Convection in a Horizontal Porous Layer

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Abstract: We consider the problem of nonlinear compositional convection in a horizontal porous layer with impermeable boundaries. Using perturbation analyses and numerical calculations, we determine the stable convective flow under certain range of the parameter values where such analyses are valid. We find, in particular, that depending on the range of values of the parameters, the effect of impermeable boundaries lead to stability of convection in the form of either subcritical up-hexagons or supercritical down-hexagons.

Keywords: compositional convection, flow stability, buoyant convection, porous layer, buoyant flow, convection, convective stability, porous medium

1 Introduction

The present study considers the problem of finite-amplitude steady convection in a horizontal porous layer with impermeable upper and lower boundaries. This is a mathematical model for convection which can be realized during directional solidification of binary alloys. The present investigation is based on perturbation analyses and some numerical integration procedures [6] to determine qualitative results about the realizable and stable form of the buoyant convection in such a porous layer.

A simplified porous-layer model with impermeable interface was introduced first in [1]. The model was based on a near–eutectic approximation, which reduces applicability of the model to more general alloy cases, in the limit of large far-field temperature and in the regime of small deviation from the classical system of convection in a horizontal porous layer of constant permeability. To construct such a single-layer model for the porous zone, the authors in [1] needed to make a number of simplifying assumptions including those stated above and the one that the thickness $\delta$ of the porous layer is small. In addition, the authors assumed that the amplitude $\varepsilon$ of convection is of the same order as $\delta$. The authors found that either sub- or supercritical steady rolls are possible and steady hexagons can be transcritical.

The work in [2] extended the analytical studies in [1] to the limit of large Stefan number $S$, which represents the latent heat release due to solidification, and the case $\varepsilon^2<<\delta<<1$. The authors applied a double-series expansion in powers of $\varepsilon$ and $\delta$ for the dependent variables and the Rayleigh number $R$. They focused on the steady modes of convection and calculated, in particular, the finite amplitude steady solutions in the form of two-dimensional rolls and hexagons.

The authors in [6] developed a new and more realistic model for the
weakly nonlinear convection in mushy layers. In this model no assumption was made on δ, and a number of simplifying assumptions made in previous theoretical studies of the problem [1-2, 4] were lifted in order to make the model more realistic.

2 Governing system

We consider a binary alloy melt that is cooled from below and solidified at a constant speed \( V_0 \). We consider a horizontal porous layer of thickness \( h \) adjacent and above the solidification front with overlying liquid and underlying solid zones. The overlying liquid is assumed to have a composition \( C_0 \) greater than the eutectic concentration \( C_e \) and temperature \( T_e > T_L(C_0) \) far above the dendrite layer, where \( T_L(C') \) is the liquidus temperature of the alloy and \( C' \) is the composition. We consider the solidification system in a moving frame of reference \( O'x'y'z' \), whose origin lies on the solidification front and translating at the speed \( V_0 \) with the solidification front in the positive \( z' \)-direction.

We consider the equations for momentum, continuity, heat and solute for the flow of melt in the porous layer in the moving frame described before. These equations are non-dimensionalized by using \( V_0, \kappa/V_0, \kappa/V_0^2, \beta \Delta C \rho g k/\kappa, \Delta C, \Delta T \) as scales for velocity, length, time, pressure, solute and temperature, respectively. Here \( \kappa \) is the thermal diffusivity, \( \rho \) is a reference (constant) density, \( \beta = \beta^* - \Gamma \alpha^* \), where \( \alpha^* \) and \( \beta^* \) are the expansion coefficients for the heat and solute, respectively, \( \Gamma \) is the slope of liquidus, \( \Delta C = C_0 - C_e \), \( \Delta T = T_L(C_0) - T_e \) and \( T_e \) is the eutectic temperature. The non-dimensional form of the equations for momentum, continuity, temperature and solute concentration in the porous layer and in the realistic limit of sufficiently large Lewis number are [8]

\[
\begin{align*}
\tilde{\mathbf{u}} &= -\nabla \tilde{p} - \mathbf{R} \mathbf{e}_z, & (1a) \\
\nabla \cdot \tilde{\mathbf{u}} &= 0, & (1b) \\
(\partial/\partial t - \partial/\partial z)(0-S(1-\chi)) + \tilde{\mathbf{u}} \cdot \nabla \theta &= \nabla^2 \theta, & (1c) \\
(\partial/\partial t - \partial/\partial z)(\chi \theta + C(1-\chi)) + \tilde{\mathbf{u}} \cdot \nabla \theta &= 0, & (1d)
\end{align*}
\]

where \( \tilde{\mathbf{u}} = \tilde{u} \mathbf{x} + \tilde{v} \mathbf{y} + \tilde{w} \mathbf{z} \) is the volume flux vector per unit area, \( \tilde{u} \) and \( \tilde{v} \) are the horizontal components of \( \tilde{\mathbf{u}} \) along \( x \) and \( y \)-directions, respectively, \( \mathbf{x} \) and \( \mathbf{y} \) are unit vectors along the positive \( x \) and \( y \)-directions, \( \tilde{w} \) is the vertical component of \( \tilde{\mathbf{u}} \) along \( z \)-direction, \( \mathbf{z} \) is a unit vector along the positive \( z \)-direction, \( \tilde{p} \) is the modified pressure, \( \theta \) is the non-dimensional composition, or equivalently temperature [9], \( \theta = [T - T_L(C_0)]/\Delta T = (\tilde{C} - C_0)/\Delta C, t \) is the time variable, \( \chi \) is the local liquid fraction or porosity, \( R = \beta \Delta C \rho g \Pi/(\nabla \nu) \) is the Rayleigh number, \( \Pi \) is the permeability of the porous medium, which is assumed to be constant, \( \nu \) is the kinematic viscosity, \( g \) is acceleration due to gravity, \( S = L/(C_m \Delta T) \) is the Stefan number, \( C_m \) is the specific heat per unit volume, \( L \) is the latent heat of solidification per unit volume, \( C = (C_e - C_0)/\Delta C \) is a concentration ratio and \( C_e \) is the composition of the solid-phase in the porous medium.

The governing equations (1a)-(1d) are solved subject to the impermeable boundary conditions

\[
\begin{align*}
\theta + 1 &= \tilde{w} = 0 & \text{at z = 0} & & \text{and} & & \
\theta = 1 - \chi &= \tilde{w} = 0 & \text{at z = 0} & & \text{and} & & \tilde{w} = 0 & \text{at z = } \delta, \quad (2a-b)
\end{align*}
\]

where \( \delta = hV/\kappa \) is a Peclet number representing the dimensionless depth of the layer.

3 Analyses and results

The deviation from the motionless basic state is measured by the small amplitude \( \varepsilon \) of convection. The
system (1)-(2) admits a motionless basic state, which is steady and horizontally uniform, is already known [6]. To determine δ we replace the basic state temperature in its expression by 0, which corresponds to z = δ. Thus, δ is determined as a function of the system parameters C, S and 0∞.

The finite-amplitude solutions for the governing system which can be realized for R beyond its critical value are determined by applying a weakly nonlinear procedure [3], based on a series expansion in powers of ε for the dependent variables and R of the steady modes. In the present analysis, the coefficient of R in the order ε^m (m≥0) of such series expansion is designated by R_m. Linear eigenvalue problem, which corresponds to the governing system to zeroth order in ε, determines R_0, its lowest value, which is the critical value of R, and the value of the horizontal wave number α at which R attains its lowest value. At order ε^1, the solvability condition for the governing system yield the expression for R_1. The expressions for the dependent variables for the finite-amplitude solutions in this order are then determined numerically. At order ε^2, the solvability condition for the governing system yield the expression for R_2.

The expressions for R_1 and R_2 can be used to study the steady finite-amplitude solutions. We shall restrict our attention to the simplest types of solutions, which include those observed in the applications. These solutions are called regular or semi-regular solutions [3]. In the case of a regular solution all angles between two neighboring wave number vectors are equal. In the more general semi-regular solution, the scalar products between any of the wave number vectors and its two neighboring wave number vectors assume the constant values α_1 and α_2. An example of a semi-regular solution is that due to rectangular cells, where α_1 = - α_2.

Simple forms of regular solutions correspond to the cases of rolls, square cells and hexagons.

To distinguish the physically realizable solution from among all the steady finite-amplitude solutions the stability of such solutions with respect to arbitrary two- and three-dimensional disturbances were investigated. Following Busse [3], we applied a series expansion in powers of ε for the dependent variables and the growth rates of such disturbances in the respective stability system and solved the systems for the disturbances to orders ε^0, ε^1 and ε^2. The leading order growth rate of the disturbances were then determined for particular range of values of the parameters.

We investigated particular parameter regimes where the finite-amplitude steady solutions were found to be stable and realizable in the sense that the growth rates of the investigated disturbances were found to be negative. The results are provided in the following three paragraphs.

First we set the values of C and S equal to 1. We then found that for stable state involving 0.1<0∞<3.0 and 3.0<0∞<8.0, stable convection is in the form of subcritical up-hexagons, where the flow is upward at the cell’ centers and downward at the cells’ boundaries, and supercritical down-hexagons, where the flow is downward at the cell’ centers and upward at the cells’ boundaries, respectively, for sufficiently small |k|.

Next, we set values of S and 0∞ equal to 1. Then for 0.2<C<0.4 and 0.4<C<10.0, we found that the stable convection is in the form of supercritical down-hexagons and subcritical up-
hexagons, respectively for sufficiently small $|\varepsilon|$.

For the last case, we set the values of $C$ and $\theta_0$ equal to 1. Then for $0.1 < S < 10.0$, we found that the stable convection is in the form of subcritical up-hexagons for sufficiently small $|\varepsilon|$.

5 Conclusion

We investigated the problem of nonlinear convection in a horizontal porous layer with constant permeability and impermeable boundaries during alloy solidification. We considered a newly developed and realistic model, and we analyzed the effects of several parameters on two- and three-dimensional steady convection patterns in the porous layer. The main results of the present study were the prediction of three-dimensional stable flow structures in the form of subcritical up-hexagons and supercritical down-hexagons. The convection in the form of down-hexagon, which is predicted in the present study under certain conditions, was, in fact, observed experimentally [7] for $R$ near its critical value.

References:


