NON-LOCAL EQUILIBRIUM FLOW WITH VISCOUS DISSIPATION IN A PLANE HORIZONTAL POROUS LAYER

A. Barletta, M. Celli
DIENCA, Alma Mater Studiorum – Università di Bologna, Via dei Colli 16, 40136 Bologna, Italy

ABSTRACT

Darcy’s flow in a horizontal porous layer with impermeable boundaries is studied. The viscous dissipation effect is taken into account and the local thermal non-equilibrium (LTNE) model for the energy balance is adopted. The upper boundary is assumed to be perfectly isothermal and the lower boundary is taken to be thermally insulated. The basic solution is expressed analytically. The case of a perfectly conducting solid phase is considered. The onset of convective roll instability is investigated by a linear analysis, with different values of the inter-phase heat transfer parameter. The eigenvalue problem is solved numerically by a Runge-Kutta method.

INTRODUCTION

Viscous dissipation can play an important role in the stability analysis of basic flow solutions in porous media. In this kind of problems, a sufficiently intense temperature gradient is needed for the onset of convective instabilities. In the absence of a thermal forcing induced by the temperature boundary conditions, the viscous dissipation effect may be the only possible cause of instability. For instance, in a horizontal porous layer with an upper isothermal boundary and a lower adiabatic boundary, a possibly unstable stratification may be induced by the frictional heating associated with a basic horizontal throughflow. In the classical Darcy-Bénard problem [1], the basic temperature gradient is forced by the boundary conditions and the viscous dissipation provides a non linear contribution or, more precisely, a second order term in the perturbations. The latter term, in a linear stability analysis, is neglected. On the other hand, if a basic throughflow is imposed, the viscous dissipation provides also a linear term in the perturbations and, thus, it may influence the onset conditions of the instability.

In the present contribution, a Darcy flow in a horizontal porous layer with impermeable boundaries is studied. The viscous dissipation term in the energy balance for the fluid phase is taken into account. Two different temperature fields for the porous solid and for the saturating fluid are assumed in order to model the local thermal non-equilibrium (LTNE) [1–4]. Two local energy balances, one for each phase, are introduced. The upper boundary is taken to be perfectly isothermal and the lower boundary is taken to be thermally insulated. In this configuration, the viscous dissipation contribution provides a source of possible instability. The basic velocity field is assumed to be stationary and uniform. The basic solution is expressed analytically and perturbed by plane waves in order to investigate the onset of convective rolls. The special case of a porous solid with a very high thermal conductivity is examined. This assumption implies a drastic reduction in the number of governing parameters. The eigenvalue problem thus obtained is solved by means of a Runge-Kutta method. The onset of the instability is described through the governing dimensionless parameters $H$ and $R$, where $H$ is the inter-phase heat transfer parameter and $R$ is the stability parameter defined as $R = Ge Pe^2$. Here, $Ge$ is the Gebhart number and $Pe$ is the Pécellet number.

NOMENCLATURE

- $a$: wave number, Eq. (32)
- $c$: heat capacity per unit mass
- $e_y$: unit vector in the $y$-direction
- $F(y, H)$: dimensionless function, Eq. (36)
- $g$: gravitational acceleration; modulus of $g$
- $Ge$: Gebhart number, Eq. (8)
- $h$: inter-phase heat transfer coefficient
- $H$: dimensionless parameter, Eq. (8)
- $k$: thermal conductivity
- $L$: layer thickness
Greek symbols
\[\alpha\] thermal diffusivity
\[\beta\] thermal expansion coefficient
\[\epsilon\] perturbation parameter
\[\eta\] dimensionless parameter, Eq. (43)
\[\theta\] fluid phase temperature disturbance
\[\Lambda\] dimensionless parameter, Eq. (10)
\[\mu\] dynamic viscosity
\[\rho\] density
\[\varphi\] porosity
\[\phi\] solid phase temperature disturbance
\[\Phi, \Theta, \Psi\] disturbance amplitudes, Eq. (32)
\[\chi\] inclination angle, Eq. (15)
\[\psi\] streamfunction disturbance
\[\Omega\] dimensionless parameter, Eq. (8)

Superscript, subscripts
- dimensional quantity
\[\text{B}\] basic flow
\[\text{cr}\] critical value
\[f\] fluid phase
\[s\] solid phase

1 Mathematical model

We study a horizontal porous layer with impermeable boundary planes \(\bar{y} = 0\) and \(\bar{y} = L\). The \(y\)-axis is oriented upward, so that \(g = -g e_y\). The lower boundary, \(\bar{y} = 0\), is thermally insulated, while the upper boundary \(\bar{y} = L\) is kept at a uniform temperature \(T_0\). Let us assume that:

- Darcy's law holds;
- the Oberbeck-Boussinesq approximation can be applied;
- the viscous dissipation cannot be neglected;
- a condition of local thermal non-equilibrium (LTNE) holds.

Then, the governing equations can be written as

\[
\nabla \cdot \mathbf{u} = 0, \tag{1}
\]

\[
\frac{\mu}{K} \nabla \times \mathbf{u} = \rho_f g \beta \nabla T_f, \tag{2}
\]

\[
(1 - \varphi) \rho c_f \frac{\partial T_f}{\partial t} = \nabla \cdot (\rho c_f \mathbf{u} \nabla T_f) + \frac{\mu}{K} \mathbf{u} \cdot \mathbf{G}, \tag{3}
\]

\[
 \varphi (\rho c_f) \frac{\partial T_f}{\partial t} + (\rho c_f) \mathbf{u} \cdot \mathbf{G} = \varphi k_f \nabla^2 T_f + \frac{\mu}{K} \mathbf{u} \cdot \mathbf{G}, \tag{4}
\]

where the curl operator has been applied to the local momentum balance equation in order to eliminate the pressure gradient term.

The boundary conditions are given by

\[
\bar{y} = 0: \quad v = 0, \quad \frac{\partial T_s}{\partial \bar{y}} = \frac{\partial T_f}{\partial \bar{y}} = 0, \tag{5}
\]

\[
\bar{y} = L: \quad v = 0, \quad T_s = T_f = T_0. \tag{6}
\]

1.1 Dimensionless formulation

Let us introduce dimensionless variables such that

\[
\mathbf{x} = x L, \quad \bar{t} = \frac{t^2}{\alpha_f}, \quad \mathbf{u} = u \frac{\alpha_f}{L}, \quad \bar{T} = \frac{T - T_f}{T_0 - T_f}, \tag{7}
\]

and the dimensionless parameters

\[
Ge = \frac{\beta g L}{c_f}, \quad \Lambda = \frac{\alpha_f}{\alpha_s}, \quad H = \frac{h L^2}{\varphi k_f}, \tag{8}
\]

Here, \(Ge\) is the Gebhart number. Then, Eqs. (1)-(4) can be rewritten as

\[
\nabla \cdot \mathbf{u} = 0, \tag{9}
\]

\[
\nabla \times \mathbf{u} = \nabla \times (T_f e_y), \tag{10}
\]

\[
\Lambda \frac{\partial T_f}{\partial \bar{t}} = \nabla^2 T_f + H \Omega (T_f - T_s), \tag{11}
\]

\[
\frac{\partial T_f}{\partial \bar{t}} + \frac{1}{\varphi} \mathbf{u} \cdot \nabla T_f = \nabla^2 T_f + \frac{Ge}{\varphi} \mathbf{u} \cdot \mathbf{u} + \frac{H (T_s - T_f)}. \tag{12}
\]

The boundary conditions are given by

\[
y = 0: \quad v = 0, \quad \frac{\partial T_s}{\partial y} = \frac{\partial T_f}{\partial y} = 0, \tag{13}
\]

\[
y = 1: \quad v = 0, \quad T_s = T_f = 0. \tag{14}
\]

2 Basic solution

We assume a basic uniform horizontal flow inclined of an angle \(\chi\) with respect to the \(x\)-direction, with purely vertical temperature gradients. Then, a stationary solution of Eqs. (9)-(14) is given by

\[
\mathbf{u}_B = \varphi \rho e \cos \chi, \quad \mathbf{v}_B = 0, \tag{15}
\]

\[
w_B = \varphi \rho e \sin \chi, \tag{16}
\]

\[
T_sB = -\frac{\varphi R \Omega e^{-y\sqrt{H(\Omega + 1)}}}{2 H (\Omega + 1)^2 \left( e^{2\sqrt{H(\Omega + 1)}} + 1 \right)} \times \left\{ e^{y\sqrt{H(\Omega + 1)}} + e^{y(\Omega + 2)\sqrt{H(\Omega + 1)}} \right\} \times \frac{1}{\Omega (\Omega + 1) \left( y^2 - 1 \right) + 2} - 2 e^{2(y+1)\sqrt{H(\Omega + 1)}} - 2 e^{y\sqrt{H(\Omega + 1)}} \right\}, \tag{17}
\]
librium (LTE) is recovered. In this limit, Eqs. (16) and (17) yield
\[
\lim_{H \to 0} T_{sb} = 0, \quad \lim_{H \to 0} T_{fb} = \frac{\varphi R}{2} (1 - y^2).
\]
When \( H \to \infty \), the condition of local thermal equilibrium (LTE) is recovered. In this limit, Eqs. (16) and (17) yield
\[
\lim_{H \to \infty} T_{sb} = \lim_{H \to \infty} T_{fb} = \frac{\varphi R \Omega}{2 (\Omega + 1)} \left( 1 - y^2 \right).
\]
One can easily show that, as a consequence of the definition of \( \Omega \), Eq. (20) is perfectly consistent with the basic temperature profile obtained for the case of LTE [5].

Fig. 1 shows that, with increasing values of \( H \), the distributions of \( T_{sb} \) and \( T_{fb} \) tend to become coincident. Moreover, the distributions of \( T_{fb} \) reveal that the lower boundary \( y = 0 \) has a higher fluid temperature than the upper boundary \( y = 1 \). Hence, the basic flow has a possibly unstable thermal stratification.

In the following, we will restrict our analysis to the case of a fluid saturated porous medium with \( k_s \gg k_f \) and \( \alpha_s \gg \alpha_f \), so that the parameters \( \Lambda \) and \( \Omega \) are vanishingly small. As a consequence, Eq. (11) becomes \( \nabla^2 T_s = 0 \) and the basic solution, Eqs. (16) and (17), simplifies to
\[
T_{sb} = 0,
\]
\[
T_{fb} = \frac{\varphi R}{2} \left[ \cosh \left( \sqrt{H} \right) - \cosh \left( y \sqrt{H} \right) \right].
\]
Fig. 2 shows the distribution of \( T_{fb} \) for different values of the parameter \( H \). One can note that, as the inter-phase heat transfer becomes more and more efficient (increasing values of \( H \)), the temperature distribution of the fluid phase tends to zero, i.e. it tends to coincide with the temperature distribution of the solid phase.

3 Linear disturbances

One can now perturb the basic state given by Eqs. (15), (21) and (22) by defining
\[
\begin{align*}
    u &= u_B + U \epsilon, \quad v = V \epsilon, \quad w = w_B + W \epsilon, \\
    T_f &= T_{fb} + \theta \epsilon, \quad T_s = T_{sb} + \phi \epsilon,
\end{align*}
\]
On substituting Eq. (23) in Eqs. (9)-(12), with \( \Lambda \to 0 \) and \( \Omega \to 0 \), and neglecting terms \( O(\epsilon^2) \), we obtain...
the linearized stability equations, namely
\[ \nabla \cdot U = 0, \]
(24)
\[ \nabla \times U = \nabla \times (\theta \mathbf{e}_y), \]
(25)
\[ \nabla^2 \phi = 0, \]
(26)
\[ \frac{\partial \theta}{\partial t} + P \left( \cos \chi \frac{\partial \phi}{\partial x} + \sin \chi \frac{\partial \phi}{\partial y} \right) + \frac{V}{\varphi} \frac{dT_{IB}}{dy} 
= \nabla^2 \theta + H (\phi - \theta) 
+ 2 Ge P \frac{U}{\varphi} (U \cos \chi + W \sin \chi), \]
(27)
where the basic temperature field, \( T_{IB} \), is defined by Eq. (22). The linearity of Eqs. (24)-(27) allows one to treat the rolls of different orientations separately. Each of these cases can thus be dealt with by using a purely 2D treatment. Two-dimensional solutions of Eqs. (24)-(27) are thus sought, such that \( W = 0 \) and the fields \( U, V, \theta, \phi \) depend only on \((x, y, t)\). Now, on assuming that the disturbances \( U, V, \theta, \phi \) are \( O(1) \), we note that the term \( V/\varphi \frac{dT_{IB}}{dy} \) is of order \( R = Ge P^2 \), while the term \( 2 Ge P \frac{U}{\varphi} \sin \chi \) is of order \( Ge P \). In physically interesting cases, the Gebhart number is so small that a conceivable condition for a flow with a non-negligible viscous dissipation is: \( Ge \ll 1, |P| \gg 1 \), with \( R = Ge P^2 \sim O(1) \). This argument was pointed out by Barletta et al. [5, 6]. In this regime, the term \( 2 Ge P \frac{U}{\varphi} \sin \chi \) can be neglected with respect to the term \( (V/\varphi) \frac{dT_{IB}}{dy} \).

It is now convenient to introduce a streamfunction, \( \psi \), such that Eq. (24) is satisfied,
\[ U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x}, \]
(28)
The system Eqs. (25)-(27) can thus be rewritten as
\[ \nabla^2 \psi + \frac{\partial \theta}{\partial x} = 0, \]
(29)
\[ \nabla^2 \phi = 0, \]
(30)
\[ \frac{\partial \theta}{\partial t} + P \cos \chi \frac{\partial \phi}{\partial x} + \frac{1}{\varphi} \frac{\partial \psi}{\partial x} \frac{dT_{IB}}{dy} 
= \nabla^2 \theta + H (\phi - \theta). \]
(31)
Solutions of Eqs. (29)-(31) are sought in the form of plane waves,
\[ \psi(x, y, t) = \Psi(y) \cos(a x - t a P \cos \chi), \]
\[ \phi(x, y, t) = \Phi(y) \sin(a x - t a P \cos \chi), \]
\[ \theta(x, y, t) = \Theta(y) \sin(a x - t a P \cos \chi), \]
(32)
where the positive real parameter \( a \) is the prescribed wave number. On substituting Eq. (32) in Eqs. (29)-(31) one obtains
\[ \Psi'' - a^2 \Psi + a \Theta = 0, \]
(33)
\[ \Phi'' - a^2 \Phi = 0, \]
(34)
\[ \Theta'' - (a^2 + H) \Theta + H \Phi 
+ a R F(y, H) \Psi = 0, \]
(35)
where the primes denote differentiation with respect to \( y \) and, on account of Eq. (22), \( F(y, H) \) is defined as
\[ F(y, H) = -\frac{1}{\varphi R} \frac{dT_{IB}}{dy} = \frac{\sinh(y \sqrt{H})}{\sqrt{H} \cosh(y \sqrt{H})}. \]
(36)
From Eqs. (13) and (14), the boundary conditions are given by
\[ y = 0 : \quad \Psi = 0, \quad \Phi' = \Theta' = 0, \]
(37)
\[ y = 1 : \quad \Psi = 0, \quad \Phi = \Theta = 0. \]
(38)
We note that Eq. (34) is uncoupled to any other differential equation. Hence, we conclude that the unique solution of Eq. (34) subject to the boundary conditions, Eqs. (37) and (38), is \( \Phi = 0 \). The self-adjoint eigenvalue problem thus reduces to
\[ \Psi'' - a^2 \Psi + a \Theta = 0, \]
(39)
\[ \Theta'' - (a^2 + H) \Theta + a R F(y, H) \Psi = 0, \]
(40)
\[ y = 0 : \quad \Psi = 0, \quad \Theta' = 0, \]
(41)
\[ y = 1 : \quad \Psi = 0, \quad \Theta = 0. \]
(42)
We note that the eigenvalue problem Eqs. (39)-(42) is independent of the inclination angle \( \chi \). Hence, we have the same response from the system with longitudinal, transverse or oblique roll disturbances. This means that roll disturbances having different inclinations with respect to the basic flow direction do not alter the onset conditions of the instability.

4 The numerical solution

Eqs. (39)-(42) are solved numerically by using a sixth-order Runge-Kutta method and the shooting method. In order to apply the Runge-Kutta method, we must complete the initial conditions on the ODEs (39)-(40) by replacing Eq. (41) with
\[ \Psi(0) = 0, \quad \Psi'(0) = 1, \quad \Theta(0) = \eta, \quad \Theta'(0) = 0, \]
(43)
where the condition \( \Psi(0) = 0 \) fixes the otherwise indeterminate scale of the eigenfunctions \( (\Psi, \Theta) \) in Eqs. (39)-(42). The parameter \( \eta \) is unknown a priori and can be evaluated by a shooting method where the target conditions are those given in Eq. (42). In fact, Eq. (42) contains two conditions that allow one to evaluate \( \eta \) and the eigenvalue \( R \), on assuming that \( (a, H) \) are prescribed. With this procedure, one may determine an eigenvalue function \( R(a) \) for any given \( H \). The minimum of this function corresponds to the critical pair \( (a_{cr}, R_{cr}) \) for the onset of the convective instability. The sixth-order Runge-Kutta method with an adaptive step-size control is easily implemented by using the Mathematica 7.0 package (©1996, Wol...
### Table 1: Values of $a_{cr}$ and $R_{cr}$ for $H = 0$: comparison between the Runge-Kutta method with fixed step-size and the Runge-Kutta method with adaptive step-size control

<table>
<thead>
<tr>
<th>Method</th>
<th>$a_{cr}$</th>
<th>$R_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed step-size 1/3</td>
<td>2.4483351</td>
<td>61.864478</td>
</tr>
<tr>
<td>Fixed step-size 1/4</td>
<td>2.4482740</td>
<td>61.866234</td>
</tr>
<tr>
<td>Fixed step-size 1/8</td>
<td>2.4482662</td>
<td>61.866563</td>
</tr>
<tr>
<td>Fixed step-size 1/16</td>
<td>2.4482661</td>
<td>61.866567</td>
</tr>
<tr>
<td>Fixed step-size 1/32</td>
<td>2.4482661</td>
<td>61.866567</td>
</tr>
<tr>
<td>Adaptive</td>
<td>2.4482661</td>
<td>61.866567</td>
</tr>
</tbody>
</table>

Table 2: Critical values $a_{cr}$ and $R_{cr}$ for different $H$

<table>
<thead>
<tr>
<th>$H$</th>
<th>$a_{cr}$</th>
<th>$R_{cr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.448266</td>
<td>61.86657</td>
</tr>
<tr>
<td>1</td>
<td>2.673854</td>
<td>99.04851</td>
</tr>
<tr>
<td>5</td>
<td>3.290168</td>
<td>307.6197</td>
</tr>
<tr>
<td>10</td>
<td>3.849151</td>
<td>691.3329</td>
</tr>
<tr>
<td>50</td>
<td>7.326780</td>
<td>7039.107</td>
</tr>
<tr>
<td>100</td>
<td>10.35456</td>
<td>19907.48</td>
</tr>
</tbody>
</table>

Figure 4: Plots of $a_{cr}$ and $R_{cr}$ versus $H$

5 Discussion of the results

Fig. 3 shows the neutral stability curves corresponding to increasing values of $H$. We note that, as the parameter $H$ increases, a more efficient inter-phase heat transfer is present. Physically, the perfectly conducting solid phase ($k_s \gg k_f$ and $\alpha_s \gg \alpha_f$) extracts efficiently the heat generated by the viscous dissipation from the fluid phase. As a consequence, on increasing $H$, the fluid phase experiences smaller and smaller temperature gradients. This implies that the system stability is increased, and hence the neutral stability curves are displaced upward as $H$ increases.

Table 2 contains values of $R_{cr}$ and $a_{cr}$ for different $H$. The behaviour $R_{cr}$ and $a_{cr}$ versus $H$ is also shown in Fig. 4. Table 2 and Fig. 4 suggest that the value of $R_{cr}$ increases indefinitely as the parameter $H$ increases. This would imply that, for an infinite inter-phase heat transfer coefficient, the system is always stable. We mention that, in the limit of LTE, $H \to \infty$, the fluid phase is forced to
CONCLUSIONS

Darcy’s flow in a horizontal porous layer bounded by impermeable boundary walls, upper isothermal boundary and lower adiabatic boundary, is studied. The local thermal non-equilibrium model for the energy balance is considered and the viscous dissipation is taken into account. An inclined basic flow with a purely vertical temperature distribution has been assumed and a streamfunction-temperature formulation has been adopted. A perfectly conducting porous solid has been considered. This assumption has allowed us to simplify drastically the problem: the temperature distribution of the solid phase has become trivial and it has been forced to be uniform and equal to the value at the isothermal boundary. A linear stability analysis for determining the onset condition of oblique convective rolls has been carried out, based on the numerical solution of the disturbance equations by means of a sixth-order Runge-Kutta method. The stability analysis has been based on the reasonable assumption that, in practical cases, the Gebhart number is rather small, so that the effect of viscous dissipation may be important only with large values of the Péclet number associated with the basic flow.

An important feature of the analysis carried out is that the linearly stable or unstable behaviour of the system in response to oblique roll disturbances is independent of the inclination angle. It has been shown that the perfectly conducting solid phase influences the fluid phase temperature profile as it produces an efficient subtraction of the heat generated by the viscous dissipation. As the heat transfer between the two phases increases in efficiency the system becomes more and more stable with respect to the onset of convective rolls. Eventually, in the limit of an infinitely efficient process of heat transfer between the phases, the system becomes absolutely stable.

REFERENCES