^2 ((\theta^*)^2 + (1 - \theta^*)^2) A^3 + O(A^4). \quad (7.2)$$

The Nusselt number is minimum for $\theta^* = 0.5$ and, for the quadratic equation of state employed in this study, $\text{Nu}$ is symmetric about $\theta^* = 0.5$. The corresponding result in the intermediate regime when $\text{Ra} = O(1/A)$ so that $R = A \text{Ra} = O(1)$ is

$$\text{Nu} = \text{Nu}_0 + A \text{Nu}_1 + O(A^2), \quad (7.3)$$

where

$$\text{Nu}_0 = \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \left( 1 + \frac{(\text{Ra})^2}{30} (\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \right) \quad (7.4)$$

and

$$\text{Nu}_1 = \frac{\partial \tau_1}{\partial \hat{x}} + \frac{R^2}{30} \left( \frac{\hat{\theta}_0 - \theta^*}{\partial \hat{x}} \right)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \partial \tau_1 + \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right) \right)$$

$$+ R(\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \frac{\partial^3 \hat{\theta}_0}{\partial \hat{x}^3} - \frac{12}{12} \frac{\partial^3 \hat{\theta}_0}{\partial \hat{x}^3} - \frac{R^2}{360} (\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4$$

$$- \frac{R^2}{360} (\hat{\theta}_0 - \theta^*) \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^4 + 2(\hat{\theta}_0 - \theta^*)^2 \left( \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \left( \frac{\partial^2 \hat{\theta}_0}{\partial \hat{x}^2} \right)$$

$$+ \frac{R^2}{30} \left( 2(\hat{\theta}_0 - \theta^*) \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \partial \tau_1 + \frac{\partial \hat{\theta}_0}{\partial \hat{x}} \right)^2 \right). \quad (7.5)$$

The value of the Nusselt number should not depend on $\hat{x}$ and Equations (5.7) and (5.10) can be used to show that $\text{Nu}_0$ and $\text{Nu}_1$ are indeed independent of $\hat{x}$. Evidence for the successful numerical solution of Equations (5.7) and (5.10) is provided by the fact that when the right-hand sides of Equations (5.7) and (5.10) are evaluated numerically, the results are indeed independent (to six significant digits) of $\hat{x}$. Numerical values of $\text{Nu}_0$ and $\text{Nu}_1$ are plotted as a function of $R$ in Fig. 18 for $\theta^* = 0.25$ and 0.5.

The Nusselt number for the numerical solutions plotted in Figs 12 and 14 have been listed in Table 1, along with the Nusselt number calculated using the conduction regime asymptotics and the intermediate
regime asymptotics. For \( R = 0.1 \) and 1, the Nusselt number is only slightly perturbed from 1 and both the conduction regime and the intermediate regime asymptotic solutions are in excellent agreement with the numerical solutions. For \( R = 10 \) and 100, the Nusselt number predicted from the intermediate regime asymptotic solutions are in good agreement with the results from the numerical solutions; however, the predictions from the conduction regime asymptotic solutions begin to vastly overestimate the Nusselt number. For \( R = 1000 \) it was not possible to calculate a Nusselt number using the intermediate regime asymptotic solutions, since it was not possible to solve Equation (5.10) for this large value of \( R \).

8. Conclusions

In this paper, we have used asymptotic analysis to examine convection near the density maximum in a shallow porous rectangular cavity. The analysis is valid in the limit as the aspect ratio \( A \to 0 \) and solutions have been obtained in the conduction regime when the Rayleigh number \( Ra = O(1) \) and in the intermediate regime when the Rayleigh number \( Ra = O(1/A) \). It has been shown that the flow is divided into two counter-rotating cells. The size of the two cells will depend on the temperature of the density maximum and the temperatures of the sidewalls. In the conduction regime the leading order temperature field varies linearly between the hot and cold walls and the first convective influence (i.e. Rayleigh number dependence) occurs at \( O(A^2) \). In the intermediate regime, however, the leading order temperature field is dependent on the Rayleigh number, with the temperature field becoming significantly non-linear as the Rayleigh number increases. It is observed that the location of the density maximum moves towards the cold (hot) wall as the Rayleigh number is increased if the dimensionless temperature of the density maximum \( \theta^* < 0.5 \) (\( > 0.5 \)). Full numerical solutions of the governing equations have been obtained and these solutions have been compared with the asymptotic solutions. It was concluded that the conduction regime solutions are only valid for \( R = A Ra < 4 \). The intermediate regime solutions derived in this paper are valid for \( 0 \leq R < 200 \), provided \( A \leq \frac{1}{52} \).
REFERENCES


Appendix A

Asymptotic matching of the core region solutions with the density maximum region solutions is performed by expressing the core region solutions in terms of
\[ \eta = x - x^*/A \]
(i.e. \( \hat{x} = A\eta + x^* \)) and imposing the limits
\[ (\hat{\psi}_c^*, \hat{\theta}_c^*) = \lim_{\eta \to -\infty} (\dot{\psi}, \dot{\theta}) \quad (A.1) \]

and
\[ (\hat{\psi}_h^*, \hat{\theta}_h^*) = \lim_{\eta \to \infty} (\dot{\psi}, \dot{\theta}) \quad (A.2) \]

The matching constants in the core region are determined by only performing matching for \( \theta \), which is all that will be considered here.

In the conduction regime (\( Ra = O(1) \)) the governing energy equations at \( O(A^n) \) for \( n = 0, 1, 2 \) and 3 are such that
\[ \int_0^1 \left( \frac{\partial^2 \dot{\theta}_n}{\partial \eta^2} + \frac{\partial^2 \dot{\theta}_n}{\partial y^2} \right) dy = 0, \quad (A.3) \]

which implies that
\[ \int_0^1 \dot{\theta}_n dy = c_n + \dot{d}_n \eta, \quad (A.4) \]

where \( \dot{c} \) and \( \dot{d} \) are numerical constants. By imposing the limits given by Equations (A.1) and (A.2), and using the core region solutions derived in Section 4.1, Equation (A.4) can be used to show that \( \dot{c}_n = c_n^c \) and \( \dot{c}_n = c_n^h \) (i.e. \( c_n^c = c_n^h \)) and similarly that \( \dot{d}_n = d_n^c = d_n^h \).

Matching the core region solutions with the density maximum region solutions in the intermediate regime (\( Ra = O(1/A) \)) follows the same procedure as in the conduction regime except that the core region solutions in the intermediate regime must be Taylor expanded about \( \hat{x} = x^* \) before matching can be performed. For example in the cold core region
\[ \dot{\theta}_0^c = \dot{\theta}_0^c(x^*) + A \frac{\partial \dot{\theta}_0^c}{\partial \hat{x}} \bigg|_{\hat{x}=x^*} \eta + A^2 \frac{\partial^2 \dot{\theta}_0^c}{\partial \hat{x}^2} \bigg|_{\hat{x}=x^*} \eta^2 + \cdots \quad (A.5) \]

with a corresponding result for \( \dot{\theta}_0^h \) in the hot core region. It can be shown that Equation (A.4) holds in the intermediate regime for \( n = 0, 1 \) and 2 and, by performing matching at these orders, it can be shown that \( \dot{\theta}_0, d\dot{\theta}_0/d\hat{x}, \tau_1 \) and \( d\tau_1/d\hat{x} \) are all continuous at \( \hat{x} = x^* \). Thus, Equations (5.7) and (5.10) can be solved as boundary value problems for \( \dot{\theta}_0 \) and \( \tau_1 \) between \( \hat{x} = 0 \) and \( \hat{x} = 1 \) once the appropriate boundary conditions are determined by matching with the hot and cold end-region solutions.