

6 Vector methods

6.1 Introduction

In this section of the unit we will introduce what a vector is together with concepts such as the length of a vector, unit vectors and addition. Thereafter we meet both the scalar and vector products of two vectors, so-named because the result of the products are a scalar and a vector, respectively. Then the scalar triple product appears. Using these concepts the idea of orthogonality may be defined, and we then apply vector theory to physical problems like (i) what is the angle between two directions in three-dimensional space, (ii) how far is a given point from a given line, (iii) how close do two lines approach each other, and (iv) how far is a point from a plane? Some of these geometrical ideas would be considerably more difficult to work out without vectors. That lot would normally take up four lectures' worth of time.

6.2 Elementary definitions

There is a simple difference between scalars and vectors:

Scalar: This has a magnitude. Examples include my mass, and the fact that London is 100 miles away.

Vector: This has a magnitude and a direction. Examples include the size and direction of the force induced on the ground due to my weight, and the fact that London is 100 miles east of the University.

Other scalar quantities include temperature, pressure, voltage, how much cash I have in the bank and the number of children I have. Other vector quantities include position relative to a fixed point (a so-called position vector), velocities, accelerations and forces. It is worth noting that there is a scientific distinction between speed and velocity: the former is a scalar while the latter is a vector. The same is true formally, at least, between mass and weight (thought perhaps not in your local grocer shop).

6.3 Notation

In the following figure we introduce the notations which are used in the context of vectors.

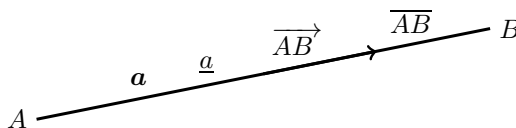


Figure 6.1. Showing a vector in the direction from point A to point B, together with some of the notations which are often used.

Some more detail on the notations:

- $\underline{a}$ is a very common notation where the underbar denotes the fact that it is a vector.
- $\mathbf{a}$ is what often appears in textbooks; it's a bold italic font. Strange, because it is quite difficult to write bold characters on a piece of paper or on a blackboard/whiteboard!
- $\overrightarrow{AB}$ or $\overline{AB}$ tells us that the vector corresponds to the position of B relative to A , or to the direction and distance that one has to travel to get from A to B .

The length of a vector also has a variety of notations: $|\underline{a}|$, $|\mathbf{a}|$, $|\overrightarrow{AB}|$ and AB . Given all of these, it is clear that $|\underline{a}| = |-\underline{a}|$ and $AB = BA$ and so on.

6.4 Addition of two vectors

This is most easily seen in the following figure:

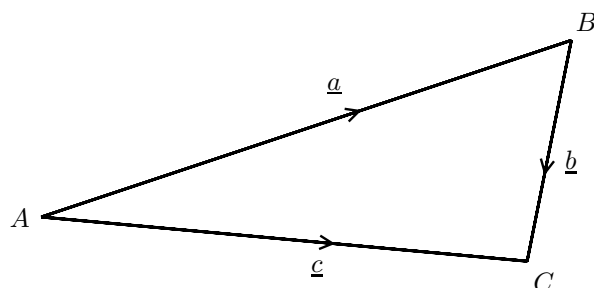


Figure 6.2. Showing vector addition diagrammatically. Here $\underline{a} + \underline{b} = \underline{c}$.

We have

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \underline{a} + \underline{b} = \underline{c}. \quad (1)$$

If we now create a point D which has a position vector $\underline{b}$ relative to A , as in the following Figure, then what is the vector denoted by $\underline{v}$?

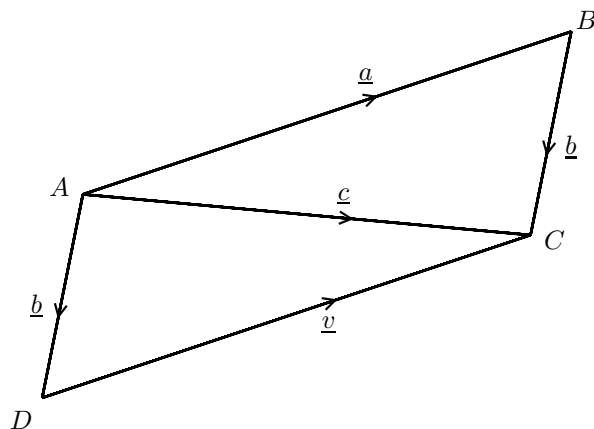


Figure 6.3. Showing that $\underline{v} = \underline{a}$.

From this Figure we see that,

$$\underline{v} = \overrightarrow{DA} + \overrightarrow{AC} = -\underline{b} + \underline{c} = -\underline{b} + \underline{a} + \underline{b} = \underline{a}. \quad (2)$$

6.5 Cartesian Coordinates and Unit Vectors

While it is possible to derive many results involving vectors without having a coordinate system in mind, it is sometimes helpful think of vectors in terms of their Cartesian components. The following Figure shows how this occurs.

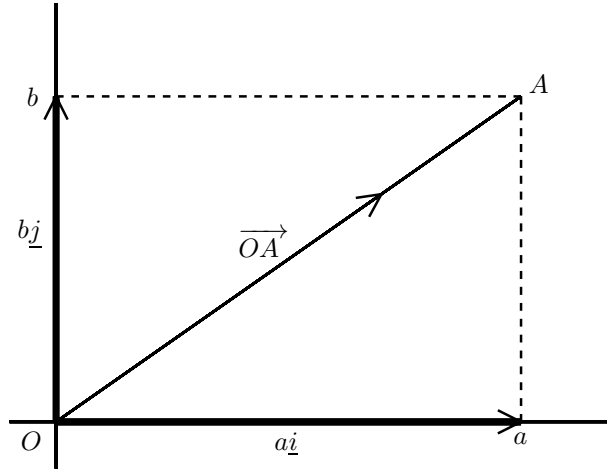


Figure 6.4. Showing how a 2D vector may be decomposed into its components in the x and y -directions, and how these components are proportional to the respective unit vectors, $\underline{i}$ and $\underline{j}$.

From this Figure we see that the x -component of the vector $\overrightarrow{OA}$ is a , while the y -component is b . As vectors which are equivalent to travelling unit distances in the x and y directions may be written in the form,

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3)$$

respectively, we may write $\overrightarrow{OA}$ in the equivalent forms,

$$\overrightarrow{OA} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\underline{i} + b\underline{j}. \quad (4)$$

Here, it is to be noted that this notation, $\begin{pmatrix} a \\ b \end{pmatrix}$, is distressingly similar that for the Binomial coefficients, but this present context is very different and therefore confusion should not arise.

Given this representation we have

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad (5)$$

and therefore vector addition is **commutative** in Cartesian coordinates, i.e. the same answer is obtained independently of the order in which two vectors are added, just as with $2 + 3 = 3 + 2 = 5$.

From this diagram we see that the magnitude of the vector $\overrightarrow{OA}$ may be obtained using Pythagoras's theorem:

$$|\overrightarrow{OA}| = \sqrt{a^2 + b^2}. \quad (6)$$

In three dimensions we have the analogous result:

$$|\overrightarrow{OA}| = \sqrt{a^2 + b^2 + c^2}, \quad (7)$$

where

$$\overrightarrow{OA} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \quad (8)$$

as shown in the following Figure.

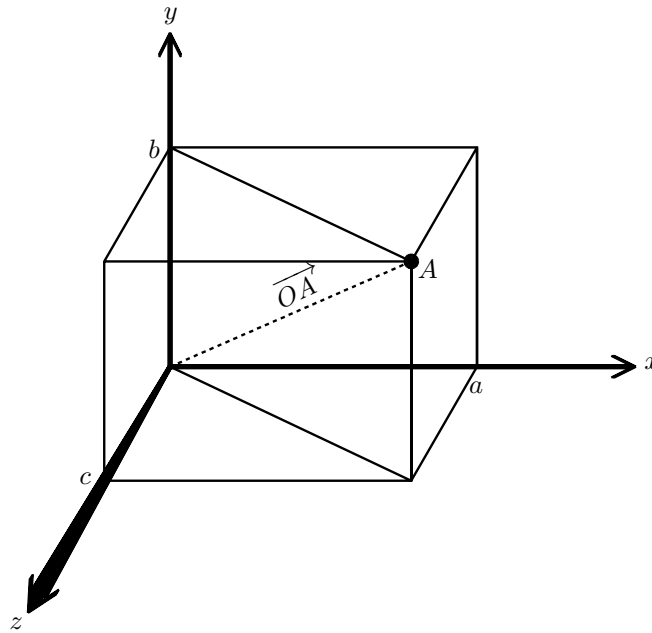


Figure 6.5 Showing how a 3D vector, $\overrightarrow{OA}$, may be understood in terms of its components in the x , y and z -directions.

6.6 The scalar product of two vectors

There are two ways which have been defined for the multiplication of two vectors, namely the scalar (or dot) product, and the vector (or cross) product.

The principal formula for the scalar product is

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta, \quad (9)$$

where θ is the angle between the directions of the two vectors, as illustrated in Figure 6.6.

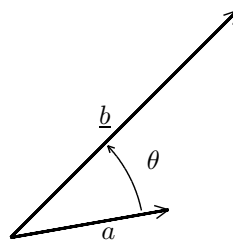


Figure 6.6 Showing two different vectors, and the angle, θ between their directions.

6.6.1 The angle between two vectors

It is possible to use the formula in Eq. (9) to calculate the scalar product without having to evaluate θ , itself. Then, we may use Eq. (9) directly so that $\cos \theta$, and hence θ itself, may be found.

Restricting ourselves to 2D for now, we need to note first that

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = 1 \quad \text{and} \quad \underline{i} \cdot \underline{j} = 0, \quad (10)$$

since the angles between the various vectors are either zero or $\frac{1}{2}\pi$. This means that any two perpendicular vectors have a zero scalar product. Now if we define

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 \underline{i} + a_2 \underline{j} \quad \text{and} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 \underline{i} + b_2 \underline{j}, \quad (11)$$

then

$$\begin{aligned} \underline{a} \cdot \underline{b} &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &= (a_1 \underline{i} + a_2 \underline{j}) \cdot (b_1 \underline{i} + b_2 \underline{j}) \\ &= a_1 b_1 \underline{i} \cdot \underline{i} + a_2 b_2 \underline{j} \cdot \underline{j} + (a_1 b_2 + a_2 b_1) \underline{i} \cdot \underline{j} \\ &= a_1 b_1 + a_2 b_2. \end{aligned} \quad (12)$$

Thus the scalar product may be found simply by multiplying corresponding components of the two vectors and adding the resulting products. Given that we now have two different formulae for the scalar product, they may be equated:

$$a_1 b_1 + a_2 b_2 = |\underline{a}| |\underline{b}| \cos \theta, \quad (13)$$

and therefore θ may be found.

A similar result applies in three dimensions, for if

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k} \quad \text{and} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}, \quad (14)$$

then

$$\begin{aligned} \underline{a} \cdot \underline{b} &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \cdot (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}) \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned} \quad (15)$$

So the angle between general 3D vectors is given by,

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = |\underline{a}| |\underline{b}| \cos \theta. \quad (16)$$

Example 6.1. What is the angle between the vectors $\underline{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$?

Using the formula Eq. (13) the left hand side gives,

$$\underline{a} \cdot \underline{b} = (2 \times 1) + (1 \times 2) = 4.$$

while the right hand side give,

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta = \sqrt{5} \sqrt{5} \cos \theta = 5 \cos \theta.$$

As these quantities are equal, then

$$5 \cos \theta = 4 \quad \Rightarrow \quad \cos \theta = \frac{4}{5} \quad \Rightarrow \quad \theta = 0.643501 \text{ (36.8699}^\circ\text{)}.$$

Example 6.2. What is the angle between the vectors $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$?

These two look as though they are quite similar in direction so my initial guess is that the angle is fairly small. The left hand side of Eq. (16) gives,

$$\underline{a} \cdot \underline{b} = (1 \times 4) + (2 \times 5) + (3 \times 6) = 32.$$

while the right hand side is

$$\underline{a} \cdot \underline{b} = \sqrt{1^2 + 2^2 + 3^2} \sqrt{4^2 + 5^2 + 6^2} \cos \theta = 32.8329 \cos \theta.$$

Therefore

$$\cos \theta = \frac{32}{32.8329} = 0.97463 \quad \Rightarrow \quad \theta = 0.22573 \text{ (12.933}^\circ\text{)}.$$

Yes, quite a small angle. Smug mode....

Example 6.3. Two vectors are called **orthogonal** if the angle between them is 90° . If $\underline{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ \alpha \end{pmatrix}$ are orthogonal, what is the value of α ?

We are told that the vectors are orthogonal, which means that $\underline{a} \cdot \underline{b} = 0$ because $\cos(\pi/2) = 0$. We find that,

$$\underline{a} \cdot \underline{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ \alpha \end{pmatrix} = (4 \times 2) + (1 \times \alpha) = 8 + \alpha \quad \Rightarrow \quad \alpha = -8.$$

6.7 The unit vector

If we have a vector such as $\underline{a} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ then its corresponding unit vector $\hat{\underline{a}}$ has a unit length but is pointed in the same direction as $\underline{a}$. Here $|\underline{a}| = \sqrt{2^2 + 3^2 + 6^2} = 7$ (woo, a Pythagorean quadruple...) and therefore,

$$\hat{\underline{a}} = \frac{1}{7} \underline{a} = \begin{pmatrix} 2/7 \\ 3/7 \\ 6/7 \end{pmatrix}. \quad (17)$$

More generally we can say that $\underline{b} = |\underline{b}| \hat{\underline{b}}$, where $|\underline{b}|$ takes care of the magnitude, and $\hat{\underline{b}}$, the direction of $\underline{b}$.

6.8 The vector product of two vectors

For the *scalar* product of two vectors we had

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta, \quad (18)$$

a scalar. The standard formula for the *vector* product is

$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \underline{\hat{p}}, \quad (19)$$

where $\underline{\hat{p}}$ is the unit vector which is perpendicular to both $\underline{a}$ and $\underline{b}$, and which points in the direction given by the right hand rule. There are various ways of illustrating this; one of these is to imagine the outward motion of a screw if the slit in the head is rotated from the direction given by $\underline{a}$ to that given by $\underline{b}$. Personally, I find that difficult to visualise easily, so a good alternative is that given by considering the thumb and fingers of a right-handed hitchhiker in action. Here the rotational direction given by the curved fingers correspond to the rotation required to move from the direction of $\underline{a}$ to the direction of $\underline{b}$, and then the direction in which the thumb is pointing gives the direction of $\underline{\hat{p}}$.

A third alternative is given by setting one's thumb and first two fingers of the right hand at right angles to one another, taking $\underline{a}$ to be in the direction of the thumb, $\underline{b}$ in the direction of the first finger, and $\underline{\hat{p}}$ in the direction of the second finger. I find that one the easiest to remember.

My hope is that this may be seen in the following Figure!

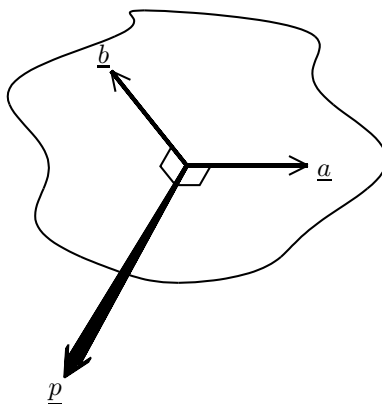


Figure 6.7 Showing the direction, $\underline{p}$, taken by the cross product, $\underline{a} \times \underline{b}$, as required by the right hand rule.

Here $\underline{a}$ and $\underline{b}$ lie in the plane defined by the wiggly closed curve, while $\underline{p}$ is perpendicular to the page/screen, depending on the age of the technology you're using to view this.

If we take $\underline{i} \times \underline{j} = \underline{k}$ as the definition of a right handed system of orthogonal unit vectors, $\underline{i}$, $\underline{j}$ and $\underline{k}$, then the application of the above right hand rule to the various unit vector combinations yields the results,

$$\begin{aligned} \underline{i} \times \underline{j} &= \underline{k}, & \underline{j} \times \underline{i} &= -\underline{k}, \\ \underline{i} \times \underline{k} &= -\underline{j}, & \underline{k} \times \underline{i} &= \underline{j}, \\ \underline{j} \times \underline{k} &= \underline{i}, & \underline{k} \times \underline{j} &= -\underline{i}. \end{aligned} \quad (20)$$

Notice here that the vector product is not commutative since, for example,

$$\underline{i} \times \underline{j} \neq \underline{j} \times \underline{i}.$$

This may be verified using one of three illustrations above for what constitutes a right-handed system. In fact, the following is true in general:

$$\underline{a} \times \underline{b} = -\underline{b} \times \underline{a}. \quad (21)$$

6.8.1 Lexicographical ordering

Note that it is easy to remember cross products of the unit vectors if we recall the order in which they are usually stated: $\underline{i}, \underline{j}, \underline{k}, \underline{i}, \underline{j}$ etc.. If we evaluate the cross product of any two neighbouring vectors in the order in which they appear in this sequence, then the result is *plus* the next vector. So we have

$$\underline{i} \times \underline{j} = \underline{k} \quad \underline{j} \times \underline{k} = \underline{i} \quad \underline{k} \times \underline{i} = \underline{j}, \quad (22)$$

and it is this is called **Lexicographical ordering**. If the two vectors being multiplied are in sequence from right to left, then the product is *minus* the next vector in the sequence. Hence

$$\underline{j} \times \underline{i} = -\underline{k} \quad \underline{k} \times \underline{j} = -\underline{i} \quad \underline{i} \times \underline{k} = -\underline{j}. \quad (23)$$

We also note that the cross product of a vector with itself is zero, i.e. that

$$\underline{a} \times \underline{a} = 0, \quad (24)$$

since $\sin \theta = 0$.

6.8.2 The determination of vector products

We have

$$\begin{aligned} \underline{a} \times \underline{b} &= (a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}) \times (b_1 \underline{i} + b_2 \underline{j} + b_3 \underline{k}) \\ &= (a_2 b_3 - a_3 b_2) \underline{i} + (a_3 b_1 - a_1 b_3) \underline{j} + (a_1 b_2 - a_2 b_1) \underline{k} \quad \text{using (22) and (23)} \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \end{aligned} \quad (25)$$

This construction is a determinant. It is likely that a substantial proportion of the class won't have come across this before — it is part of next's semester content on matrices — but for now I offer the following way to evaluate it.

The secret to remembering the formulae is to note how the coefficients of the unit vectors in the penultimate line of Eq. (25) are obtained from the determinant notation on the last line. For the $\underline{i}$ coefficient we have,

$$\left(a_2b_3 - a_3b_2\right)\underline{i} + \left(a_3b_1 - a_1b_3\right)\underline{j} + \left(a_1b_2 - a_2b_1\right)\underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \quad (26)$$

Note how the terms typeset in red are multiplied out. This is called cross-multiplication. For the $\underline{j}$ coefficient we have,

$$\left(a_2b_3 - a_3b_2\right)\underline{i} + \left(a_3b_1 - a_1b_3\right)\underline{j} + \left(a_1b_2 - a_2b_1\right)\underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} \quad (27)$$

where I have written ‘ghostly’ terms to the right of the determinant purely in order to preserve the multiplying-out pattern used in Eq. (26). And finally, for the $\underline{k}$ coefficient, we have,

$$\left(a_2b_3 - a_3b_2\right)\underline{i} + \left(a_3b_1 - a_1b_3\right)\underline{j} + \left(a_1b_2 - a_2b_1\right)\underline{k} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} a_1 & a_2 \\ b_1 & b_2 \end{matrix} \quad (28)$$

I hope that this is clear. There are other ways of doing this, but I prefer this as a particularly safe way to proceed.

Example 6.4. If $\underline{a} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, then find a unit vector which is perpendicular to them both.

This is easily done by applying the determinantal formula for the vector product. We have

$$\underline{p} = \underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}.$$

We’ll test out the answer by taking the dot products of both $\underline{a}$ and $\underline{b}$ with $\underline{p}$:

$$\underline{a} \cdot \underline{p} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = 6 - 5 - 1 = 0.$$

$$\underline{b} \cdot \underline{p} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix} = 3 + 0 - 3 = 0.$$

This confirms that $\underline{p}$ is perpendicular to both $\underline{a}$ and $\underline{b}$.

The length of the vector is $\sqrt{3^2 + 5^2 + 1^2} = \sqrt{35}$. Hence,

$$\hat{\underline{p}} = \frac{1}{\sqrt{35}} \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}.$$

Given the phrasing of the Exercise (Find a unit vector), then one could legitimately also write down,

$$\hat{\underline{p}} = \frac{1}{\sqrt{35}} \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}.$$

Example 6.5. Prove that the vectors $\underline{a} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$ all lie in the same plane.

Note that this question doesn't ask about three points lying in a plane because three points always lie in a plane, hence the widespread use of tripods! When three vectors lie in a plane it is assumed that they all come from a common origin, and therefore we are really considering four points. It is illustrated below.



Figure 6.8. Showing three vectors lying in a plane.

In general four points don't lie in a plane; this could be illustrated by raising, say, the end of the red pen above the piece of paper the pens are on. In such a case, three points lie on a plane and the fourth doesn't.

So the question is: is there a general condition that will be satisfied when three vectors lie in a plane?

If $\underline{a}$, $\underline{b}$ and $\underline{c}$ all lie in the same plane, then one approach could be to find values of α and β which are such that

$$\underline{c} = \alpha \underline{a} + \beta \underline{b}. \quad (29)$$

This approach corresponds to using $\underline{a}$ and $\underline{b}$ like Cartesian coordinates, albeit ones that are not necessarily at right angles. If one can find these values, then the three vectors lie in the same plane, but if there is no solution (always likely because this represents three equations for two unknowns), then they don't. However, this approach can be fairly time consuming. A better approach is to realise that, if the three vectors lie in the same plane, then $\underline{a} \times \underline{b}$ must be perpendicular to $\underline{c}$. In mathematical terms this **co-planarity condition** is

$$\underline{a} \times \underline{b} \cdot \underline{c} = 0. \quad (30)$$

This is called a **scalar triple product**. For the specific vectors in this exercise, we can show that,

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 2 & 1 & -4 \end{vmatrix} = \begin{pmatrix} 7 \\ 14 \\ 7 \end{pmatrix},$$

and therefore

$$\underline{a} \times \underline{b} \cdot \underline{c} = \begin{pmatrix} 7 \\ 14 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = 7 - 42 + 35 = 0.$$

So these three vectors lie in the same plane.

Note: The condition $\underline{a} \times \underline{b} \cdot \underline{c} = 0$ is the condition for coplanarity. When this is satisfied then the vectors are also called **linearly dependent**, and in such a case it is possible to find the values of α and β in Eq. (29). By

linearly dependent we mean that any one of the three vectors may be written in terms of the other two. If the three vectors do not satisfy co-planarity condition then they are **linearly independent** and may then be used as a 3D coordinate system — this means that any point in space may be reached using a unique combination of these vectors. Of course the unit vectors, $\underline{i}$, $\underline{j}$ and $\underline{k}$, are linearly independent vectors.

Example 6.6. Determine if four points are co-planar.

This is a variation on the previous example. These are points, not vectors. If the four points are labelled, A , B , C and D , then we could choose A , say, as a reference point and create three vectors:

$$\overrightarrow{AB}, \overrightarrow{AC} \text{ and } \overrightarrow{AD}.$$

Then we may apply the analysis of Example 6.5 to these.

6.9 Area of a triangle

This is the first section of what might be called applications of vectors.

The finding of the area of a triangle is also a rather unusual application. Here is the diagram.

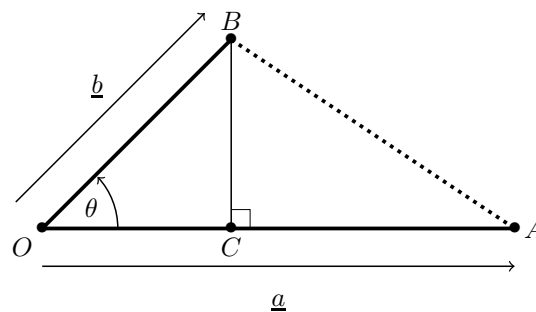


Figure 6.9. Showing how to find the area of the triangle OAB using vector methods.

The aim is find the area of the triangle, OAB , and all the quantities that we need are shown in Fig. 6.9. For simplicity we shall assume that the triangle lies in the (x, y) -plane. We start with the standard verbal description of the area of a triangle and gradually coerce that into an overtly mathematical form.

$$\begin{aligned}
 \text{Area} &= \frac{1}{2} \text{ base} \times \text{the perpendicular height} \\
 &= \frac{1}{2} OA \times CB && \text{scalar multiplication} \\
 &= \frac{1}{2} |\underline{a}| \times |\underline{b}| \sin \theta && \text{resolving the angle} \\
 &= \frac{1}{2} (\underline{a} \times \underline{b}) \cdot \underline{k} \quad \text{or} \quad \frac{1}{2} |\underline{a} \times \underline{b}|. && \text{vector multiplication}
 \end{aligned} \tag{31}$$

I have provided two possible solutions. The first, in red, takes account of the fact that $\underline{a} \times \underline{b}$ is in the $\underline{k}$ direction. The dot product with $\underline{k}$ merely reveals the coefficient of $\underline{k}$. Given the way that Fig. 6.9 has been drawn with the relative placement of $\underline{a}$ and $\underline{b}$ as shown, then $\sin \theta > 0$ and this answer will be positive. There could well be scenarios when a negative value might be a legitimate answer. The second answer, in purple, merely gives the absolute value of the cross product and hence it is always positive.

6.10 The distance of points from lines

We'll off with a general line. The theory isn't too tricky, he says somewhat optimistically, and relies on translating the words into a diagram. After that it is a case of using what we know that vectors can do. Let's see how we do....

Example 6.7. A straight line passes through a point with position vector, $\underline{a}$, and in the direction, $\underline{b}$. How close does the line get to the point? Here's the diagram which will help us.

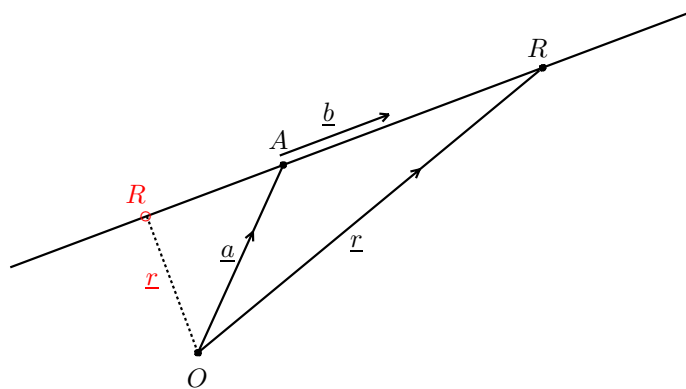


Figure 6.10. Showing the line which passes through the point with position vector, $\underline{a}$, and in the direction, $\underline{b}$. The general point on the line is depicted as $\underline{r}$. The dotted line shows where the line is closest to the origin.

Starting with the origin, O , the vector, $\underline{a}$, has been drawn with the end of the vector defining the point, A . The line from A to what I have labelled as R is in the direction of $\underline{b}$. Therefore, if we wish to travel from O to the arbitrary point, R , on this line, then we start from O , travel to A and then on to R . The vector taking us from A to R is an arbitrary multiple of $\underline{b}$. Mathematically that gives us,

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR} \quad \text{or} \quad \underline{r} = \underline{a} + \lambda \underline{b}, \quad (32)$$

where λ is the parameter which tells us how far along the line we are from A .

The red labels on Figure 6.10 represent the closest approach of the line and the origin. The essential criterion is that $\overrightarrow{OR}$ and $\underline{b}$ must be perpendicular. As perpendicular vectors have a zero dot product, this means that,

$$\underline{r} \cdot \underline{b} = 0. \quad (33)$$

Given the expression for $\underline{r}$ in Eq. (32), Eq. (33) gives,

$$(\underline{a} + \lambda \underline{b}) \cdot \underline{b} = 0 \quad \Rightarrow \quad \underline{a} \cdot \underline{b} + \lambda \underline{b} \cdot \underline{b} = 0. \quad (34)$$

We may now complete this general case by writing, $\lambda = -(\underline{a} \cdot \underline{b})/(\underline{b} \cdot \underline{b})$, then finding a final expression for r using this value for λ , and ultimately declaring that the required distance we seek is $|\underline{r}|$. However, when one has a numerical exercise, the dot products have already been done and the final answer obtained more easily. So we had better do one.

Example 6.8. A straight line passes through a point with position vector, $(2, 1, 1)^T$, and in the direction, $(1, 0, 1)^T$. How close does this get to the origin?

Ah, I have used the *Transpose* notation here. Formally, in matrix theory (that'll happen next semester), the rows and columns are swapped around when the transpose is taken. So here a normal 3-vector, which is written as a column of numbers, is transposed into a form where the three numbers aligned horizontally as a row. There's nothing special about this apart from the fact that the typesetting is much easier with evenly-spaced lines. So

please don't get freaked out by this — I am simply trying to make the typesetting look nicer, and the presence of the T shows the rest of the world that I am not being careless but I truly know what I doing! I generally do use the transpose version when within a body of text, but not in the displayed equations.

OK, we'll power through this. I would normally draw roughly the same diagram as above but would populate it with the numerical data. Hence

$$\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

At nearest approach the line joining the origin to the given line is perpendicular to the direction of the line. In other words, we have

$$\begin{aligned} \underline{r} \cdot \underline{b} &= 0 \\ \Rightarrow \left[\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} &= 0 \\ \Rightarrow 3 + 2\lambda &= 0 \\ \Rightarrow \lambda &= -\frac{3}{2}. \end{aligned} \tag{35}$$

Hence $\underline{r} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix}$, and therefore the distance of the point from the line is $|\underline{r}| = \sqrt{\frac{3}{2}}$.

Example 6.9. We have $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \underline{c}$. A straight line passes through points A and B . How close does the line get to C ? Here,

$$\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Here is the diagram:

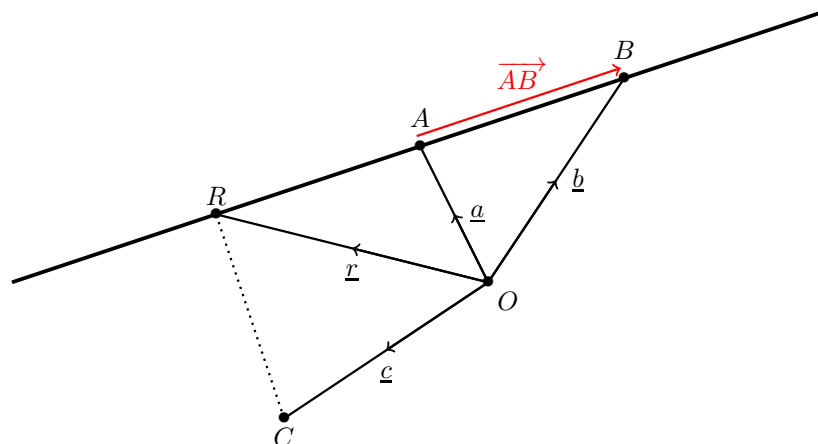


Figure 6.10. The shortest distance from C to a line which passes through both A and B .

This example is a bit of a variation on the previous example. This time we don't have the direction of the line but we'll have to work it out. And we are looking of the closest approach to C rather than to the origin, O .

First we need to write down a vector which takes us from O to a general point on the line. This will be,

$$\overrightarrow{OR} = \overrightarrow{OA} + \lambda \overrightarrow{AB} \quad \text{or} \quad \underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) \quad \text{or} \quad \underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix}.$$

We now need $\overrightarrow{CR}$ to be perpendicular to $\overrightarrow{AB}$, and so

$$(\underline{r} - \underline{c}) \cdot (\underline{b} - \underline{a}) = 0. \quad (36)$$

Hence,

$$\begin{aligned} 0 &= \left[\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \\ &= -9 + 18\lambda \\ \Rightarrow \lambda &= \frac{1}{2}. \end{aligned}$$

Hence,

$$\underline{r} - \underline{c} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad \Rightarrow \quad |\underline{r} - \underline{c}| = \sqrt{1/2}$$

is the distance of closest approach.

Example 6.10. One line passes through a point with position vector $\underline{a}_1$ in the direction of $\underline{b}_1$, while a second passes through a point with position vector $\underline{a}_2$ in the direction of $\underline{b}_2$. How close do they pass? We have

$$\underline{a}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{b}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \underline{b}_2 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

A general diagram is given by,

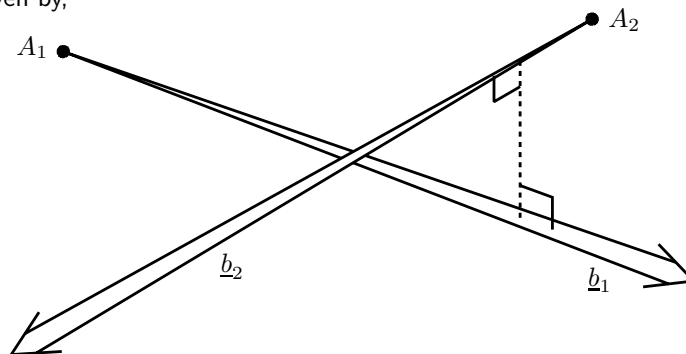


Figure 6.12. Showing two lines travelling in different directions but which do not cross. The dotted line indicates where the lines have their closest approach to one another.

Arbitrary points on the two lines are given by,

$$\underline{r}_1 = \underline{a}_1 + \lambda \underline{b}_1 \quad \text{and} \quad \underline{r}_2 = \underline{a}_2 + \sigma \underline{b}_2, \quad (37)$$

where λ and σ are parameters. The following is then a vector which takes us from an arbitrary point on one of the lines to an arbitrary point on the other:

$$\underline{r}_1 - \underline{r}_2 = \underline{a}_2 - \underline{a}_1 + \sigma \underline{b}_2 - \lambda \underline{b}_1. \quad (38)$$

At closest approach $\underline{r}_1 - \underline{r}_2$ will be the dotted line in Fig. 6.12 and it must be perpendicular to the directions of both the lines, $\underline{b}_1$ and $\underline{b}_2$. If this is difficult to visualise, then consider the line through A_1 to be on a table and the line through A_2 being horizontal above the table. Then the dotted line must be vertical.

We'll proceed in general terms for a while. There are two ways of doing this.

Method 1. Given the red text above, its mathematical translation is that we must have,

$$(\underline{r}_1 - \underline{r}_2) \cdot \underline{b}_1 = 0 \quad \text{and} \quad (\underline{r}_1 - \underline{r}_2) \cdot \underline{b}_2 = 0, \quad (39)$$

where

$$\underline{r}_1 - \underline{r}_2 = \underline{a}_2 - \underline{a}_1 + \sigma \underline{b}_2 - \lambda \underline{b}_1.$$

Hence,

$$0 = \underline{b}_1 \cdot (\underline{r}_1 - \underline{r}_2) = \underline{b}_1 \cdot (\underline{a}_2 - \underline{a}_1) + \sigma \underline{b}_1 \cdot \underline{b}_2 - \lambda \underline{b}_1 \cdot \underline{b}_1,$$

$$0 = \underline{b}_2 \cdot (\underline{r}_1 - \underline{r}_2) = \underline{b}_2 \cdot (\underline{a}_2 - \underline{a}_1) + \sigma \underline{b}_2 \cdot \underline{b}_2 - \lambda \underline{b}_1 \cdot \underline{b}_2.$$

All the dot products will be known values, given the specific vectors stated in the question, and this leaves us with a pair of simultaneous equations to solve for λ and σ . Once these are found we can determine $\underline{r}_1 - \underline{r}_2$, and the modulus of that is the required distance, $|\underline{r}_1 - \underline{r}_2|$.

Method 2. This begins with Eq. (38). If $\underline{r}_1 - \underline{r}_2$ is to be perpendicular to the directions of both the lines, then it also means that it will be parallel to the vector, $\underline{b}_1 \times \underline{b}_2$, because that, by definition, must also be perpendicular to both $\underline{b}_1$ and $\underline{b}_2$. If we let $\underline{p} = \underline{b}_1 \times \underline{b}_2$ and subsequently find its corresponding unit vector, $\hat{\underline{p}}$, then $(\underline{r}_1 - \underline{r}_2) \cdot \hat{\underline{p}}$ will be the distance between the two lines.

If you are like me, then your first reaction to that last paragraph might well have been a "what?" followed by a shaking of the head! Apologies.

Here's a slightly different argument. The dotted line in Fig. 6.12, which illustrates the closest approach of the lines, will correspond to a unique pair of values of λ and σ . So if we have precisely the right values, then the dot product of $(\underline{r}_1 - \underline{r}_2)$ with $\hat{\underline{p}}$ must give the distance because these two vectors are in precisely the same direction.

A third possible argument, this time a little more mathematical. If instead, we take the dot product of Eq. (38) with $\hat{\underline{p}}$, then we get,

$$\begin{aligned} (\underline{r}_1 - \underline{r}_2) \cdot \hat{\underline{p}} &= (\underline{a}_2 - \underline{a}_1) \cdot \hat{\underline{p}} + \sigma \underline{b}_2 \cdot \hat{\underline{p}} - \lambda \underline{b}_1 \cdot \hat{\underline{p}} \\ &= (\underline{a}_2 - \underline{a}_1) \cdot \hat{\underline{p}} \end{aligned} \quad (40)$$

given that $\underline{b}_1 \cdot \hat{\underline{p}} = 0$ and $\underline{b}_2 \cdot \hat{\underline{p}} = 0$ by definition of $\hat{\underline{p}}$. This means that the value of $(\underline{r}_1 - \underline{r}_2) \cdot \hat{\underline{p}}$ is independent of the values of λ and σ , and so

$$(\underline{a}_2 - \underline{a}_1) \cdot \hat{\underline{p}} \quad (41)$$

is the shortest distance between the two lines. The reasonableness of Eq. (41) as the final answer may be understood if we again consider the two lines to be horizontal, and so $\hat{\underline{p}}$ is now vertical. Hence Eq. (41) merely tells us the vertical distance between these two points.

Now we'll use the given numbers.

The equations for the lines are,

$$\underline{r}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{r}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix},$$

and so

$$\underline{r}_2 - \underline{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \sigma \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} - \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}. \quad (42)$$

Method 1. We need $\underline{r}_2 - \underline{r}_1$ to be perpendicular to both $(0, 2, -1)^T$ and $(2, 1, 0)^T$. Therefore we need to take the appropriate dot products:

$$0 = (\underline{r}_2 - \underline{r}_1) \cdot \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = -2 + 5\sigma - 2\lambda,$$

$$0 = (\underline{r}_2 - \underline{r}_1) \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = -2 + 2\sigma - 5\lambda.$$

This yields $\lambda = \sigma = \frac{2}{3}$. After some arithmetic we obtain,

$$\underline{r}_2 - \underline{r}_1 = \begin{pmatrix} -1/3 \\ 2/3 \\ 4/3 \end{pmatrix} \Rightarrow |\underline{r}_2 - \underline{r}_1| = \sqrt{7/3}.$$

Method 2. The vector which is perpendicular to the direction of the lines is,

$$\begin{aligned} \underline{p} = \underline{b}_1 \times \underline{b}_2 &= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 0 \\ 0 & 2 & -1 \end{vmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}. \end{aligned}$$

Hence $\hat{\underline{p}} = \frac{1}{\sqrt{21}} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$. And finally we obtain,

$$(\underline{r}_2 - \underline{r}_1) \cdot \hat{\underline{p}} = (\underline{a}_2 - \underline{a}_1) \cdot \hat{\underline{p}} = \frac{1}{\sqrt{21}} \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \frac{7}{\sqrt{21}} = \sqrt{\frac{7}{3}}.$$

6.11 The distance of a plane from the origin

The final topic is to determine the shortest distance from a plane to the origin. Figure 6.13 gives the general set-up where three points, A , B and C , define the plane. Use

$$\overrightarrow{OA} = \underline{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{OB} = \underline{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{OC} = \underline{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

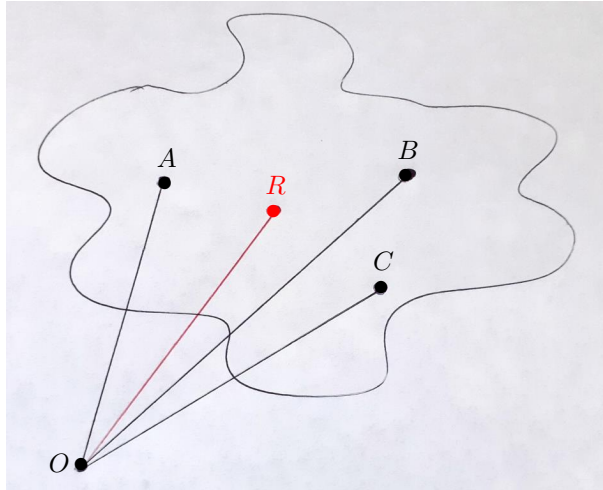


Figure 6.13. Showing a plane defined by three points.

Any point in the plane may be found by first travelling from O to the plane, and then by travelling different distances in two different directions. We will choose the following route:

$$\overrightarrow{OR} = \overrightarrow{OA} + \lambda \overrightarrow{AB} + \sigma \overrightarrow{AC},$$

where I have used A as a proxy origin for within the plane, and the vectors which are multiplied by λ and σ are vectors relative to that point. One could use any of the other possible choices of proxy origin. Therefore we have,

$$\underline{r} = \underline{a} + \lambda(\underline{b} - \underline{a}) + \sigma(\underline{c} - \underline{a}). \quad (43)$$

Once more there are two possible methods to follow from here.

Method 1. The shortest distance to the plane arises when $\underline{r}$ is perpendicular to the plane. Therefore $\underline{r}$ needs to be perpendicular to both of our vectors in the plane, i.e. $(\underline{b} - \underline{a})$ and $(\underline{c} - \underline{a})$. Mathematically, this means that both of the following need to be satisfied:

$$\begin{aligned} 0 &= (\underline{b} - \underline{a}) \cdot \underline{r} = (\underline{b} - \underline{a}) \cdot [\underline{a} + \lambda(\underline{b} - \underline{a}) + \sigma(\underline{c} - \underline{a})] \\ 0 &= (\underline{c} - \underline{a}) \cdot \underline{r} = (\underline{c} - \underline{a}) \cdot [\underline{a} + \lambda(\underline{b} - \underline{a}) + \sigma(\underline{c} - \underline{a})] \end{aligned}$$

When the numerical data is substituted into this we obtain,

$$0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = -1 + 2\lambda + \sigma,$$

$$0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] = 0 + \lambda + 2\sigma.$$

Hence $\lambda = \frac{2}{3}$ and $\sigma = -\frac{1}{3}$. Therefore,

$$\underline{r} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix} \Rightarrow |\underline{r}| = \sqrt{1/3}.$$

Method 2. Again bearing in mind that the shortest distance corresponds to a vector which is perpendicular to the plane. Given that $(\underline{b} - \underline{a})$ and $(\underline{c} - \underline{a})$ are in the plane, then

$$\underline{p} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a})$$

is perpendicular to it. Thus $\underline{r} \cdot \underline{\hat{p}}$ will yield the required distance by isolating the component of a general $\underline{r}$ in that direction.

Therefore,

$$\begin{aligned} \underline{p} = (\underline{b} - \underline{a}) \times (\underline{c} - \underline{a}) &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

$$\text{Hence } \underline{\hat{p}} = \begin{pmatrix} 1/3 \\ -1/3 \\ 1/3 \end{pmatrix}.$$

If we take the dot product of Eq. (43) with $\underline{\hat{p}}$ then we obtain,

$$\begin{aligned} \underline{r} \cdot \underline{\hat{p}} &= \underline{a} \cdot \underline{\hat{p}} + \lambda(\underline{b} - \underline{a}) \cdot \underline{\hat{p}} + \sigma(\underline{c} - \underline{a}) \cdot \underline{\hat{p}} \\ &= \underline{a} \cdot \underline{\hat{p}} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}. \end{aligned}$$

So this matches with Method 1. This second method shows explicitly that the final result is independent of the values of λ and σ , and therefore we have found the component of $\underline{a}$ which is perpendicular to the plane. Given

that we chose $\underline{a}$ arbitrarily at the start of our analysis to represent a point on the plane, then we should obtain the same distance for the other two main choices. So we'll check this:

$$\underline{b} \cdot \underline{\hat{p}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}},$$

$$\underline{c} \cdot \underline{\hat{p}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}.$$

If this all feel rather dodgy, then consider a horizontal plane (a sheet of paper, for example) which is above a table and parallel with it. There is an origin somewhere on the table. Then, whatever point is chosen on the plane has the same vertical component.

Note: The above analyses have told us that the two-parameter equation for the plane is Eq. (43). If we multiply this by the unit perpendicular vector, then it gives

$$\underline{r} \cdot \underline{\hat{p}} = \underline{a} \cdot \underline{\hat{p}}.$$

If we set $\underline{r} = (x, y, z)^T$, then the above numerical case this gives us the following version for the equation for the plane:

$$\frac{x}{\sqrt{3}} - \frac{y}{\sqrt{3}} - \frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

By its construction the coefficients of x , y and z are the coefficients of the unit perpendicular vector and the right hand side is the nearest distance. Therefore if we were to have the following as the equation of the plane,

$$ax + by + cz = d,$$

then we may use this to find the perpendicular vector to the plane and its closest distance to the origin. The perpendicular vector is,

$$\underline{\hat{p}} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

and the closest approach is $d/\sqrt{a^2 + b^2 + c^2}$.

To fix ideas, if we were given $x + 2y + 2z = 6$ as an equation for the plane, then $\underline{p} = (1, 2, 2)^T$. Given that $|\underline{p}| = 3$, we may divide the equation for the plane by 3 to yield,

$$\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z = 2.$$

Hence $\underline{\hat{p}} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$ and the closest distance from the origin is 2.