4 Integration

4.5 Integration of products

- Although it is generally straightforward to differentiate products of functions, the process of integrating products is more intricate. This is the so-called integration by parts method.
- Future topics such as Fourier Series, Laplace Transforms and Fourier Transforms, will require a rapid facility with this method.
- My view is that, while the standard formula is a basic recipe to follow, it is nevertheless often applied incorrectly.
- There is a better way.....
- .....but we'll start with the classic textbook method and then transform it into something better.

The standard derivation uses the well–known result for the derivative of the product of two functions, $u(x)$ and $v(x)$:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

This is rearranged slightly and integrated to obtain the result,

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx,$$

or the other way around,

$$\int v\frac{du}{dx}\,dx = uv - \int u\frac{dv}{dx}\,dx.$$

$$\boxed{\int v \frac{du}{dx}\, dx = uv - \int u \frac{dv}{dx}\, dx.}$$

Example 4.18. Integrate $\int x\, e^{ax}\, dx$ which involves the product of x and e^{ax}.

The overall aim of the above formula is to replace the integral on the left with a right hand one which may be solved or at least is easier to solve.

$$\text{Let} \quad \left\{ \begin{array}{l} \dfrac{du}{dx} = e^{ax} \\ \\ v = x \end{array} \right. \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} u = e^{ax}/a \\ \\ \dfrac{dv}{dx} = 1. \end{array} \right.$$

Hence

$$\int x\, e^{ax}\, dx = x \times \frac{e^{ax}}{a} - \int 1 \times \frac{e^{ax}}{a}\, dx.$$

Hence the final result is,

$$\int x\, e^{ax}\, dx = \frac{x\, e^{ax}}{a} - \frac{e^{ax}}{a^2} = \frac{1}{a^2}(ax - 1)e^{ax},$$

after a little tidying-up. Of course there should be an arbitrary constant in there too.

So what is the problem?

Answer:

- An integral such as $\int \boldsymbol{x^5 e^{ax}\, dx}$ often requires more than a page of writing.
- The workings cannot be checked easily.
- A sizeable minority of examination students commit sign errors.

There are some quicker ways, including the preparation of a Table of data (called the DI method — see all those videos on youtube!), but even that takes twice as long to perform than the minimum possible time.

A new way for integration by parts is needed, one which effectively uses just one line of workings, is safe and may be checked easily afterwards....

....but a little preparation/derivation is needed first.

Recap:

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}. \tag{1}$$

Introduce new variables into the above:

$$u(x) = f(x) \qquad \text{and} \qquad \frac{dv}{dx} = g(x)$$

For convenience we will need to use a superscript notation for differentiation and integration:

$$f^{(1)} = \frac{df}{dx}, \qquad f^{(2)} = \frac{d^2 f}{dx^2} \qquad \text{and} \qquad g^{(-1)} = \int g\,dx, \qquad g^{(-2)} = \int g^{(-1)}\,dx.$$

Equation (1) now becomes,

$$\boxed{\int fg\,dx = fg^{(-1)} - \int f^{(1)}g^{(-1)}\,dx}.$$

As before, we choose one function to be differentiated (called f here) and one to be integrated (called g here).

Thus far we have only achieved a change in notation.

Let us apply this new formula to itself.

$$\int \boldsymbol{fg}\,\boldsymbol{dx} = \boldsymbol{fg}^{(-1)} - \int \boldsymbol{f}^{(1)}\boldsymbol{g}^{(-1)}\,\boldsymbol{dx}$$

$\Rightarrow$

$$\int \boldsymbol{f}^{(1)}\boldsymbol{g}^{(-1)}\,\boldsymbol{dx} = \boldsymbol{f}^{(1)}\boldsymbol{g}^{(-2)} - \int \boldsymbol{f}^{(2)}\boldsymbol{g}^{(-2)}\,\boldsymbol{dx}.$$

Hence,

$$\boxed{\int \boldsymbol{fg}\,\boldsymbol{dx} = \boldsymbol{fg}^{(-1)} - \boldsymbol{f}^{(1)}\boldsymbol{g}^{(-2)} + \int \boldsymbol{f}^{(2)}\boldsymbol{g}^{(-2)}\,\boldsymbol{dx}},$$

which represents two integrations by parts.

Why stop there?

So the formulae for five and six integrations by parts are,

$$\int fg\,dx = fg^{(-1)} - f^{(1)}g^{(-2)} + f^{(2)}g^{(-3)} - f^{(3)}g^{(-4)} + f^{(4)}g^{(-5)} - \int f^{(5)}g^{(-5)}\,dx$$

and

$$\int fg\,dx = fg^{(-1)} - f^{(1)}g^{(-2)} + f^{(2)}g^{(-3)} - f^{(3)}g^{(-4)} + f^{(4)}g^{(-5)} - f^{(5)}g^{(-6)} + \int f^{(6)}g^{(-6)}\,dx.$$

I do not wish you to remember all of these new formulae. It is better to follow a set of rules. Here they are:

1. The first product on the right hand side, $fg^{(-1)}$, has the g-term integrated but the f-term fas been left alone. So we start by integrating, which feels natural given that the overall aim is to find an integral!

2. Each successive product which appears is related to the previous product by having the term called f being differentiated and the term called g being integrated.

3. The signs alternate between the terms on the right hand side including for the final integral.

4. The final integral (should it be required) uses a final differentiation of the preceding f term.

The following two examples won't need to use the final integral because the power of x will eventually become zero after a sufficient number of differentiations.

Example 4.19. Find the integral, $\displaystyle\int x^2 e^{-ax}\,dx$.

We will choose to differentiate the x^2 terms and to integrate the exponential. We obtain the following,

$$\int \underbrace{x^2}_{D}\ \underbrace{e^{-ax}}_{I}\,dx = \underbrace{\left[x^2\right]}_{D_0}\underbrace{\left[\frac{e^{-ax}}{-a}\right]}_{I_1} - \underbrace{\left[2x\right]}_{D_1}\underbrace{\left[\frac{e^{-ax}}{a^2}\right]}_{I_2} + \underbrace{\left[2\right]}_{D_2}\underbrace{\left[\frac{e^{-ax}}{-a^3}\right]}_{I_3} - \underbrace{\left[0\right]}_{D_3}\underbrace{\left[\frac{e^{-ax}}{a^4}\right]}_{I_4} + c.$$

The beauty of the solution for this Example is that one may hop through all the D terms to check that all the differentiations are correct, and through all the I terms to check the integrations.

The final point to make is that all of the D and I terms are sitting cosily inside their own pair of brackets. Do not try to process any minus signs until the next line of working.

$$\int x^2\, e^{-ax}\,dx \;=\; -\frac{(a^2x^2 + 2ax + 2)}{a^3}e^{-ax} + c.$$

Example 4.20. Find the integral, $\displaystyle\int x^5 e^{ax}\,dx$.

This one would normally be the stuff of nightmares, but we can take it in our stride....

$$\begin{aligned}
\int \underbrace{x^5}_{D}\ \underbrace{e^{ax}}_{I}\,dx = {} & \underbrace{\left[x^5\right]}_{D_0}\underbrace{\left[\frac{e^{ax}}{a}\right]}_{I_1} - \underbrace{\left[5x^4\right]}_{D_1}\underbrace{\left[\frac{e^{ax}}{a^2}\right]}_{I_2} + \underbrace{\left[20x^3\right]}_{D_2}\underbrace{\left[\frac{e^{ax}}{a^3}\right]}_{I_3} - \underbrace{\left[60x^2\right]}_{D_3}\underbrace{\left[\frac{e^{ax}}{a^4}\right]}_{I_4} \\
& + \underbrace{\left[120x\right]}_{D_4}\underbrace{\left[\frac{e^{ax}}{a^5}\right]}_{I_5} - \underbrace{\left[120\right]}_{D_5}\underbrace{\left[\frac{e^{ax}}{a^6}\right]}_{I_6} + c \\
= {} & \frac{(a^5x^5 - 5a^4x^4 + 20a^3x^3 - 60a^2a^2 + 120ax - 120)}{a^6}e^{ax} + c.
\end{aligned}$$

Example 4.20b. Find the integral, $\int_0^{\infty} x^5 e^{-ax}\,dx$.

$$\int_0^{\infty} \underbrace{x^5}_{D}\ \underbrace{e^{-ax}}_{I}\,dx = \Big[\underbrace{x^5}_{D_0}\Big]\underbrace{\left[\frac{e^{-ax}}{-a}\right]}_{I_1}{}_0^{\infty} - \Big[\underbrace{5x^4}_{D_1}\Big]\underbrace{\left[\frac{e^{-ax}}{a^2}\right]}_{I_2}{}_0^{\infty} + \Big[\underbrace{20x^3}_{D_2}\Big]\underbrace{\left[\frac{e^{-ax}}{-a^3}\right]}_{I_3}{}_0^{\infty}$$

$$-\Big[\underbrace{60x^2}_{D_3}\Big]\underbrace{\left[\frac{e^{-ax}}{a^4}\right]}_{I_4}{}_0^{\infty} + \Big[\underbrace{120x}_{D_4}\Big]\underbrace{\left[\frac{e^{-ax}}{-a^5}\right]}_{I_5}{}_0^{\infty} - \Big[\underbrace{120}_{D_5}\Big]\underbrace{\left[\frac{e^{-ax}}{a^6}\right]}_{I_6}{}_0^{\infty}$$

$$= \frac{120}{a^6}.$$

Example 4.21. Find the integral, $\int x^2 \cos ax \, dx$.

We have,

$$I = \int \underbrace{x^2}_{D} \underbrace{\cos ax}_{I} \, dx = \underbrace{\left[x^2\right]}_{D_0} \underbrace{\left[\frac{\sin ax}{a}\right]}_{I_1} - \underbrace{\left[2x\right]}_{D_1} \underbrace{\left[\frac{-\cos ax}{a^2}\right]}_{I_2} + \underbrace{\left[2\right]}_{D_2} \underbrace{\left[\frac{-\sin ax}{a^3}\right]}_{I_3},$$

which may, of course, be tidied up:

$$I = \frac{1}{a^3}\Big[(a^2x^2 - 2)\sin ax + 2ax\cos ax\Big] + c.$$

Example 4.22. Find $\int e^{ax} \sin bx \, dx$.

This one is a little different because the integration by parts process doesn't terminate.

We integrate by parts twice which yields an integral that is proportional to the one we started with. Nice.

I will choose to integrate the sine, but it is worth attempting the other way around for practice.

$$\begin{aligned}
\text{Let } I = \int \underbrace{e^{ax}}_{D} \underbrace{\sin bx}_{I} \, dx &= \underbrace{\left[e^{ax}\right]}_{D_0} \underbrace{\left[\frac{-\cos bx}{b}\right]}_{I_1} - \underbrace{\left[ae^{ax}\right]}_{D_1} \underbrace{\left[\frac{-\sin bx}{b^2}\right]}_{I_2} + \int \underbrace{\left[a^2 e^{ax}\right]}_{D_2} \underbrace{\left[\frac{-\sin bx}{b^2}\right]}_{I_2} dx, \\
&= \frac{e^{ax}}{b^2}\left[-b\cos bx + a \sin bx\right] - \frac{a^2}{b^2} I.
\end{aligned}$$

The terms involving I may be brought together on the left to obtain,

$$\left[1 + \frac{a^2}{b^2}\right] I = \frac{e^{ax}}{b^2}\left[a \sin bx - b \cos bx\right]$$

and therefore we obtain,

$$I = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right].$$

Example 4.23. Find $\int \ln x \, dx$.

Another classic, and although it doesn't look like a product we can make it so: $\int 1 \times \ln x \, dx$. Once you know the trick, you know the trick. We will differentiate the $\ln x$ because we can! We will only integrate by parts once because it isn't entirely obvious what will happen, so we'll take stock at that point.

$$\begin{aligned}
\int \underbrace{1}_{I} \times \underbrace{\ln x}_{D} \, dx &= \underbrace{\left[x\right]}_{I_1} \underbrace{\left[\ln x\right]}_{D_0} - \int \underbrace{\left[x\right]}_{I_1} \underbrace{\left[\frac{1}{x}\right]}_{D_1} dx && \text{one integration by parts} \\
&= x \ln x - \int 1 \, dx && \text{tidied up the integral} \\
&= x \ln x - x + c.
\end{aligned}$$

Note: An additional rule — if in doubt, do it once!

Note: This integral may also be obtained using the substitution, $x = e^y$.