A numerical investigation of the nonlinear wave stability of vertical thermal boundary layer flow in a porous medium

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Introduction

The subject of boundary layer stability in fluid-saturated porous media has received relatively little attention compared with many other areas of boundary layer stability theory such as flow on curved surfaces (Görtler vortices) or thermal boundary layers in fluids. The methodology used in papers dealing with the porous medium problem has followed the traditional parallel-theoretic approach, which reduces the disturbance equations to ordinary differential equations, and which results in neutral curves for the onset of vortex disturbances [1–3]. To the author's knowledge similar papers dealing with the onset of convection in the form of waves for either horizontal, inclined, or thermal boundary layers do not exist. However, a recent paper [4] has described the use of a time-dependent, nonlinear, spatially elliptic numerical code in investigating wave instabilities in horizontal boundary layer flow. It was found that there is not an abrupt transition to convection (as is implied by the use of the concept of a neutral curve) and that convection is time-dependent but not periodic. In this paper the same numerical code has been modified to investigate the propagation of wave disturbances in convection induced by an isothermally heated vertical surface.

In view of the results presented below, it is worthwhile at the outset to consider briefly the known stability characteristics of the analogous problem of flow in a differentially heated porous layer of constant width and infinite extent. When the layer is inclined to the horizontal and heated from below, the most unstable mode takes the form of longitudinal vortices and the critical Darcy-Rayleigh number is \(4\pi^2/\cos \phi\), where \(\phi\) is the inclination above the horizontal [5]. In general, the onset of waves (i.e. two-dimensional disturbances) takes place at a higher Darcy-Rayleigh number than this, but small-amplitude wave disturbances always decay for inclinations above about 31 degrees [6]. However, when the layer is vertical, not
only is the basic flow linearly stable to all disturbances [7], but Squire's theorem applies implying that the most slowly decaying disturbance is in fact two-dimensional [8]. Without performing detailed three-dimensional calculations it is difficult to predict whether the qualitative behaviour of disturbances in this spatially uniform channel flow applies to the spatially developing boundary layer flow which is the subject of this paper. Should the analogy apply, then the above-quoted analyses of vortex disturbances for both horizontal and inclined boundary layers give the definitive account of the onset of vortex instability. Again, assuming the analogy holds, the most dangerous mode for vertical boundary layers is two-dimensional and this forms a motivation for the present study. It is hoped that the evolution of vortex disturbances will be addressed in a later paper.

**Governing equations**

The nondimensional equations governing unsteady two-dimensional Darcy-Boussinesq convection in a porous medium are

\[
\psi_{xx} + \psi_{yy} = \theta_y, \tag{1}
\]

\[
\theta_t = \theta_{xx} + \theta_{yy} + \psi_x \theta_y - \psi_y \theta_x, \tag{2}
\]

where the \(x\)-axis is orientated vertically and the Darcy-Rayleigh number has been scaled out of the equations (see Riley and Rees [9] for details of the nondimensionalisation). The boundary conditions required to complete the specification of the problem are that \(\psi = 0\) and \(\theta = 1\) on the positive \(x\)-axis, \(\psi = 0\) and \(\theta_y = 0\) on the negative \(x\)-axis and the ambient temperature is zero far from the \(x\)-axis. In this cartesian coordinate system it is now well-known that the vertical boundary layer thickness increases with increasing \(x\), and is asymptotically proportional to \(x^{1/2}\) for large \(x\). The introduction of parabolic coordinates defined by

\[
x = \frac{1}{2}(\xi^2 - \eta^2), \quad y = \frac{1}{2}\xi \eta, \tag{3}
\]

yields the equations

\[
\psi_{\xi \xi} + \psi_{\eta \eta} = \frac{1}{2} [\xi \theta_{\eta} + \eta \theta_{\xi}], \tag{4}
\]

\[
\theta_t = \frac{4}{\xi^2 + \eta^2} [\theta_{\xi \xi} + \theta_{\eta \eta} + \psi_{\xi} \theta_{\eta} - \psi_{\eta} \theta_{\xi}]. \tag{5}
\]

The leading order boundary layer profile is given by

\[
\psi = \frac{1}{2} \xi f(\eta), \quad \theta = g(\eta), \tag{6}
\]

where \(f(\eta)\) and \(g(\eta)\) satisfy

\[
f'' - g' = 0, \quad g'' + \frac{1}{2} fg' = 0, \tag{7}
\]
subject to the boundary conditions \( f(0) = 0, \ g(0) = 1, \) and \( f', \ g \to 0 \) as \( \eta \to \infty. \) In this new coordinate system the boundary layer thickness is constant, which facilitates the use of a fully numerical scheme. It is worth reiterating a point made in Rees and Bassom [10] that (6) yields the exact solution of (4) in the whole flowfield if the porous medium is bounded by the \( x \)-axis (here the positive \( x \)-axis is heated, and the remainder is insulated, as given above). As relative simplicity of programming is afforded by the use of such a semi-infinite configuration, the results presented below correspond entirely to this particular configuration.

### Numerical solutions

The full equations (4) and (5) were solved using an implicit finite-difference discretisation. The resulting nonlinear difference equations were solved by pointwise Gauss-Seidel iteration accelerated by multigrid \( V \)-cycling. Further details of the method and its implementation may be found in reference 4. The finite-difference grid used, as well as the streamlines and isotherms of the basic flow given by (6), are shown in Fig. 1. The computational mesh is uniform in the \( \xi \)-direction and the coordinate-stretching transformation used in the \( \eta \)-direction is also given in reference 4.

![Figure 1](image)

(a) the computational grid, (b) the streamlines of the basic flow plotted at an interval of 2, (c) the isotherms of the basic flow plotted at an interval of 0.05.
Values of $\xi_{\text{max}} = 64$ and $\eta_{\text{max}} = 15$ were taken on a $256 \times 32$ grid. These values are such that the boundary layer is contained well within the computational domain, as may be seen in Fig. 1c.

Only disturbances to the basic temperature field were considered since the streamfunction, $\psi$, adjusts instantaneously to variations in the temperature. Two types of disturbance were examined. Type 1 consists of an isolated cell given by

$$\delta \theta = \frac{A}{4} \left(1 + \cos \frac{2\pi(\xi - c)}{a}\right)\left(1 - \cos \frac{2\pi \eta}{b}\right),$$

$$-a/2 \leq \xi - c \leq a/2, \quad 0 \leq \eta \leq b$$

(8)

where $\delta \theta$ is the temperature perturbation, $A$ its amplitude, $a$, its streamwise extent, $b$, its width and $c$, the streamwise location of its mid point. Type 2 comprises a set of counter-rotating cells given by

$$\delta \theta = \frac{A}{2} \cos \frac{2\pi \xi}{a} \left(1 - \cos \frac{2\pi \eta}{b}\right), \quad 0 \leq \eta \leq b$$

(9)

where $a$ is now the period of the disturbance in terms of $\xi$.

As the numerical scheme is fully nonlinear the evolution of large-amplitude disturbances can be followed. Indeed, in all the computations presented here the amplitude $A = 1$ has been chosen. Solutions for smaller amplitudes are qualitatively very similar. For each figure displaying perturbation isotherms, contours are at respective intervals of $|\delta \theta|_{\text{max}}/10$; the overriding reason for this choice is because perturbations tend to decay quickly through many orders of magnitude.

Figure 2 shows the instantaneous perturbations to the temperature field at various times after the imposition of a Type 1 disturbance with $A = 1$,
\(a = 2, \ b = 2\) and \(c = 12\). Soon after the introduction of the disturbance, which represents a region of elevated temperature, a relatively small region in which the temperature perturbation is negative is induced upstream of the main disturbance. As it evolves the disturbance decays markedly in strength and advects downstream. Somewhat surprisingly the evolution is also marked by a substantial growth in size in both the streamwise and cross-streamwise directions. The streamwise growth could possibly be without limit for it has been found always to fill a substantial part of the computational domain. For the analogous problem of flow in a uniform, differentially heated vertical channel it is known that the most slowly decaying mode has zero wavenumber. If the analogy holds and if it is possible to view the perturbation shown in Fig. 2a as a sum of discrete modes of different wavenumber, or perhaps, as an integral over a continuous spectrum, then such a streamwise growth could be explained in terms of modes with larger wavenumbers decaying more quickly than those with small wavelengths. It must be said, however, that the concept of ‘mode’ in this context, where the basic flow is spatially varying and nonperiodic, needs to be defined rigorously, but it certainly won’t be as straightforward as, say, in the Bénard problem. On comparing with Fig. 1c, it can be seen that the disturbance also spreads a substantial distance outside of the boundary layer. Qualitatively the same behaviour has been found for disturbances placed at other locations and with differential values of \(a\) but the initial rates of decay are dependent on both these factors. For instance, a disturbance placed near to the leading edge of the heated surface decays more quickly than an otherwise identical one (i.e. one with a different value of \(c\) in (7)) placed further downstream; see Fig. 3. Likewise, a disturbance with a larger streamwise extent, \(a\), decays more slowly than one with a smaller value of \(a\); see Fig. 4.

Figure 5 shows the evolution of a Type 2 disturbance with \(A = 1, \ a = 2, \ b = 2\). As this disturbance evolves it too decays more quickly nearer the leading edge. However, the dynamics of the evolution are somewhat different because the spatial form of the initial disturbance is substantially different. As evolution proceeds cell-merging takes place the back of the main disturbance field. Such merging turns out to be qualitatively different from that observed in reference 4 for there the flow is strongly nonlinear, whereas here the disturbance is weak and can be regarded as evolving linearly. The overall effect of cell-merging is to reduce the ‘wavenumber’ of the disturbance and thereby to reduce the decay rate. Further computations indicate that, for initial disturbances with larger wavelengths than that shown in Fig. 5, the initial decay rate is slower; cell merging takes place as in Fig. 5, but occurs later and increasingly further downstream. For smaller wavelength disturbances the opposite conclusions hold. Nevertheless, in both cases the dynamics are qualitatively the same.
Conclusions

A time-dependent, spatially elliptic, numerical method has been employed to follow the evolution of disturbances in vertical thermal boundary layer flow in a porous medium. In all cases considered disturbances have been found to decay. Since the method of solution necessarily uses a finite computational domain, it cannot be concluded categorically that vertical thermal boundary layer flow is stable, as is the flow in the analogous problem of convection in a uniform vertical porous channel heated from the...
Perturbation isotherms for a Type 2 disturbance given by $A = 1$, $a = 2$ and $b = 2$. Isotherms plotted at an interval of $|\delta \theta|_{\text{max}}/10$. (a) $t = 0$, $|\delta \theta|_{\text{max}} = 1$; (b) $t = 100$, $|\delta \theta|_{\text{max}} = 3.067 \times 10^{-1}$; (c) $t = 200$, $|\delta \theta|_{\text{max}} = 1.287 \times 10^{-1}$; (d) $t = 300$, $|\delta \theta|_{\text{max}} = 3.383 \times 10^{-2}$; (e) $t = 400$, $|\delta \theta|_{\text{max}} = 1.071 \times 10^{-2}$; (f) $t = 500$, $|\delta \theta|_{\text{max}} = 5.678 \times 10^{-3}$; (g) $t = 600$, $|\delta \theta|_{\text{max}} = 3.565 \times 10^{-3}$; (h) $t = 850$, $|\delta \theta|_{\text{max}} = 1.147 \times 10^{-3}$.

side. However, it has been demonstrated that the basic boundary layer flow is nonlinearly stable in the region $\xi < 64$ (or $x < 1024$). Indeed the code was also used to study the starting problem (i.e. where $\theta = 0$ initially, and the temperature of the positive $x$-axis is raised suddenly to 1); it was found that the flow evolved to the steady boundary layer profile. In view of the stability of the basic flow, at least for the range of values of $x$ used here, we can conclude that the precise heat transfer result given in reference 10 is valid.
For larger values of $x$, the boundary layer approximation to the exact solution (6) is good, and a parallel-theoretic method should therefore supply accurate results, but such an analysis is outside the scope of this paper.

A considerable qualitative difference in stability characteristics has now been found to exist between the vertical and horizontal buoyancy-induced boundary layer flows in porous media. Here, the vertical case has been shown to be stable (subject to the comments made above), and, in reference 4, the horizontal case admits aperiodic convection. It is now a task of some importance to determine how these dissimilar characteristics can be reconciled by examining the upward-facing inclined boundary layer flow using similar numerical methods. Work on this aspect is currently in progress.

References


Abstract

A two-dimensional, nonlinear, time-dependent, elliptic, numerical method coupled with an appropriate coordinate transformation is used to investigate the stability of free convection induced by an isothermally heated semi-infinite surface embedded in a fluid-saturated porous medium. It is found that the basic boundary layer flow is stable even to large amplitude disturbance for nondimensional distances of up to 1024 from the leading edge of the heated surface.

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