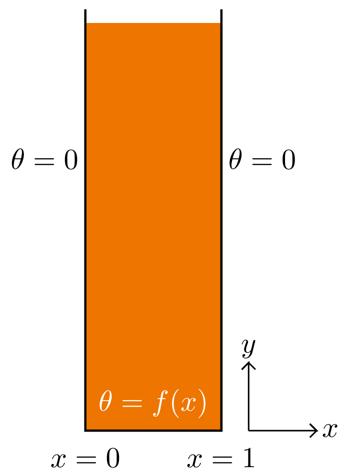
Fourier Transforms for solving Partial Differential Equations

1 Introduction

1.1 From Fourier series to Fourier transforms

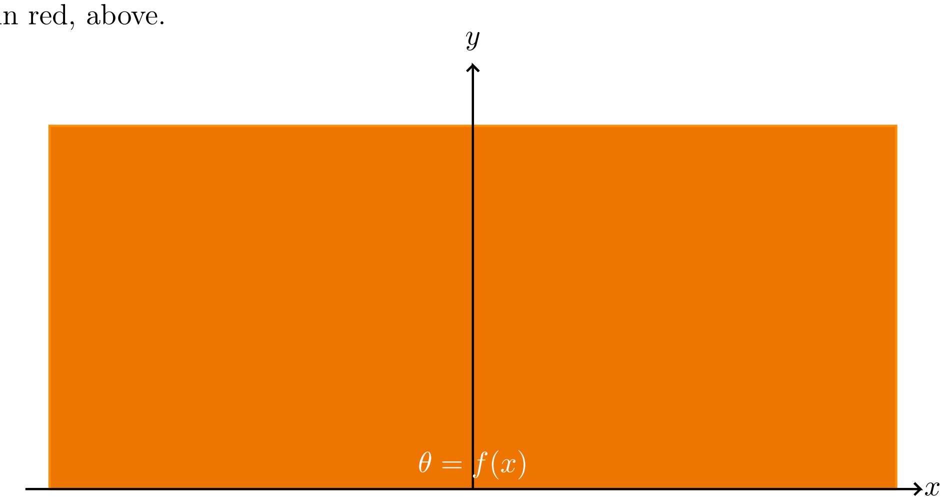
We have already seen how Fourier Series may be used to solve PDEs. Crudely speaking, the way we did it was to use a suitable selection of solutions of the form $B \sin kx \, e^{-ky}$ — this expression works for Laplace's equation — adding them all together, and finally finding the values of the arbitrary constants using Fourier Series methods. Therefore a problem of the following type,



has a solution of the form,

$$\theta = \sum_{n=1}^{\infty} B_n \sin n\pi x \, e^{-n\pi y}.$$

The above Figure is finite in the x-direction, and therefore Fourier Series may be used in that direction. However, when the domain is infinitely wide, i.e. $-\infty < x < \infty$, then this is when Fourier Transforms are used. Solutions will now be an integral over k of functions of the type displayed in red, above.



As we will see later, the solution of Laplace's equation in this case takes the form,

$$\theta = \frac{2}{\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} e^{-|\omega| y} \, d\omega,$$

i.e. an integral over a continuous set of frequencies, ω, rather than a sum over a discrete set of frequencies, $n\pi$. We will study this case in detail in another lecture.

1.2 Definition of the Fourier Transform and its inverse.

The so-called Fourier Transform pair is defined this way,

$$\mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} \, dx = F(\omega), \tag{1}$$

$$\mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} \, d\omega = f(x). \tag{2}$$

In Eq. (1) note that the presence of the complex exponential in the integral means that the Fourier Transform of a real $f(x)$ will usually return a complex-valued $F(\omega)$.

Note also that ω is a spatial frequency, and therefore the Fourier Transform of a function gives us its frequency content. I will illustrate this in much more detail later.

The formula for the inverse Fourier Transform bears a lot of similarity to the definition of the Fourier Transform itself. The two differences are (i) the sign of the exponent in the complex exponential, (ii) the different constants multiplying the integrals, although some textbooks use $1/\sqrt{2\pi}$ in both cases, something I don't like because one carries a lot of instances of $\sqrt{\pi}$ everywhere.

These formulae will be quoted on the exam paper.

1.3 Comparison with the Laplace Transform

Last year I made the very strange statement that the Laplace Transform variable, s, could be interpreted as an imaginary frequency. In practical terms this makes no sense! But when we look at the two formulae:

$$\text{Laplace Transform:} \quad \mathcal{L}[f(x)] = \int_{0}^{\infty} f(x) e^{-sx} \, dx$$

$$\text{Fourier Transform:} \quad \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} \, dx,$$

we see that s plays the same sort of role in Laplace transforms as does $j\omega$ in Fourier Transforms. Given that one always thinks of s as being real, then in a sense this is equivalent to ω being purely imaginary.

Having said all of that, you may now forget it!

1.4 Existence of the Fourier Transform

The main difference between the Laplace Transform and the Fourier Transform is that the range of functions for which the Fourier Transform integral gives us a finite value, i.e. a well-defined function, is smaller than for a Laplace Transform. We can find $\mathcal{L}[x]$ but not $\mathcal{F}[x]$. The former converges to $1/s^2$ (assuming that $s > 0$), but the latter does not converge. This is all due to the fact that e^{-sx} decays exponentially as x becomes large, while the real and imaginary parts of $e^{-j\omega x}$ oscillate between +1 and −1. Therefore there is a set of rules to determine in advance whether or not the Fourier Transform integral converges to something sensible. Here they are....

1. $f(x) \to 0$ as $x \to \pm\infty$
2. $f(x)$ must be finite everywhere.

Note the strange wording: these are sufficient conditions. This means that if both are satisfied then there is a Fourier Transform, but there remain some exotic functions which violate one or other of these conditions but yet still have a Fourier Transform.

The following Table gives some example functions.

$f(x)$	✓/✗	$F(\omega)$	Comment	
e^{-x^2}	✓	$\sqrt{\pi}e^{-\omega^2/4}$		
$e^{-\|x\|}$	✓	$2/(1+\omega^2)$		
$1/(1+x^2)$	✓	$\pi e^{-\|\omega\|}$		
unit pulse	✓			
e^{-x}	✗		violates (i)	$\longrightarrow \infty$ as $x \longrightarrow -\infty$
$1/x^2$	✗		violates (ii)	infinite at $x = 0$
1	✓	$2\pi\delta(\omega)$	violates (i)	
$\cos ax$	✓	$\pi[\delta(\omega-a)+\delta(\omega+a)]$	violates (i)	
$\|x\|^{-1/2}$	✓	$\sqrt{2\pi}\|\omega\|^{-1/2}$	violates (ii)	
$\cos(x^2)$	✓	$\sqrt{\pi}\cos(\frac{1}{4}\omega^2-\frac{1}{4}\pi)$	violates (i)	

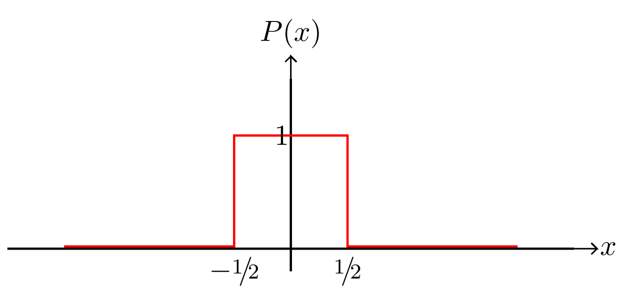
The first set of functions satisfy both conditions and therefore have Fourier Transforms. The transform of the unit pulse is not given because it depends on its duration and where it is centred. The second set, a pair, violates one condition each and don't have transforms. The third set is in that fuzzy region between the first two where a sufficient condition is violated but there is, nevertheless, a transform. It is interesting to note that, if those transforms were converted back to functions of x simply by replacing ω with x, then they too will violate one of the conditions.

In many ways this subsection is really just for information only. However, we will play a little with the unit impulse later.

2 Examples of Fourier Transforms

2.1 Example 1.

We shall find the Fourier Transform of the unit pulse of duration 1 which is centred at $x = 0$.



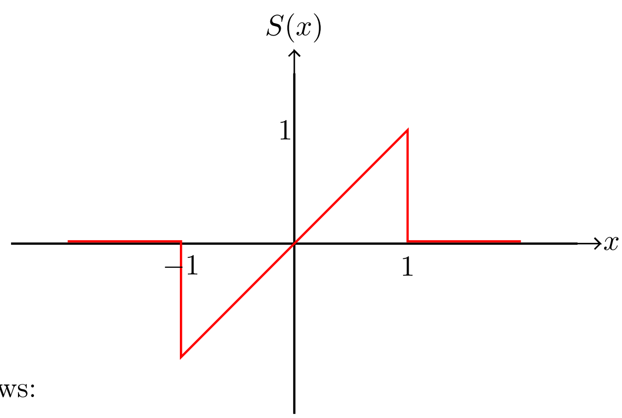
In the following derivation of the Fourier Transform of this unit pulse, use is made of the fact that it is an even function, and that integrals of odd functions (over symmetric intervals) are zero.

$$\begin{aligned}
\mathcal{F}\left[P(x)\right] &= \int_{-\infty}^{\infty} P(x)e^{-j\omega x}\,dx && \text{by definition}\\
&= \int_{-\infty}^{-1/2} 0\,e^{-j\omega x}\,dx + \int_{-1/2}^{1/2} 1\,e^{-j\omega x}\,dx + \int_{1/2}^{\infty} 0\,e^{-j\omega x}\,dx && \text{splitting into three regions}\\
&= \int_{-1/2}^{1/2} 1\Big(\cos\omega x - \cancel{j\sin\omega x}\Big)\,dx && \text{expanding the complex exponential}\\
&= \int_{-1/2}^{1/2} \cos\omega x\,dx = \frac{\sin(\omega/2)}{\omega/2}. && \text{since sines are odd}
\end{aligned}$$

We note that the transform of this function is real; this is because the given unit pulse is even, therefore the product of the pulse and $\sin\omega x$ is odd, and hence its integral is zero. This will always be true for even functions.

2.2 Example 2.

We shall now find the Fourier Transform of a single sawtooth shape, $S(x)$, as shown below.



The Transform follows:

$$\begin{aligned}
\mathcal{F}\left[S(x)\right] &= \int_{-\infty}^{\infty} S(x)e^{-j\omega x}\,dx && \text{by definition}\\
&= \int_{-\infty}^{-1} 0\,e^{-j\omega x}\,dx + \int_{-1}^{1} x\,e^{-j\omega x}\,dx + \int_{1}^{\infty} 0\,e^{-j\omega x}\,dx && \text{splitting into three regions}\\
&= \int_{-1}^{1} x\Big(\cos\omega x - j\sin\omega x\Big)\,dx && \text{expanding the complex exponential}\\
&= \int_{-1}^{1} \Big[\cancel{x\cos\omega x} - j\,x\sin\omega x\Big]\,dx && \text{Cancelled integrand is odd}\\
&= -j\int_{-1}^{1} x\sin\omega x\,dx \quad = \quad 2j\left(\frac{\omega\cos\omega - \sin\omega}{\omega^2}\right). && \text{using integration by parts}
\end{aligned}$$

In this case the sawtooth function is odd and therefore its product with $\cos\omega x$ is also odd. Hence the integral of that component is zero. The consequence is that the Fourier Transform is purely imaginary, a result that is true for all odd functions.

We have the general result:

> If $f(x)$ is even then $\mathcal{F}[f(x)]$ is real.
>
> If $f(x)$ is odd then $\mathcal{F}[f(x)]$ is imaginary.
>
> If $f(x)$ is neither even nor odd then the real part of $\mathcal{F}[f(x)]$ corresponds to the even component of $f(x)$ and the odd part corresponds to the odd component.

2.3 Example 3

Let us find the Fourier Transform of the unit impulse at $x = a$. We have,

$$\mathcal{F}[\delta(x-a)] \quad = \quad \int_{-\infty}^{\infty} \delta(x-a)e^{-j\omega x}\,dx \quad = \quad e^{-j\omega a}.$$

This integral has used the general result for integrals involving the unit impulse:

$$\int_{-\infty}^{\infty} \delta(x-a)g(x)\,dx = g(a),$$

i.e. that the integral is given by the rest of the integrand, $g(x)$, being evaluated where the impulse occurs.

We may play with this result a little. Given the above, then we have

$$\mathcal{F}[\delta(x+a)] \quad = \quad e^{j\omega a}.$$

This follows either by integrating this new transform from scratch, or else by replacing a by $-a$ in the previous result.

We can add these two results to get

$$\mathcal{F}[\delta(x+a)+\delta(x-a)] \quad = \quad e^{j\omega a}+e^{-j\omega a} \quad = \quad 2\cos\omega a, \tag{3}$$

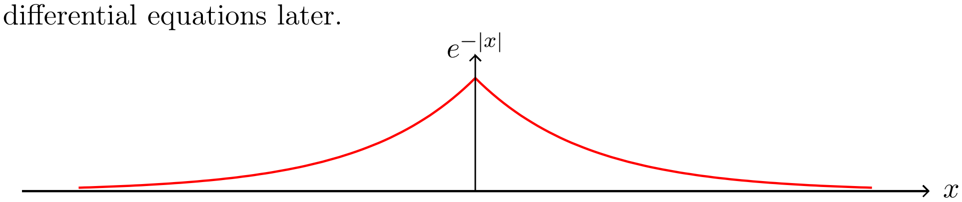
or we may subtract them to get,

$$\mathcal{F}[\delta(x+a)-\delta(x-a)] \quad = \quad e^{j\omega a}-e^{-j\omega a} \quad = \quad 2j\sin\omega a. \tag{4}$$

We will also use these results later.

2.4 Example 4

We will consider the Fourier Transform of $e^{-|x|}$. This seems like a bit of strange function to choose for it doesn't arise too often in engineering, but it too will prove useful when we solve some partial differential equations later.



First we notice that this function is even, and therefore we expect the Fourier Transform to be real. In what follows we will get to a point where I either have to use integration by parts or else to take the real part of a complex integral; I have decided to choose the latter route because that is my favoured way.

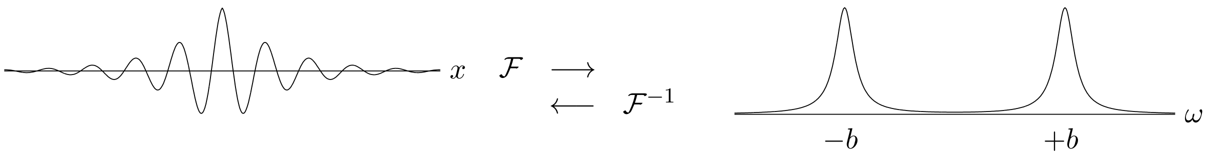
$$
\begin{aligned}
\mathcal{F}\left[e^{-|x|}\right] &= \int_{-\infty}^{\infty} e^{-|x|} e^{-j\omega x}\, dx && \text{by definition} \\
&= \int_{-\infty}^{\infty} \underbrace{e^{-|x|}}_{\text{even}} \big(\underbrace{\cos\omega x}_{\text{even}} - \underbrace{j\sin\omega x}_{\text{odd}} \big)\, dx \\
&= \int_{-\infty}^{\infty} e^{-|x|} \cos\omega x\, dx && \text{but this integrand is even, so...} \\
&= 2\int_{0}^{\infty} e^{-|x|} \cos\omega x\, dx && \text{note the lower limit change} \\
&= 2\int_{0}^{\infty} e^{-x} \cos\omega x\, dx && e^{-|x|} = e^{-x} \text{ when } x \geq 0 \\
&= 2\,\text{Real} \int_{0}^{\infty} e^{-x} e^{j\omega x}\, dx && \text{Added an imaginary term for convenience} \\
&= 2\,\text{Real} \int_{0}^{\infty} e^{-(1-j\omega)x}\, dx \\
&= 2\,\text{Real} \left[\frac{e^{-(1-j\omega)x}}{-(1-j\omega)} \right]_{0}^{\infty} \\
&= 2\,\text{Real} \left[\frac{1}{1-j\omega} \right] \\
&= 2\,\text{Real} \left[\frac{1+j\omega}{1+\omega^2} \right] \\
&= \frac{2}{1+\omega^2}.
\end{aligned}
$$

I have performed the above integration in quite pedantic detail. If you can speed that up then that would be excellent!

3 Physical meaning of the Fourier Transform

Having experienced four examples of Fourier Transforms, it is quite likely that the first thought is that these are merely integrals, and that the whole idea is that it is a mathematical trick. Well, it is a trick but it is also a meaningful trick. Here are some examples of Fourier Transform pairs. In all cases I have used even functions to transform because these have real transforms.

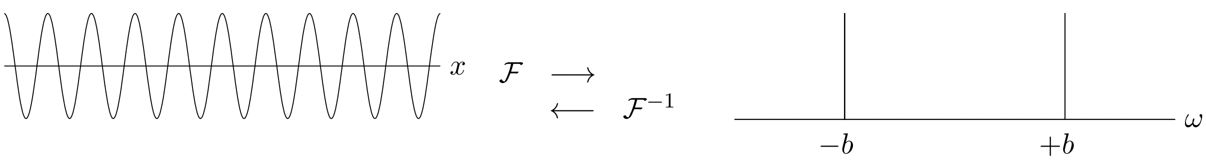
Example 5. $\mathcal{F}\left[e^{-a|x|}\cos bx\right] = \dfrac{a}{a^2+(b+\omega)^2} + \dfrac{a}{a^2+(b-\omega)^2}$



In this case the peaks of the Fourier Transform occur at $\omega = \pm b$. This means that the transform is telling us that the original signal, $e^{-a|x|}\cos bx$, comtains a strong component with frequency, b, as is quite obvious!

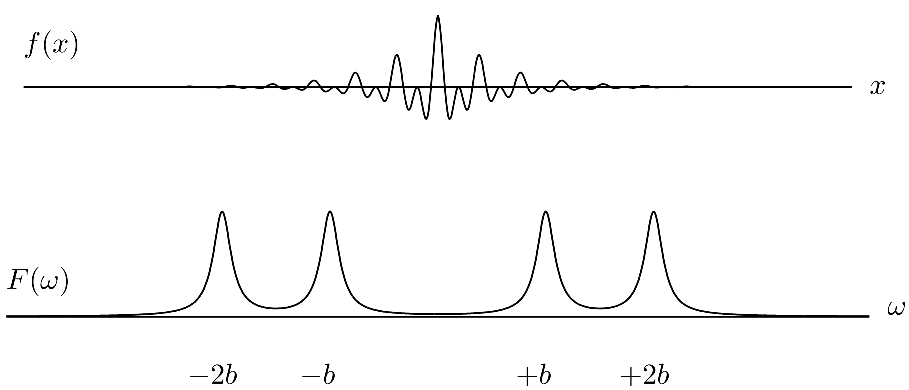
In the above, if we were to decrease the value of a, then the exponential decays more slowly than is depicted here. The consequence on the transform is that the two peaks will narrow and the maximum values ($\simeq 1/a$) will increase. In the limit as $a \to 0$ we obtain the following situation where the transform of $\cos bx$ is the sum of two delta functions.

Example 6. $\mathcal{F}\left[\cos bx\right] = \frac{1}{2}\Big[\delta(b+\omega)+\delta(b-\omega)\Big]$



Example 7.

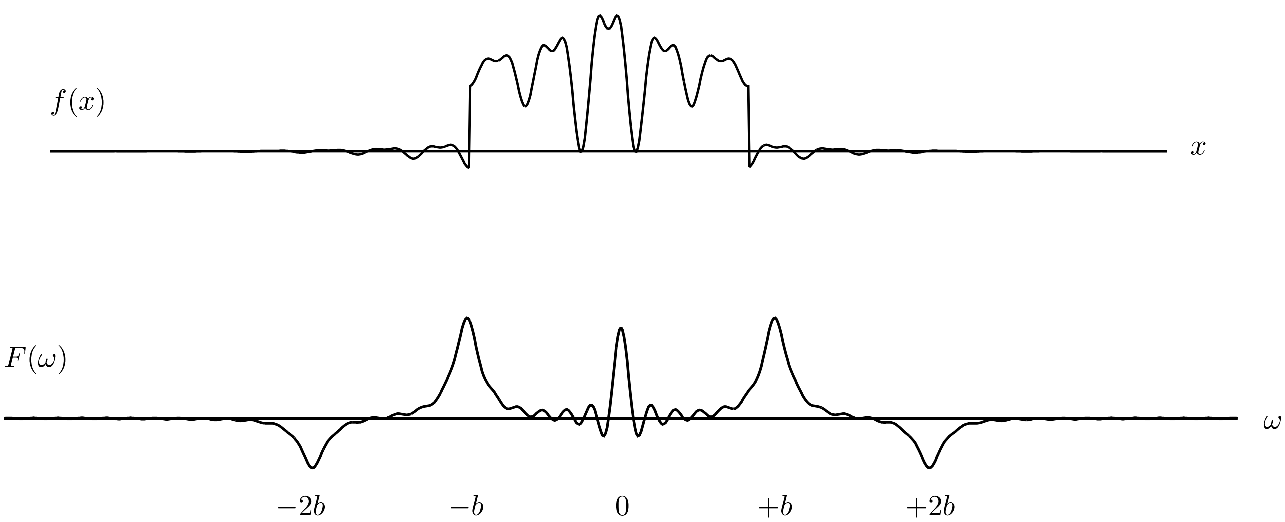
$$\mathcal{F}\left[e^{-a|x|}\cos bx + e^{-a|x|}\cos 2bx\right] = \frac{a}{a^2+(b+\omega)^2} + \frac{a}{a^2+(b-\omega)^2} + \frac{a}{a^2+(2b+\omega)^2} + \frac{a}{a^2+(2b-\omega)^2}$$



In this example we have found the transform of a sum of two functions and this is the same as the sum of the individual transforms. A nice result. Even nicer is the fact that the Fourier Transform clearly has well-defined peaks at $\omega = \pm b$ and $\pm 2b$. While these frequencies are seen clearly in the red and blue formulae making up $f(x)$, they are not so obvious in the graph of $f(x)$. It is this property that is used in CAT scans and other scientific applications.

Example 8. $\mathcal{F}\left[e^{-a|x|}\cos bx - 0.5e^{-a|x|}\cos 2bx + P(x)\right] =$

$$\frac{a}{a^2+(b+\omega)^2} + \frac{a}{a^2+(b-\omega)^2} + \frac{-0.5a}{a^2+(2b+\omega)^2} + \frac{-0.5a}{a^2+(2b-\omega)^2} + \frac{\sin(\omega/2)}{(\omega/2)}.$$



In the above the green colour represents both the unit pulse (see Example 3) and its Fourier Transform in their respective places.

The function, $f(x)$, now looks very strange but the Fourier Transform has uncovered its essential character. We can see the peaks at $\omega = \pm b$ and $\omega = \pm 2b$, although the amplitude of the latter is negative and has half the magnitude — this is consistent with the corresponding blue and red functions in $f(x)$. In addition, there is a peak at $\omega = 0$ and this corresponds to the unit pulse. Thus the transform provides us with a lot of information about the frequency content of the original signal.

4 Fourier Transforms of derivatives

The main aim of this part of the `ME20021` unit is the solution of PDEs using Fourier Transforms, and therefore we need to find out a few things about the transforms of derivatives.

4.1 The Fourier Transform of a single derivative

We shall find the Fourier Transform of $f'(x)$:

$$\begin{aligned}
\mathcal{F}\left[f'(x)\right] &= \int_{-\infty}^{\infty} f'(x)\, e^{-j\omega x}\, dx && \text{we'll integrate the } f' \\
&= \underbrace{\cancel{\Big[f\Big]\Big[e^{-j\omega x}\Big]_{-\infty}^{\infty}}}_{f \to 0 \text{ as } |x| \to \infty} - \int_{-\infty}^{\infty} \Big[f\Big]\Big[-j\omega e^{-j\omega x}\Big]\, dx && \text{by parts once} \\
&= j\omega \int_{-\infty}^{\infty} f\, e^{-j\omega x}\, dx \quad = \quad j\omega F(\omega).
\end{aligned}$$

The most important aspect of this analysis was the assumption that f tends to zero as x tends to $\pm\infty$. This was one of the sufficient conditions that were stated in §1.4.

Subject to having $f \to 0$ as $|x| \to \infty$, an x-derivative in the spatial domain is equivalent to multiplication by $j\omega$ in the frequency domain.

4.2 The Fourier Transform of a second derivative

We shall follow the same idea but will integrate by parts twice.

$$\begin{aligned}
\mathcal{F}\left[f''(x)\right] &= \int_{-\infty}^{\infty} f''(x)\, e^{-j\omega x}\, dx \\
&= \underbrace{\cancel{\left[f'\right]\left[e^{-j\omega x}\right]_{-\infty}^{\infty}}}_{f' \to 0 \text{ as } |x| \to \infty} - \underbrace{\cancel{\left[f\right]\left[-j\omega e^{-j\omega x}\right]_{-\infty}^{\infty}}}_{f \to 0 \text{ as } |x| \to \infty} + \int_{-\infty}^{\infty} \left[f\right]\left[j^2\omega^2 e^{-j\omega x}\right] dx \\
&= (j\omega)^2 \int_{-\infty}^{\infty} f\, e^{-j\omega x}\, dx \quad = \quad -\omega^2 F(\omega).
\end{aligned}$$

Thus we need both f and f' to tend to zero as $x \to \pm\infty$ for this result to be valid. The manner in which this analysis proceeds tells us that there is a simple rule for yet higher derivatives, namely that

$$\mathcal{F}\left[\frac{d^n f}{dx^n}\right] = (j\omega)^n F(\omega),$$

provided that f and its first $n-1$ derivatives tend to zero when x is large.

5 Useful Theorems

As with Laplace Transforms there is a small set of useful theorems that may occasionally be used to assist in solving PDEs. Thus we have two shift theorms, a symmetry theorem and the convolution theorem. I will cover these in turn below.

5.1 The Shift Theorem in x

We shall start with defining $\mathcal{F}[f(x)] = F(\omega)$. Now we shall shift the origin in x and attempt to find the transform of that function in terms of $F(\omega)$.

$$\begin{aligned}
\mathcal{F}[f(x-a)] &= \int_{-\infty}^{\infty} f(x-a)\, e^{-j\omega x}\, dx && \text{By definition} \\
& && x-a \text{ is awkward, so let } \xi = x-a \text{ hence } d\xi = dx \\
&= \int_{-\infty}^{\infty} f(\xi) e^{-j\omega(\xi+a)}\, d\xi && \text{Limits unchanged} \\
&= e^{-j\omega a} \underbrace{\int_{-\infty}^{\infty} f(\xi) e^{-j\omega\xi}\, d\xi}_{\mathcal{F}[f(x)]} \\
&= e^{-j\omega a} F(\omega).
\end{aligned}$$

An example of the use of this is the following. We know that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$, then the shift theorem tells us that the Fourier Transform of $e^{-|x-5|}$ is $2e^{-5j\omega}/(1+\omega^2)$.

5.2 The Shift Theorem in ω

Although the above shift theorem was quite quick to prove, this one is even faster!

$$\begin{aligned}
\mathcal{F}\left[f(x)\,e^{jax}\right] &= \int_{-\infty}^{\infty} f(x)\,e^{jax}e^{-j\omega x}\,dx && \text{By definition} \\
&= \int_{-\infty}^{\infty} f(x)e^{-j(\omega-a)x}\,dx \\
&= F(\omega-a). && \text{after noting that } \int_{-\infty}^{\infty} f(x)e^{-j\omega x}\,dx = F(\omega)
\end{aligned}$$

An example of the use of this theorem is the following. Given that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$ then $\mathcal{F}\left[e^{-|x|+6jx}\right] = 2/(1+(\omega-6)^2)$.

5.3 The Symmetry Theorem

The similarity between the above two shift theorems (namely that an origin shift in either x or ω is equivalent to multiplication by a complex exponential in either ω or x) is not an accident, but is based on the very great similarity between the definitions of the Fourier Transform and of its inverse given in Eqs. (1) and (2). This similarity motivates questions like the following:

Can we use the fact that $\mathcal{F}\left[e^{-|x|}\right] = 2/(1+\omega^2)$, which is a fairly straightforward integral to perform, to help us find $\mathcal{F}\left[2/(1+x^2)\right]$, which is considerably more difficult?

The answer is yes; here is the Symmetry Theorem:

$$\text{If} \quad \mathcal{F}\left[f(t)\right] = F(\omega) \quad \text{then} \quad \mathcal{F}\left[F(x)\right] = 2\pi f(-\omega). \tag{5}$$

The proof of this may be found in the standard online typeset notes, but this is not examinable and will not be included here.

5.3.1 Example 9.

We shall answer the question in red which motivated the theorem. If we let $f(x) = e^{-|x|}$, then $F(\omega) = 2/(1+\omega^2)$. Hence the theorm gives us

$$\mathcal{F}\Big[\underbrace{\frac{2}{1+x^2}}_{F(x)}\Big] \quad = \quad 2\pi\underbrace{e^{-|-\omega|}}_{f(-\omega)} = \quad = \quad 2\pi e^{-|\omega|}.$$

5.3.2 Example 10.

The aim is to find the Fourier Transform of 1, which is a case that is not guaranteed to have a transform. If we let $f(x) = \delta(x)$, then $F(\omega) = 1$ (see Example 3 with $a = 0$). Then the Symmetry Theorem states that,

$$\mathcal{F}\Big[\underbrace{\;1\;}_{F(x)}\Big] \quad = \quad 2\pi\underbrace{\delta(-\omega)}_{f(-\omega)} = \quad = \quad 2\pi\delta(\omega).$$

This result means that all of the frequency content of a constant signal is concentrated at $\omega = 0$.

5.3.3 Example 11.

We have already seen in Example 3 (specifically Eq. (3)) that,

$$\mathcal{F}\left[\delta(x+a)+\delta(x-a)\right] = 2\cos a\omega.$$

If we divide both sides by 2 then we have

$$\mathcal{F}\left[\frac{\delta(x+a)+\delta(x-a)}{2}\right] = \cos a\omega.$$

If we define

$$f(x) = \Big[\delta(x+a)+\delta(x-a)\Big]/2$$

and

$$F(\omega) = \cos a\omega,$$

then the Symmetry Theorem tells us that,

$$\mathcal{F}\,[\underbrace{\cos ax}_{F(x)}] = 2\pi \times \underbrace{\tfrac{1}{2}\Big[\delta(-\omega+a)+\delta(-\omega-a)\Big]}_{f(-\omega)} = \pi\Big[\delta(\omega-a)+\delta(\omega+a)\Big],$$

since $\delta(-b) = \delta(b)$.

5.4 The Convolution Theorem

This operates in the same sort way as for the Laplace Transform except that the convolution integral itself has different limits. For the Fourier Transform, the convolution of two functions, $f(x)$ and $g(x)$ is defined to be,

$$f * g \quad = \quad \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi \quad = \quad \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi.$$

The Laplace Trasform version has $\xi = 0$ and $\xi = x$ as its limits.

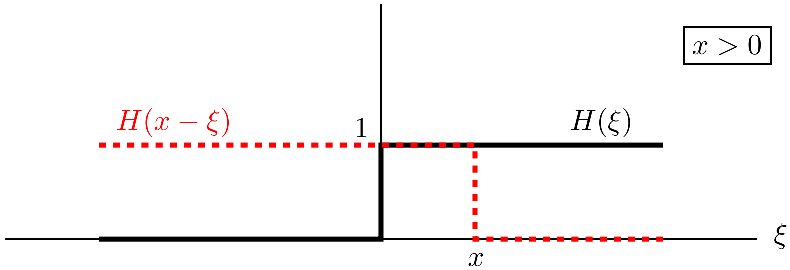
For the Fourier Transform, the Convolution Theorem is

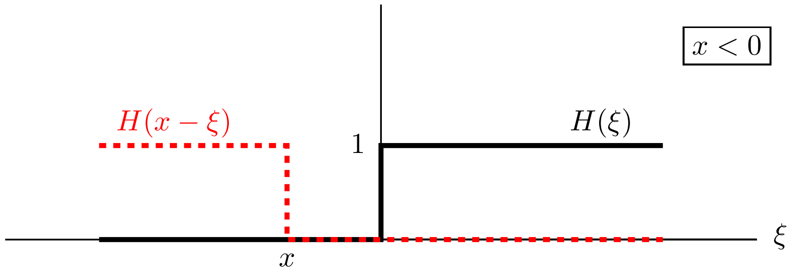
$$\mathcal{F}\,[f * g] = F(\omega)G(\omega).$$

So the transform of the convolution is equal to the product of the transforms. The proof of this is also contained in the standard online notes and this too is very definitely not needed for the exams.

5.4.1 A convolution of unit step functions

We shall find the convolution of the unit step function, $H(x)$, with itself: $H(x) * H(x)$. The chief difficulty with this example is with the manipulation of the limits of integration. It is best to sketch a couple of diagrams first.





The convolution is given by

$$\int_{-\infty}^{\infty} H(\xi)H(x-\xi)\,d\xi.$$

We may use the above diagrams to evaluate this integral. The integrand is given by the product of the values of the black line and those of the red line. For the first case, $x > 0$, the product of the step functions is equal to 1 in the range $0 < \xi < x$ and is zero otherwise. For the second case the product is equal to zero everywhere. Hence,

$$\int_{-\infty}^{\infty} H(\xi)H(x-\xi)\,d\xi = \left\{ \begin{array}{ll} \displaystyle\int_0^x 1\,d\xi = x & (x>0), \\ & \\ 0 & (x<0) \end{array} \right\} = xH(x).$$

6 An ODE example

As a final preparation for solving PDEs, we shall solve the following ODE,

$$\frac{dy}{dx} + y = \left\{ \begin{array}{ll} e^{-2x} & (x>0), \\ & \\ 0 & (x<0) \end{array} \right\} = e^{-2x}H(x).$$

We solved this in `ME10305` using the Particular Integal and Complementary Function approach, and also using Laplace Transforms. Admittedly, the above equation looks odd especially since I have given no initial condition. One could assume that $y = 0$ is an initial condition at a negatively infinite value of x. Alternatively one can see that the Complementary Function part of the solution is e^{-x}, which decays exponentially, and therefore any nonzero initial condition can be imposed at such a large but negative value of x that y will have decayed virtually to zero by the time $x = 0$ is reached which is when the right hand side suddenly becomes nonzero. However, my intention with this is that $y = 0$ is assumed when $x < 0$ and that the above problem is essentially the same as the problem, $y' + y = e^{-2x}$ subject to $y(0) = 0$.

We start by taking the Fourier Transform of the ODE. If we define $Y(\omega) = \mathcal{F}[y(x)]$ then §4.1 gives $\mathcal{F}[y'] = j\omega Y$. The transform of the right hand side of the ODE is

$$\begin{aligned}
\mathcal{F}\left[e^{-2x}\right] &= \int_{-\infty}^{\infty} e^{-2x} H(x) e^{-j\omega x}\,dx && \text{By definition}\\
&= \int_{0}^{\infty} e^{-2x} \times 1 \times e^{-\omega x}\,dx && H = 0 \text{ when } x < 0\\
&= \int_{0}^{\infty} e^{-(2+j\omega)x}\,dx\\
&= \frac{1}{2+j\omega}.
\end{aligned}$$

Now we can transform the given equation:

$$\begin{aligned}
(\omega j)Y + Y &= \frac{1}{2+\omega j}\\
\implies \quad (1+\omega j)Y &= \frac{1}{2+\omega j}\\
\implies \quad Y &= \frac{1}{(1+\omega j)(2+\omega j)}\\
\implies \quad Y &= \frac{1}{1+\omega j} - \frac{1}{2+\omega j} && \text{partial fractions}
\end{aligned}$$

Hence

$$y = \left(e^{-x} - e^{-2x}\right) H(x).$$

This completes the preparation and background on Fourier Transforms. The rest of the Fourier Transform lecture material consists of solving PDEs.

D.A.S.R. 24/3/2020