1.7 Solution of inhomogeneous, linear, constant-coefficient ODEs

The most general n^{th} order inhomogeneous or heterogeneous constant-coefficient ODE is

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = \boxed{F(t).}$$

Analytical progress is assured when the forcing function, $F(t)$, takes exponential, sinusoidal or polynomial form.

The good news: We've already completed half of this!

The bad news: We still have to complete the other half.

The solution consists of two parts: **Complementary Function** and the **Particular Integral**.

The Complementary Function (CF) is the full solution of the corresponding homogeneous equation. It is entirely independent of $F(t)$.

The Particular Integral (PI) is any solution of the full equation (but it is generally that part which is intimately associated with the presence of $F(t)$), and doesn't include any part of the Complementary Function.

Example 1.13: Solve the ODE, $\boxed{y' + y = e^{2t}.}$

For now, we shall solve this ODE by means of the method used for First Order Linear equations.

The integrating factor is $\boxed{e^{\int 1\,dt} = e^t.}$ Hence,

$$e^t(y' + y) = e^{3t} \quad \Longrightarrow \quad e^t y = \tfrac{1}{3}e^{3t} + c \quad \Longrightarrow \quad y = \underbrace{ce^{-t}}_{\text{CF}} + \underbrace{\tfrac{1}{3}e^{2t}}_{\text{PI}}.$$

Note: The part labelled, CF, is the general solution of $y' + y = 0$, while the part labelled, PI, arises because of the forcing term.

We will eventually see that the CF should always obtained first, while the PI is found afterwards.

Although this example is simple, there are other cases where the form for the PI will depend on the form of the CF.

Example 1.13 revisited: Solve the ODE, $\boxed{y' + y = e^{2t}.}$

We'll do this in the CF/PI way.

- Solve $y' + y = 0$.

  Let $y_{\text{cf}} = Ae^{\lambda t}$. Hence $\boxed{\lambda + 1 = 0} \Rightarrow \boxed{\lambda = -1} \Rightarrow \boxed{y_{\text{cf}} = Ae^{-t}}$.

- Solve the full equation, $y' + y = e^{2t}$.

  Let $y_{\text{pi}} = Be^{2t}$. Hence $\boxed{3Be^{2t} = e^{2t}} \Rightarrow \boxed{B = \frac{1}{3}} \Rightarrow \boxed{y_{\text{pi}} = \frac{1}{3}e^{2t}.}$

- Find the general solution by adding the CF and the PI.

  Hence $\boxed{y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + \frac{1}{3}e^{2t}.}$

- If an initial condition were given, then now is the time to apply it.

  An example: if we had $\boxed{y(0) = 1}$ then $\boxed{A = \frac{2}{3}}$ and hence $\boxed{y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.}$

Example 1.14: Solve the equation, $\boxed{y'' + 3y' + 2y = e^{at}.}$ The value of a is an unspecified constant.

The CF is found by solving $y''+3y'+2y = 0$. Hence λ satisfies $\lambda^2+3\lambda+2 = 0$, and so $\lambda = -1, -2$.

Hence the CF is,

$$\boxed{y_{\text{cf}} = Ae^{-t} + Be^{-2t}.}$$

The PI is found by solving the full ODE using $y_{\text{pi}} = Ce^{at}$. Hence,

$$Ce^{at}\Big[a^2 + 3a + 2\Big] = e^{at} \qquad \Longrightarrow \qquad C = \frac{1}{a^2 + 3a + 2} = \frac{1}{(a+1)(a+2)},$$

and hence,

$$\boxed{y_{\text{pi}} = \frac{e^{at}}{(a+1)(a+2)}}$$

The general solution is

$$\boxed{y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + Be^{-2t} + \frac{e^{at}}{(a+1)(a+2)}.}$$

$$y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + Be^{-2t} + \frac{e^{at}}{(a+1)(a+2)}.$$

Let us consider a few different values of a.

$$a = -3 \implies y'' + 3y' + 2y = e^{-3t} \implies y = Ae^{-t} + Be^{-2t} + \tfrac{1}{2}e^{-3t}$$

$$a = 10 \implies y'' + 3y' + 2y = e^{10t} \implies y = Ae^{-t} + Be^{-2t} + \tfrac{1}{132}e^{10t}$$

$$a = 0 \implies y'' + 3y' + 2y = 1 \implies y = Ae^{-t} + Be^{-2t} + \tfrac{1}{2}$$

$$a = -1.01 \implies y'' + 3y' + 2y = e^{-1.01t} \implies y = Ae^{-t} + Be^{-2t} - \tfrac{10000}{99}e^{-1.01t}$$

$$a = -2.01 \implies y'' + 3y' + 2y = e^{-2.01t} \implies y = Ae^{-t} + Be^{-2t} + \tfrac{10000}{101}e^{-2.01t}$$

These are fine, but what about when $a = -1$ or $a = -2$? The PI is infinite in these cases.

The two troublesome cases are precisely when a is one of the roots of the auxiliary equation for the CF.

Could it be due to a repetition of a λ-value, perhaps?

Example 1.15: Solve the equation, $\boxed{y' + 2y = e^{-2t}}$.

A naive application of the CF/PI method works fine for the CF. The auxiliary equation gives $\lambda = -2$ and hence $y_{\text{cf}} = Ae^{-2t}$.

However, if we use $y_{\text{pi}} = Be^{-2t}$ then we obtain $0 = e^{-2t}$, which clearly indicates that something has gone wrong! So what do we do to find the PI?

This ODE is a First Order Linear ODE, so we'll take that approach for now.

The Integrating Factor is $e^{\int 2\,dt} = e^{2t}$. The analysis follows:

$$e^{2t}(y' + 2y) = 1 \quad \Longrightarrow \quad e^{2t}y = t + A \quad \Longrightarrow \quad y = (\underbrace{A}_{\text{CF}} + \underbrace{t}_{\text{PI}})e^{-2t}.$$

So we should have used the substitution, $y_{\text{pi}} = Bte^{-2t}$, in order to account for a second appearance of $\lambda = -2$, even though it was via the forcing term.

Now let us rerun this Example using the CF/PI approach. So we are solving,

$$\underbrace{y' + 2y}_{\lambda = -2} = \underbrace{e^{-2t}}_{\lambda = -2},$$

We have $\lambda = -2$ for the CF and hence $y_{\text{cf}} = Ae^{-2t}$.

The forcing term is also equivalent to $\lambda = -2$ and therefore we let $y_{\text{pi}} = Bte^{-2t}$ and then find the value of B by substituting it into the full ODE:

$$\underbrace{Be^{-2t}(1 - 2t)}_{y_{\text{pi}}'} + \underbrace{2Bte^{-2t}}_{2y_{\text{pi}}} = e^{-2t}$$

$$\Longrightarrow \quad Be^{-2t} = e^{-2t} \qquad \text{all the terms involving } te^{-2t} \text{ cancel}$$

$$\Longrightarrow \quad B = 1.$$

Hence $y_{\text{pi}} = te^{-2t}$ and therefore we recover the solution given using the First Order Linear method.

Note: This is a different sort of λ-repetition situation from when we considered homogeneous equations, but the consequence is the same: includes extra powers of t for every repetition irrespective of its source.

Example 1.16: Solve $\boxed{y'' + 3y' + 2y = e^{-2t}.}$ This is the $a = -2$ instance of Ex. 1.14.

Guided by Example 1.15, we'll write out the ODE and classify it according its λ-values. We have,

$$\underbrace{y'' + 3y' + 2y}_{\lambda = -1, -2} = \underbrace{e^{-2t}}_{\lambda = -2},$$

The Complementary Function is $y_{\text{cf}} = Ae^{-t} + Be^{-2t}$, as found previously.

For the PI we note that forcing term corresponds to a second instance of $\lambda = -2$ and therefore we set $y_{\text{cf}} = Cte^{-2t}$. We get,

$$y'' + 3y' + 2y = e^{-2t}$$

$$\Longrightarrow \quad Ce^{-2t}\Big[\underbrace{-4 + \cancel{4t}}_{y''} + \underbrace{3(1 - \cancel{2t})}_{3y'} + \underbrace{\cancel{2t}}_{2y}\Big] = e^{-2t}$$

$$\Longrightarrow \quad -Ce^{-2t} = e^{-2t} \quad \Longrightarrow \quad C = -1.$$

Hence $y_{\text{pi}} = -te^{-2t}$. The general solution is,

$$\boxed{y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + Be^{-2t} - te^{-2t}.}$$

Example 1.17: Solve the ODE, $\boxed{y'' + 3y' + 2y = te^{-2t}.}$

The LHS of this ODE is the same as for Examples 1.14 and 1.16, and therefore the CF yields $\lambda = -1, -2$, as before.

But what should we make of the present forcing term?

We'll label the ODE as follows, $$\underbrace{y'' + 3y' + 2y}_{\lambda = -1, -2} = \underbrace{te^{-2t}}_{\lambda = -2, -2}.$$

As before the Complementary Function is $\boxed{y_{\text{cf}} = Ae^{-t} + Be^{-2t}.}$

Now te^{-2t} represents 2nd and 3rd instances of $\lambda = -2$, so let $\boxed{y_{\text{pi}} = (Ct + Dt^2)e^{-2t}}$.

Eventually we get to,

$$y'' + 3y' + 2y$$

$$= \left[\left(3C + 2D - 4C\right) + t\left(\cancel{2C} + 6D - \cancel{6C} - 4D + \cancel{4C} - 4D\right) + t^2\left(\cancel{2D} - \cancel{6D} + \cancel{4D}\right)\right] e^{-2t}$$

$$= (2D - C - 2Dt)e^{-2t} \quad = \quad te^{-2t}.$$

The Particular Integral is $\boxed{y_{\text{pi}} = (-t - \frac{1}{2}t^2)e^{-2t},}$ and therefore the general solution is,

$$\boxed{y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + \underbrace{Be^{-2t}}_{\text{1st}} + (\underbrace{-t}_{\text{2nd}} - \underbrace{\tfrac{1}{2}t^2}_{\text{3rd}})e^{-2t}.}$$

Example 1.18: Solve the ODE, $\boxed{y''' + 5y'' + 8y' + 4y = te^{-2t}.}$

This has been contrived to get a certain pattern of λ-values.

Labelling the ODE:

$$\underbrace{y''' + 5y'' + 8y' + 4y}_{\lambda = -1, -2, -2} = \underbrace{te^{-2t}.}_{\lambda = -2, -2}$$

Hence we use:

$$\boxed{y_{\text{cf}} = Ae^{-t} + (B + Ct)e^{-2t} \qquad \text{and} \qquad y_{\text{pi}} = (Dt^2 + Et^3)e^{-2t},}$$

where A, B and C are arbitrary, while D and E may be found by substitution into the full ODE.

Eventually we obtain,

$$\boxed{y = y_{\text{cf}} + y_{\text{pi}} = Ae^{-t} + (B + Ct)e^{-2t} + (-\tfrac{1}{2}t^2 - \tfrac{1}{6}t^3)e^{-2t}.}$$

The amount of algebra that is required to find the Particular Integral is horrendous, but there is an easier route for this ODE.

Given that there are so many instances of $\boldsymbol{\lambda = -2}$, we may factor an $\boldsymbol{e^{-2t}}$ out using $\boldsymbol{y = e^{-2t}z(t)}$:

So

$$\boldsymbol{y''' + 5y'' + 8y' + 4y = te^{-2t}} \qquad \text{becomes} \qquad \boldsymbol{z''' - z'' = t.}$$

Later we will find out how to deal with powers of $\boldsymbol{t}$ on the right hand side. This will be in Example 1.26, the very last one in the ODEs section.

Example 1.19: Find the solutions of $y''' + 3y'' + 3y' + y = t^5 e^{-t}$.

Yes, this is a very very seriously extreme example. Let us label it:

$$\underbrace{y''' + 3y'' + 3y' + y}_{\lambda = -1, -1, -1} \qquad = \qquad \underbrace{t^5 e^{-t}}_{\lambda = -1 \text{ six times.}}$$

Hence $y_{\text{cf}} = (A + Bt + Ct^2)e^{-t}$ where A, B and C are arbitrary.

The forcing term has six instances of $\lambda = -1$ and therefore we need to substitute

$$y_{\text{pi}} = (Dt^3 + Et^4 + Ft^5 + Gt^6 + Ht^7 + Jt^8)e^{-t}.$$

For this very extreme case we merely have repetitions of $\lambda = -1$ and therefore we may factor it out: let $y = z(t)e^{-t}$. Hence,

$$z''' = t^5$$

$$\Longrightarrow \qquad z = A + Bt + Ct^2 + \tfrac{1}{336}t^8 \qquad \text{using three integrations}$$

$$\Longrightarrow \qquad \boxed{y = (A + Bt + Ct^2 + \tfrac{1}{336}t^8)e^{-t}.}$$

Note: The choice of substitution for the PI depends on what the CF is. It is important to seek the pattern of $\boldsymbol{\lambda}$-values.

This is why the CF must be found first.

A checklist.

ODE	λ(CF)	λ(PI)	CF	PI
$y' - 3y = e^{2t}$	3	2	Ae^{3t}	Be^{2t}
$y' - 2y = e^{3t}$	2	3	Ae^{2t}	Be^{3t}
$y' - 2y = e^{2t}$	2	2	Ae^{2t}	Bte^{2t}
$y' - 2y = te^{2t}$	2	$2, 2$	Ae^{2t}	$(Bt + Ct^2)e^{2t}$
$y' - 2y = t^2e^{2t}$	2	$2, 2, 2$	Ae^{2t}	$(Bt + Ct^2 + Dt^3)e^{2t}$
$y'' - 4y' + 3y = e^{2t}$	$1,\ 3$	2	$Ae^{t} + Be^{3t}$	Ce^{2t}
$y'' - 3y' + 2y = e^{3t}$	$1,\ 2$	3	$Ae^{t} + Be^{2t}$	Ce^{3t}
$y'' - 3y' + 2y = e^{2t}$	$1,\ 2$	2	$Ae^{t} + Be^{2t}$	Cte^{2t}
$y'' - 3y' + 2y = te^{2t}$	$1,\ 2$	$2,\ 2$	$Ae^{t} + Be^{2t}$	$(Ct + Dt^2)e^{2t}$
$y'' - 3y' + 2y = t^2e^{2t}$	$1,\ 2$	$2,\ 2,\ 2$	$Ae^{t} + Be^{2t}$	$(Ct + Dt^2 + Et^3)e^{2t}$

ODE	λ(CF)	λ(PI)	CF	PI
$y'' - 4y' + 4y = e^{3t}$	$2, 2$	3	$(A + Bt)e^{2t}$	Ce^{3t}
$y'' - 4y' + 4y = e^{2t}$	$2, 2$	2	$(A + Bt)e^{2t}$	Ct^2e^{2t}
$y'' - 4y' + 4y = te^{2t}$	$2, 2$	$2, 2$	$(A + Bt)e^{2t}$	$(Ct^2 + Dt^3)e^{2t}$
$y'' - 4y' + 4y = t^2e^{2t}$	$2, 2$	$2,2,2$	$(A + Bt)e^{2t}$	$(Ct^2+Dt^3+Et^4)e^{2t}$
$y''' - 6y'' + 12y' - 8y = e^{3t}$	$2, 2, 2$	3	$(A + Bt + Ct^2)e^{2t}$	De^{3t}
$y''' - 6y'' + 12y' - 8y = e^{2t}$	$2, 2, 2$	2	$(A + Bt + Ct^2)e^{2t}$	Dt^3e^{2t}
$y''' - 6y'' + 12y' - 8y = te^{2t}$	$2, 2, 2$	$2, 2$	$(A + Bt + Ct^2)e^{2t}$	$(Dt^3 + Et^4)e^{2t}$
$y''' - 6y'' + 12y' - 8y = t^2e^{2t}$	$2, 2, 2$	$2,2,2$	$(A + Bt + Ct^2)e^{2t}$	$(Dt^3+Et^4+Ft^5)e^{2t}$
$y''' - 3y' - 2y = te^{2t}$	$-1, -1, 2$	$2, 2$	$(A + Bt)e^{-t} + Ce^{2t}$	$(Dt + Et^2)e^{2t}$

The final part of the ODEs section will discuss how to deal with polynomial and sinusoidal forcing terms.