University of Bath
Formulae and Statistical Tables

Departments of Physics and Mathematical Sciences

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Book A

Formulae

1 Algebraic and Trigonometrical Formulae

1.1 Algebraic formulae

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

$a^2 + b^2$ has no real factors

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$$

where ${}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{n(n-1)\cdots(n-r+1)}{r!}$. Also written as $\dbinom{n}{r}$.

$$a^x a^y = a^{x+y}$$

$$a^{-y} = \frac{1}{a^y} \quad \text{so} \quad a^x a^{-y} = a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = a^{xy}$$

$\log_a(xy) = \log_a x + \log_a y$ (a, x and y are positive and real)

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^y) = y \log_a x$$

$\log_a 1 = 0, \quad \log_a a = 1, \quad \log_a 0 = -\infty,$ (for any base $a > 0$)

$\ln(n!) \approx n \ln n - n, \quad n \gg 1$ (Stirling's approximation)

$$n! = n(n-1)(n-2)\ldots 1$$

$$0! = 1, \qquad (-\tfrac{1}{2})! = \sqrt{\pi}, \qquad (\tfrac{1}{2})! = \frac{\sqrt{\pi}}{2}$$

$\Gamma(p+1) = \displaystyle\int_0^\infty x^p e^{-x} dx = p \int_0^\infty x^{p-1} e^{-x} dx = p!$ (for $p > -1$)

$\displaystyle\int_0^1 x^p (1-x)^q dx = \frac{p!q!}{(p+q+1)!}$ (for any $p, q > -1$)

$$\frac{P}{Q} = \frac{P}{(x+x_1)(x+x_2)\cdots} = \frac{A_1}{(x+x_1)} + \frac{A_2}{(x+x_2)} + \cdots + \frac{A_r}{(x+x_r)} + \cdots$$

where $A_r = \left[(x+x_r)\dfrac{P}{Q}\right]_{x=-x_r} = \left[\dfrac{P}{Q'}\right]_{x=-x_r}$

$$\frac{P}{Q} = \frac{P}{(x+x_1)^2(x+x_2)\cdots} = \frac{A_1^1}{(x+x_1)} + \frac{A_1}{(x+x_1)^2} + \frac{A_2}{(x+x_2)} + \cdots$$

where $A_r = [(x+x_r)^2\dfrac{P}{Q}]_{x=-x_r}$ and $A_1^1 = \left[\dfrac{d}{dx}(x+x_1)^2\dfrac{P}{Q}\right]_{x=-x_1}$

1.2 Trigonometric and hyperbolic formulae

$$\tan A = \frac{\sin A}{\cos A}$$

$$\cot A = \frac{1}{\tan A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{cosec} A = \frac{1}{\sin A}$$

$$\operatorname{sinc} A = \begin{cases} 1, & \text{for } A = 0 \\ \dfrac{\sin A}{A}, & \text{otherwise} \end{cases}$$

Sine Rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$

$$\cos^2 A + \sin^2 A = 1$$

$$1 + \tan^2 A = \sec^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A \;=\; \cos^2 A - \sin^2 A \;=\; 2\cos^2 A - 1 \;=\; 1 - 2\sin^2 A$$

$$\tan 2A \;=\; \frac{2\tan A}{1-\tan^2 A}$$

$$\cos^2 A \;=\; \frac{1+\cos 2A}{2}$$

$$\sin^2 A \;=\; \frac{1-\cos 2A}{2}$$

$$\sin\theta \;=\; \frac{2t}{1+t^2}; \quad \cos\theta \;=\; \frac{1-t^2}{1+t^2} \quad \text{where} \quad t \;=\; \tan\frac{\theta}{2}.$$

$$2\sin A\cos B \;=\; \sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B \;=\; \cos(A+B)+\cos(A-B)$$

$$2\sin A\sin B \;=\; \cos(A-B)-\cos(A+B)$$

$$\sin A \pm \sin B \;=\; 2\sin\left(\frac{A\pm B}{2}\right)\cos\left(\frac{A\mp B}{2}\right)$$

$$\cos A + \cos B \;=\; 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B \;=\; -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos x \;=\; \frac{1}{2}(e^{\mathrm{i}\,x}+e^{-\mathrm{i}\,x})$$

$$\sin x \;=\; \frac{1}{2\,\mathrm{i}}(e^{\mathrm{i}\,x}-e^{-\mathrm{i}\,x})$$

$$\cosh x \;=\; \frac{1}{2}(e^{x}+e^{-x})$$

$$\sinh x \;=\; \frac{1}{2}(e^{x}-e^{-x})$$

$$\tanh x \;=\; \frac{\sinh x}{\cosh x}$$

$$\coth x \;=\; \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x \;=\; \frac{1}{\cosh x}$$

$$\operatorname{cosech} x \;=\; \frac{1}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x \;=\; 1$$

$$1 - \tanh^2 x \;=\; \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cosh(\mathrm{i}x) = \cos x$$

$$\sinh(\mathrm{i}x) = \mathrm{i}\sin x$$

$$\cos(\mathrm{i}x) = \cosh x$$

$$\sin(\mathrm{i}x) = \mathrm{i}\sinh x$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh(2x) = 2\sinh x \cosh x$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad (x \geq 1)$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left\{\frac{1+x}{1-x}\right\} \qquad (-1 < x < 1)$$

2 Limits

$n^c x^n \to 0$ as $n \to \infty$ if $|x| < 1,$ (any fixed c)

$\dfrac{x^n}{n!} \to 0$ as $n \to \infty$ (any fixed x)

$\left(1 + \dfrac{x}{n}\right)^n \to e^x$ as $n \to \infty$

$x \ln x \to 0$ as $x \to 0$

If $f(a) = g(a) = 0$ or $\pm\infty$ then $\lim_{x\to a}\left(\dfrac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\dfrac{f'(x)}{g'(x)}\right)$ (l'Hôpital's rule)

3 Derivatives

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx} = f'g + fg'$$

$$\frac{d}{dx}(fg)^n = f^{(n)}g + nf^{(n-1)}g^{(1)} + \cdots + {}^nC_r\, f^{(n-r)}g^{(r)} + \cdots + fg^{(n)}, \quad \text{where } {}^nC_r \equiv \binom{n}{r}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}(g(f(x))) = \frac{dg}{df}\frac{df}{dx}$$

Total differential:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Partial differentiation:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}.$$

3.1 Table of derivatives

y	$\frac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$

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y	$\frac{dy}{dx}$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{cosech}^{-1} x$	$\frac{-1}{x\sqrt{1+x^2}}$
$\coth^{-1} x$	$\frac{-1}{x^2-1}$

4 Integrals

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

4.1 Table of integrals

$f(x)$	$\int f(x)\, dx$
$\ln\lvert x\rvert$	$x\ln(x) - x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert = \ln\tan\left\lvert\frac{x}{2} + \frac{\pi}{4}\right\rvert = \frac{1}{2}\ln\left\lvert\frac{1+\sin x}{1-\sin x}\right\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert = \ln\left\lvert\tan\frac{x}{2}\right\rvert = \frac{1}{2}\ln\left\lvert\frac{1-\cos x}{1+\cos x}\right\rvert$
$\operatorname{sech} x$	$2\tan^{-1}(e^x)$
$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\frac{1}{a^2 - x^2}$	$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ for $\lvert x\rvert < \lvert a\rvert$
$\frac{1}{x^2 - a^2}$	$-\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)$ for $\lvert x\rvert > \lvert a\rvert$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right)$ for $\lvert x\rvert > \lvert a\rvert$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)$ for $\lvert x\rvert > \lvert a\rvert$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x+\sqrt{a^2+x^2}}{a}\right)$
$\sqrt{a^2 - x^2}$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$ for $\lvert x\rvert < \lvert a\rvert$
$\sqrt{x^2 - a^2}$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$ for $\lvert x\rvert > \lvert a\rvert$
$\sqrt{a^2 + x^2}$	$\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)$

4.2 Trigonometric and exponential integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \frac{(n-1)!!}{n!!} \times \begin{cases} \frac{\pi}{2}, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$$

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{(m-1)!!(n-1)!!}{(m+n)!!} \times \begin{cases} \frac{\pi}{2}, & m \text{ and } n \text{ both even} \\ 1, & \text{otherwise} \end{cases}$$

where $p!! = p(p-2)(p-4)\cdots 2$ or 1 and $0!! = 1$.

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(b \sin bx + a \cos bx)$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \qquad (n \geq 0 \text{ and } a \geq 0)$$

$$\int_0^{\infty} x^n e^{-x^2} dx = \frac{1}{2}\left(\frac{n-1}{2}\right)!$$

$$\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

4.3 Differentiation under the Integral Sign

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t) \, dt = \frac{dv}{dx} f\left(x, v(x)\right) - \frac{du}{dx} f\left(x, u(x)\right) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x}\left\{f(x,t)\right\} dt.$$

5 Coordinate Geometry (Two Dimensions)

Straight line: $y = mx + c$, gradient m, intercept c on y axis.

Conic Section	Cartesian Equation	Eccentricity (e)	Foci	
Circle	$(x-a)^2 + (y-b)^2 = R^2$	$e = 0$	(a, b)	Centre (a, b), radius R
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$0 < e < 1$	$(\pm ae, 0)$	$b^2 = a^2(1-e^2)$, $(a > b)$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$e > 1$	$(\pm ae, 0)$	$b^2 = a^2(e^2-1)$, asymptotes $y = \pm\dfrac{b}{a}x$
Rect. Hyperbola	$xy = c^2$ (constant)	$e = \sqrt{2}$	$(\pm c\sqrt{2}, \pm c\sqrt{2})$	asymptotes $x = 0, y = 0$
Parabola	$y^2 = 4ax$	$e = 1$	$(a, 0)$	Vertex $(0, 0)$

Polar equation for all conic sections $\ell = r(1 + e\cos\theta)$

6 Series

$$a + (a+d) + (a+2d) + \cdots + (a + |n-1|\,d) = \frac{n}{2}(2a + |n-1|\,d)$$

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$$

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$	for $\lvert x\rvert < 1$
$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$	for $\lvert x\rvert < 1$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	for all x
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	for $\lvert x\rvert < 1$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	for all x
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	for all x
$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$	for $\lvert x\rvert < \frac{\pi}{2}$
$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$	for $\lvert x\rvert < 1$
$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$	
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	for $\lvert x\rvert < 1$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	for all x
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	for all x
$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17}{315}x^7 + \cdots$	for $\lvert x\rvert < \frac{\pi}{2}$
$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$	for $\lvert x\rvert < 1$
$\cosh^{-1} x = \ln 2x - \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{6x^6} - \cdots$	for $x > 1$
$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$	for $\lvert x\rvert < 1$

6.1 Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n f^{(n)}(0)}{n!} + R_{n+1}$$

where $R_{n+1} = x^{n+1}\frac{f^{(n+1)}(\theta x)}{(n+1)!}$ for $0 < \theta < 1$

6.2 Taylor Series

$$f(x) \;=\; f(x_0+\delta) \;=\; f(x_0) \;+\; \delta f'(x_0) \;+\; \frac{\delta^2}{2!}f''(x_0) \;+\; \cdots \;+\; \frac{\delta^n f^{(n)}(x_0)}{n!} \;+\; R_{n+1}$$

where $x \;=\; x_0+\delta$ and

$$\begin{aligned} R_{n+1} \;&=\; \frac{1}{n!}\int_{x_0}^{x}(x-s)^n f^{(n+1)}(s)\; ds \\ &=\; \delta^{n+1}\frac{f^{(n+1)}(x_0+\theta\delta)}{(n+1)!} \qquad (0<\theta<1) \end{aligned}$$

6.3 Taylor Series for Two Variables

$$\begin{aligned} f(x,y) \;&=\; f(x_0+\delta, y_0+\epsilon) \\ &=\; f(x_0,y_0) + \left(\delta\frac{\partial f}{\partial x}+\epsilon\frac{\partial f}{\partial y}\right) \;+\; \frac{1}{2!}\left(\delta^2\frac{\partial^2 f}{\partial x^2}+2\delta\epsilon\frac{\partial^2 f}{\partial x \partial y}+\epsilon^2\frac{\partial^2 f}{\partial y^2}\right) \\ &\quad + \;\cdots\; + \;\frac{1}{n!}\left(\delta\frac{\partial}{\partial x}+\epsilon\frac{\partial}{\partial y}\right)^n f + \cdots \end{aligned}$$

where $x = x_0+\delta$, $y = y_0+\epsilon$ and the differential coefficients are evaluated at (x_0, y_0).

6.4 Sums of series

$$\sum_{k=0}^{K} x^k \;=\; \frac{1-x^{K+1}}{1-x} \qquad \text{and} \qquad \sum_{k=m}^{K} x^k \;=\; \frac{x^m - x^{K+1}}{1-x}$$

$$\frac{\partial}{\partial x}\sum_{k=0}^{K} x^k \;=\; \sum_{k=0}^{K} k x^{(k-1)} \;=\; \frac{1-x^{K+1}}{(1-x)^2} - \frac{(K+1)x^K}{(1-x)}$$

$$\sum_{k=0}^{n-1}(a+kr) \;=\; \frac{n}{2}\left[2a+(n-1)r\right]$$

$$\sum_{k=1}^{n} a q^{k-1} \;=\; \frac{a\,(q^n-1)}{q-1} \qquad (q \neq 1)$$

$$\sum_{k=0}^{\infty} a q^k \;=\; \frac{a}{1-q} \qquad (|q|<1)$$

$$\sum_{k=0}^{\infty} a k q^k \;=\; \frac{a\; q}{(1-q)^2} \qquad (|q|<1)$$

$$\sum_{k=0}^{\infty} a k^2 q^k \;=\; \frac{a\; q(1+q)}{(1-q)^3} \qquad (|q|<1)$$

$$\sum_{k=0}^{\infty} a k^3 q^k \;=\; \frac{a\; q(1+4q+q^2)}{(1-q)^4} \qquad (|q|<1)$$

7 Numerical Formulae

7.1 Trapezium rule

$$\int_a^{a+h} f(x)\,dx \;=\; \frac{h}{2}(f_0 + f_1) + E$$

where $E = -\frac{1}{12}h^3 f''(\xi)$, for $a < \xi < a + h$.

For $b = a + nh$,

$$\int_a^b f(x)\,dx \;=\; h\left(\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{n-1} + \frac{1}{2}f_n\right) + E$$

where $E = -\frac{1}{12}h^2(b-a)f''(\xi)$, for $a < \xi < b$.

7.2 Simpson's rule

$$\int_a^{a+2h} f(x)\,dx \;=\; \frac{h}{3}(f_0 + 4f_1 + f_2) + E$$

where $E = -\frac{1}{90}h^5 f^{(4)}(\xi)$, for $a < \xi < a + 2h$.

For $b = a + 2nh$,

$$\int_a^b f(x)\,dx \;=\; \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \cdots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\right) + E$$

where $E = -\frac{1}{180}h^4(b-a)f^{(4)}(\xi)$, for $a < \xi < b$.

7.3 Newton's formula for roots of equations $f(x) = 0$

$$x_{n+1} \;=\; x_n - \frac{f(x_n)}{f'(x_n)}$$

7.4 Numerical solution of differential equations (Modified Euler)

$$
\begin{aligned}
y_1^{(P)} &= y_0 + hy_0' \\
y_{n+1}^{(P)} &= y_{n-1} + 2hy_n'; && \text{Error } \frac{1}{3}h^3 y_n''' + \text{ higher order terms} \\
y_{n+1}^{(C)} &= y_n + \frac{h}{2}\left(y_{n+1}' + y_n'\right); && \text{Error } -\frac{1}{12}h^3 y_n''' + \text{ higher order terms}
\end{aligned}
$$

7.5 The Lagrange interpolation formula

If $f \in C^{(n+1)}[a,b]$ and $a \le x_0 < x_1 < \cdots < x_n \le b$ then for $x \in [a,b]$

$$
f(x) = \sum_{j=0}^{n} \ell_{j,n}(x) f(x_j) + \frac{P_n(x)}{(n+1)!} f^{n+1}(\xi)
$$

where

$$
\begin{aligned}
\ell_{j,n}(x) &= \prod_{\substack{k=0 \\ k\neq j}}^{n} \left[\frac{x - x_k}{x_j - x_k}\right] = \frac{P_n(x)}{(x - x_j)P_n'(x_j)} \\
P_n(x) &= \prod_{k=0}^{n} [x - x_k]
\end{aligned}
$$

and $a \le \xi \le b$.

7.6 The (modified) Hermite interpolation formula

If $f \in C^{(n+r+2)}[a,b]$ and $a \le x_0 < x_1 < \cdots < x_n \le b$ then for $x \in [a,b]$

$$
f(x) = y(x) + E(x)
$$

where $y(x) = \sum_{j=0}^{n} h_j(x) f(x_j) + \sum_{j=0}^{r} \overline{h}_j(x) f'(x_j)$ for $r \le n$, with

$$
\begin{aligned}
h_j(x) &= \begin{cases} \left[1 - (x - x_j)\{\ell_{j,n}'(x_j) + \ell_{j,r}'(x_j)\}\right] \ell_{j,n}(x)\ell_{j,r}(x), & j = 0, 1, \cdots, r; \\ \ell_{j,n}(x) P_r(x)/P_r(x_j), & j = r+1, r+2, \cdots, n. \end{cases} \\
\overline{h}_j(x) &= (x - x_j)\ell_{j,n}(x)\ell_{j,r}(x), \quad j = 0, 1, \cdots, r
\end{aligned}
$$

and $E(x) = \dfrac{P_n(x)P_r(x)}{(n+r+2)!} f^{(n+r+2)}(\xi)$ for $a \le \xi \le b$.

8 Fourier Series

$f(t)$ periodic, period $T,$ fundamental frequency $\omega_0 :\ \omega_0 = \dfrac{2\pi}{T}$

(i) real form:

$$
\begin{aligned}
f(t) &= c_0 + \sum_{n=1}^{\infty} c_n \sin(n\omega_0 t + \phi_n) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \\
a_n &= \frac{2}{T} \int_{\theta}^{\theta+T} f(t) \cos(n\omega_0 t)\ dt \\
&= \text{twice mean value of } f(t)\cos(n\omega_0 t) \text{ over a period } (\theta \text{ arbitrary}). \\
b_n &= \frac{2}{T} \int_{\theta}^{\theta+T} f(t) \sin(n\omega_0 t)\ dt \\
&= \text{twice mean value of } f(t)\sin(n\omega_0 t) \text{ over a period } (\theta \text{ arbitrary}).
\end{aligned}
$$

(ii) complex form:

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \\
C_n &= \frac{1}{T} \int_{\theta}^{\theta+T} f(t) e^{-in\omega_0 t}\ dt \\
&= \text{mean value of } f(t)e^{-in\omega_0 t} \text{ over a period } (\theta \text{ arbitrary}).
\end{aligned}
$$

8.1 Finite range (half-range) Fourier series

$g(x)$ defined for $0 < x < \ell$.

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) \quad \text{where} \quad b_n = \frac{2}{\ell} \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right)\ dx$$

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) \quad \text{where} \quad a_n = \frac{2}{\ell} \int_0^{\ell} g(x) \cos\left(\frac{n\pi x}{\ell}\right)\ dx$$

8.2 Fourier Transforms

If $\int_{-\infty}^{\infty} |g(t)|\, dt \;<\; \infty$

then $\mathcal{F}[g(t)] \;=\; \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\, dt \;=\; G(\omega)$

and $\mathcal{F}^{-1}[G(\omega)] \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\, d\omega \;=\; g(t)$

	$g(t)$	$G(\omega) \;=\; \mathcal{F}\,[g(t)]$
Even function	$g(t) \;=\; g(-t)$	$G(\omega) \;=\; G(-\omega) \;=\; 2\int_0^{\infty} g(t)\;\cos\,\omega t\; dt$
Odd function	$g(t) \;=\; -g(-t)$	$G(\omega) \;=\; -G(-\omega) \;=\; -2i\int_0^{\infty} g(t)\;\sin\,\omega t\; dt$
Symmetry	$G(t)$	$2\pi g(-\omega)$
Reflection	$g(-t)$	$G(-\omega)$
Conjugate	$g^*(t)$	$G^*(-\omega)$
Scale change	$g\left(\frac{t}{T}\right)$, $(T > 0)$	$TG(\omega T)$
Derivative	$\frac{dg(t)}{dt}$	$i\;\omega\;G(\omega)$
	$tg(t)$	$i\;\frac{dG(\omega)}{d\omega}$
Time Shift	$g(t\;+\;\tau)$	$e^{i\omega\tau}G(\omega)$
Frequency Shift	$g(t)e^{i\omega_0\, t}$	$G(\omega - \omega_0)$
Convolution	$(f * g)(t)$	$F(\omega)G(\omega)$
Frequency convolution	$f(t)g(t)$	$\frac{1}{2\pi}(F * G)(\omega)$

where $(f\;*\;g)\;(x) \;=\; \int_{-\infty}^{\infty} f(y)g(x\;-\;y)\;dy \;=\; \int_{-\infty}^{\infty} f(x\;-\;y)g(y)\;dy$

8.3 Parseval's theorem

$$\int_{-\infty}^{\infty} f(t)g(t)\ dt \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(-\omega)\ d\omega$$

and

$$\int_{-\infty}^{\infty} |g(t)|^2\ dt \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2\ d\omega$$

8.4 Fourier transform pairs

$g(t)$	$G(\omega) \ = \ \mathcal{F}\ [g(t)]$
$\delta\ (t)$	1
$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$sgn\ (t) \ = \begin{cases} 1, & t \ > \ 0 \\ -1, & t \ < \ 0 \end{cases}$	$2/i\omega$
$H(t)$	$\pi\delta(\omega) \ + \ \dfrac{1}{i\omega}$
$\dfrac{\sin\omega_0 t}{\pi t}$	$H(\omega \ + \ \omega_0) - H(\omega \ - \ \omega_0)$
$e^{-at^2}, \quad (a > 0)$	$\sqrt{\dfrac{\pi}{a}} e^{-\omega^2/4a}$
$\dfrac{1}{a^2 \ + \ t^2}, \quad (a > 0)$	$\dfrac{\pi}{a} e^{-a\|\omega\|}$

8.5 Connection to Laplace transforms

If $g(t) = 0$ for $t < 0$ and $\overline{g}(s)$ has no poles in $Re(s) \geqslant 0$ then

$$G(\omega) \ = \ \mathcal{F}[g(t)] \ = \ \mathcal{L}[g(t)]_{s\,=\,i\omega} \ = \ \overline{g}(i\omega)$$

9 Laplace Transforms

$$\mathcal{L}f(t) \;=\; \overline{f}(s) \;=\; \int_0^\infty f(t)e^{-st}\,dt$$

9.1 Operational Form

$$f(t)H(t) \;=\; \overline{f}(s)\delta(t); \qquad \frac{1}{s} \;\equiv\; \int_0^t (\;)\,dt; \qquad s \;\equiv\; \frac{d}{dt}$$

9.2 Functional Relationships

	$f(t)$	$\overline{f}(s)$
	$f'(t)$	$s\overline{f}(s) - f(0)$
	$f''(t)$	$s^2\overline{f}(s) - [sf(0) + f'(0)]$
	$f^{(n)}(t)$	$s^n\overline{f}(s) - \left[s^{n-1}f(0) + s^{n-2}f'(0) + \cdots + f^{(n-1)}(0)\right]$
	$\int_0^t f(t)dt$	$\frac{1}{s}\overline{f}(s)$
Damping	$e^{-kt}f(t)$	$\overline{f}(s\;+\;k)$
Delay	$f(t-T)H(t-T)$	$e^{-sT}\overline{f}(s)$
Scale change	$f(kt)$	$\frac{1}{k}\overline{f}\,(s/k)$
Periodic, period T	$f(t)$	$\overline{f}(s) = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$
Convolution	$f(t) * g(t) = \int_0^t f(r)g(t-r)dr = \int_0^t f(t-r)g(r)dr$	$\overline{f}(s)\overline{g}(s)$
	$\int_0^t \cdots \int_0^t f(t)(dt)^n$	$\frac{1}{s^n}\overline{f}(s)$
	$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n}\left(\overline{f}(s)\right)$
	$\frac{1}{t^n}f(t)$	$\int_s^\infty \cdots \int_s^\infty \overline{f}(s)(ds)^n$

9.3 A second independent variable

$f(t,x)$	$\overline{f}(s,x) = \int_0^\infty f(t,x)e^{-st}\,dt$
$\dfrac{\partial}{\partial t}f(t,x)$	$s\overline{f}(s,x) - f(0,x)$
$\dfrac{\partial^2}{\partial t^2}f(t,x)$	$s^2\overline{f}(s,x) - \left[sf(0,x) + \dfrac{\partial f}{\partial t}(0,x)\right]$
$\dfrac{\partial}{\partial x}f(t,x)$	$\dfrac{\partial}{\partial x}\overline{f}(s,x)$
$\int_{t=0}^{t} f(t,x)\,dt$	$\dfrac{1}{s}\overline{f}(s,x)$
$\int_{x=a}^{b} f(t,x)\,dx$	$\int_{x=a}^{b} \overline{f}(s,x)\,dx$

9.4 Limiting Values

$$\left.\begin{aligned} \lim_{t\to+0} f(t) &= \lim_{s\to+\infty} s\overline{f}(s) \\ \lim_{t\to+\infty} f(t) &= \lim_{s\to+0} s\overline{f}(s) \\ \int_0^\infty f(t)\,dt &= \lim_{s\to+0} \overline{f}(s) \end{aligned}\right\} \quad \text{(If limits and integral exist)}$$

9.5 Inversion Integral

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\overline{f}(s) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{f}(s)e^{st}\,ds \\ &= \sum \operatorname{Res} \overline{f}(s)e^{st} \end{aligned}$$

if $\overline{f}(s)$ analytic except for poles in LH half-plane.

9.6 Laplace Transforms Of Simple Functions

$f(t)$	$\overline{f}(s)$
$\delta(t) = u_0(t)$	1
$H(t) = u_{-1}(t) = U(t)$	$\dfrac{1}{s}$
$tU(t) = u_{-2}(t)$	$\dfrac{1}{s^2}$
t^n	$\dfrac{\Gamma(n+1)}{s^{n+1}}$ for $n > -1$ and $\dfrac{n!}{s^{n+1}}$ for n positive integer
e^{-kt}	$\dfrac{1}{s+k}$
te^{-kt}	$\dfrac{1}{(s+k)^2}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$e^{-kt}\sin \omega t$	$\dfrac{\omega}{(s+k)^2+\omega^2}$
$e^{-kt}\cos \omega t$	$\dfrac{s+k}{(s+k)^2+\omega^2}$
Square Wave Period $2T$ $f(t) = \begin{cases} 1 & 0 < t < T \\ -1 & T < t < 2T \end{cases}$	$\dfrac{1}{s}\dfrac{1-e^{-sT}}{1+e^{-sT}} = \dfrac{1}{s}\tanh sT/2$
Triangular Period $2T$ $f(t) = \begin{cases} t/T & 0 < t < T \\ \dfrac{-(t-2T)}{T} & T < t < 2T \end{cases}$	$\dfrac{1}{Ts^2}\dfrac{1-e^{-sT}}{1+e^{-sT}} = \dfrac{1}{Ts^2}\tanh sT/2$
Saw Tooth Period T $f(t) = t/T \qquad 0 < t < T$	$\dfrac{1}{Ts^2} - \dfrac{e^{-sT}}{s(1-e^{-sT})}$
Rectified Waves $f(t) = \lvert\sin \omega t\rvert$	$\dfrac{\omega}{s^2+\omega^2}\dfrac{1+e^{-s\pi/\omega}}{1-e^{-s\pi/\omega}} = \dfrac{\omega}{s^2+\omega^2}\coth\dfrac{s\pi}{2\omega}$
Angular Frequency ω $f(t) = \begin{cases} \sin \omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$	$\dfrac{\omega}{s^2+\omega^2}\dfrac{1}{1-e^{-s\pi/\omega}}$

9.7 Inverse Laplace Transforms

$\overline{f}(s)$	$f(t)$
$\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$
$\dfrac{1}{(s+a)^2}$	$t\,e^{-at}$
$\dfrac{s}{s^2+\omega^2}$	$\cos\omega t$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{1}{\omega}\sin\omega t$
$\dfrac{1}{s^2(s+a)}$	$\dfrac{1}{a^2}(at-1+e^{-at})$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1}{\omega^2}(1-\cos\omega t)$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{1}{\omega^3}(\omega t-\sin\omega t)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}t\sin\omega t$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(\sin\omega t-\omega t\cos\omega t)$
$\dfrac{1}{s(s^2+\omega^2)^2}$	$\dfrac{1}{\omega^4}\left(1-\cos\omega t-\dfrac{\omega t}{2}\sin\omega t\right)$
$\dfrac{s}{(s^2+a^2)(s^2+\omega^2)}$	$\dfrac{1}{a^2-\omega^2}(\cos\omega t-\cos at)$
$\dfrac{1}{(s^2+a^2)(s^2+\omega^2)}$	$\dfrac{1}{a\omega(a^2-\omega^2)}(a\sin\omega t-\omega\sin at)$
$\dfrac{1}{s(s^2+a^2)(s^2+\omega^2)}$	$\dfrac{1}{a^2\omega^2}\left\{1-\dfrac{1}{a^2-\omega^2}(a^2\cos\omega t-\omega^2\cos at)\right\}$

(the following formulae only apply $w^2=c^2-k^2<0$).

$\dfrac{s}{s^2+2ks+c^2}$	$e^{-kt}\left(\cos\omega t-\dfrac{k}{\omega}\sin\omega t\right)$
$\dfrac{1}{s^2+2ks+c^2}$	$\dfrac{1}{\omega}e^{-kt}\sin\omega t$
$\dfrac{1}{s(s^2+2ks+c^2)}$	$\dfrac{1}{c^2}\left\{1-e^{-kt}\left(\cos\omega t+\dfrac{k}{\omega}\sin\omega t\right)\right\}$
$\dfrac{1}{s^2(s^2+2ks+c^2)}$	$\dfrac{1}{c^4}\left\{c^2t-2k+e^{-kt}\left(2k\cos\omega t+\dfrac{k^2-\omega^2}{\omega}\sin\omega t\right)\right\}$
$\dfrac{s}{(s+a)(s^2+2ks+c^2)}$	$\dfrac{1}{A}\left\{-ae^{-at}+e^{-kt}\left(a\cos\omega t+\dfrac{c^2-ak}{\omega}\sin\omega t\right)\right\}$

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$\overline{f}(s)$	$f(t)$
$\dfrac{1}{(s+a)(s^2+2ks+c^2)}$	$\dfrac{1}{A}\left\{e^{-at}-e^{-kt}\left(\cos\omega t+\dfrac{k-a}{\omega}\sin\omega t\right)\right\}$
$\dfrac{1}{s(s+a)(s^2+2ks+c^2)}$	$\dfrac{1}{ac^2}+\dfrac{1}{A}\left\{-\dfrac{e^{-at}}{a}-e^{-kt}\left(B\cos\omega t+\dfrac{kB+1}{\omega}\sin\omega t\right)\right\}$
	where $A=(a-k)^2+\omega^2$ and $B=(a-2k)/c^2$

9.8 Laplace Transforms Of Special Functions

$\overline{f}(s)$	$f(t)$	
$e^{-k\sqrt{s}}$	$\dfrac{k}{2\sqrt{\pi t^3}}\exp\left(\dfrac{-k^2}{4t}\right)$	$(k>0)$
$\dfrac{1}{s}e^{-k\sqrt{s}}$	$\operatorname{erfc}\left(\dfrac{k}{2\sqrt{t}}\right)$	$(k\geq 0)$
$\dfrac{1}{\sqrt{s}}e^{-k\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}\exp\left(\dfrac{-k^2}{4t}\right)$	$(k\geq 0)$
$\dfrac{1}{\sqrt{s^3}}e^{-k\sqrt{s}}$	$2\sqrt{\dfrac{t}{\pi}}\exp\left(\dfrac{-k^2}{4t}\right)-k\ \operatorname{erfc}\left(\dfrac{k}{2\sqrt{t}}\right)$	$(k\geq 0)$
$\dfrac{e^{-b\sqrt{s+a^2}}}{s}$	$\dfrac{1}{2}\left[e^{-ab}\operatorname{erfc}\left(\dfrac{b-2at}{2\sqrt{t}}\right)+e^{ab}\operatorname{erfc}\left(\dfrac{b+2at}{2\sqrt{t}}\right)\right]$	
$\dfrac{e^{-k\sqrt{s}}}{b+\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}\exp\left(\dfrac{-k^2}{4t}\right)-be^{kb+b^2t}\operatorname{erfc}\left\{\dfrac{k+2bt}{2\sqrt{t}}\right\}$	$(k\geqslant 0, b\geqslant 0)$
$\dfrac{e^{-k\sqrt{s}}}{s+b\sqrt{s}}$	$e^{kb+b^2t}\operatorname{erfc}\left\{\dfrac{k+2bt}{2\sqrt{t}}\right\}$	$(k\geqslant 0, b\geqslant 0)$
$\dfrac{e^{-k\sqrt{s}}}{s(b+\sqrt{s})}$	$\dfrac{1}{b}\operatorname{erfc}\left(\dfrac{k}{2\sqrt{t}}\right)-\dfrac{1}{b}e^{kb+b^2t}\operatorname{erfc}\left\{\dfrac{k+2bt}{2\sqrt{t}}\right\}$	$(k\geqslant 0, b\geqslant 0)$
$K_0(a\sqrt{s})$	$\dfrac{1}{2t}\exp\left(\dfrac{-a^2}{4t}\right)$	
$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	
$\dfrac{\left\{\sqrt{s^2+a^2}-s\right\}^n}{a^n\sqrt{s^2+a^2}}$	$J_n(at), \qquad (n>-1)$	
$\dfrac{1}{\sqrt{s^2-a^2}}$	$I_0(at)$	
$\dfrac{\left\{s-\sqrt{s^2-a^2}\right\}^n}{a^n\sqrt{s^2-a^2}}$	$I_n(at), \qquad (n>-1)$	
$\dfrac{b^n e^{-b/s}}{s^{n+1}}$	$(bt)^{n/2}J_n(2\sqrt{bt}) \qquad (n>-1)$	

10 Vector Formulae

10.1 Scalar Product (Dot product)

$$\mathbf{a} \cdot \mathbf{b} \;=\; ab\cos\theta \;=\; a_1 b_1 + a_2 b_2 + a_3 b_3$$

10.2 Vector Product (Cross product)

$$\mathbf{a} \times \mathbf{b} \;=\; ab\sin\theta\,\widehat{\mathbf{n}} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

10.3 Triple Products

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \\ &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

10.4 Vector Calculus

$$\nabla \;\equiv\; \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \qquad \text{(for cartesian coordinates)}$$

$$\mathrm{grad}\;\phi \;\equiv\; \nabla\phi$$

$$\mathrm{div}\;\mathbf{A} \;\equiv\; \nabla \cdot \mathbf{A}$$

$$\mathrm{curl}\;\mathbf{A} \;\equiv\; \nabla \times \mathbf{A}$$

$$\nabla(\phi\psi) \;=\; \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\phi\mathbf{A}) \;=\; \phi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\phi$$

$$\nabla\times(\phi\mathbf{A}) \;=\; \phi\nabla\times\mathbf{A} + (\nabla\phi)\times\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) \;=\; \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) \;=\; (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) \;=\; \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\nabla\cdot\nabla\phi \;\equiv\; \nabla^2\phi$$

$$\nabla\cdot(\nabla\times\mathbf{A}) \;=\; 0$$

$$\nabla\times(\nabla\phi) \;=\; \mathbf{0}$$

$$\nabla\times(\nabla\times\mathbf{A}) \;=\; \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\mathbf{A}\times(\nabla\times\mathbf{A}) \;=\; \nabla\left(\frac{1}{2}\mathbf{A}^2\right) - (\mathbf{A}\cdot\nabla)\mathbf{A}$$

10.5 Integral Theorems

Divergence Theorem

$$\begin{aligned}\int_{\mathbf{S}}\mathbf{A}\cdot d\mathbf{S} \;&=\; \int_V \nabla\cdot\mathbf{A}\; dV\\ \int_{\mathbf{S}}\phi\; d\mathbf{S} \;&=\; \int_V \nabla\phi\; dV\end{aligned}$$

Stoke's Theorem

$$\begin{aligned}\oint_C \mathbf{A}\cdot d\mathbf{r} \;&=\; \int_{\mathbf{S}}(\nabla\times\mathbf{A})\cdot d\mathbf{S}\\ \oint_C \phi\; d\mathbf{r} \;&=\; \int_{\mathbf{S}} d\mathbf{S}\times\nabla\phi\end{aligned}$$

Green's Theorems

$$\begin{aligned}\int_V (\nabla\phi)\cdot(\nabla\psi)dV \;+\; \int_V \phi\nabla^2\psi\; dV \;&=\; \int_{\mathbf{S}}\phi\nabla\psi\cdot d\mathbf{S}\\ &=\; \int_S \phi\frac{\partial\psi}{\partial n}\; dS\end{aligned}$$

$$\int_V \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right) dV \;=\; \int_{\mathbf{S}}(\psi\nabla\phi - \phi\nabla\psi)\cdot d\mathbf{S}$$

11 Curvilinear Coordinates

General orthogonal co-ordinates (u_1, u_2, u_3)

$$\nabla\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\right)$$

$$\text{div } \mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right\}$$

$$\text{curl } \mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right\}$$

Line element: $\delta s_1 = h_1\delta u_1$, $\delta s_2 = h_2\delta u_2$, $\delta s_3 = h_3\delta u_3$.

Surface element: $\delta S_1 = h_2h_3\delta u_2\delta u_3$, $\delta S_2 = h_3h_1\delta u_3\delta u_1$, $\delta S_3 = h_1h_2\delta u_1\delta u_2$.

Volume element: $\delta V = h_1h_2h_3\delta u_1\delta u_2\delta u_3$.

Co-ordinates	u_1	u_2	u_3	h_1	h_2	h_3	Cartesian/polar relation		
Rectangular	x	y	z	1	1	1	x	y	z
Cylindrical	ρ	ϕ	z	1	ρ	1	$\rho\cos\phi$	$\rho\sin\phi$	z
Spherical	r	θ	ϕ	1	r	$r\sin\theta$	$r\sin\theta\cos\phi$	$r\sin\theta\sin\phi$	$r\cos\theta$

Form for $\nabla^2 U$ (**scalars only**);

Cylindrical Polars: $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2} + \frac{\partial^2 U}{\partial z^2}$$

Spherical Polars: $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

11.1 Index Notation Formulae

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & (ijk) \text{ cyclic in } (123) \\ -1 & (ijk) \text{ anticyclic in } (123) \\ 0 & \text{otherwise} \end{cases}$$

$$\epsilon_{kij}\epsilon_{kpq} \equiv \epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$$

12 Legendre Polynomials

12.1 Under standard normalisation $P_n(1) = 1$

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}, \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x. \\ &\vdots \\ (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n \in \mathbb{N}. \end{aligned}$$

12.2 Orthogonality properties on the interval $[-1,\, 1]$:

$$\langle P_n,\, P_m \rangle = 0, \qquad m \neq n.$$

where $\qquad \langle f,\, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx,$

12.3 Legendre polynomials with orthogonal normalisation

$$
\begin{aligned}
\phi_0(x) &= \frac{1}{\sqrt{2}} \\
\phi_1(x) &= \sqrt{\frac{3}{2}}x \\
\phi_2(x) &= \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) \\
&\vdots \\
\phi_n &= \frac{P_n(x)}{\|P_n\|}
\end{aligned}
$$

13 Orthogonal Polynomials (for equidistant abscissae)

n	3		4			5				6				
f_i	f_1	f_2	f_1	f_2	f_3	f_1	f_2	f_3	f_4	f_1	f_2	f_3	f_4	f_5
										-5	$+5$	-5	$+1$	-1
						-2	$+2$	-1	$+1$					
			-3	$+1$	-1					-3	-1	$+7$	-3	$+5$
	-1	$+1$				-1	-1	$+2$	-4					
			-1	-1	$+3$					-1	-4	$+4$	$+2$	-10
	0	-2				0	-2	0	$+6$					
			$+1$	-1	-3					$+1$	-4	-4	$+2$	$+10$
	$+1$	$+1$				$+1$	-1	-2	-4					
			$+3$	$+1$	$+1$					$+3$	-1	-7	-3	-5
						$+2$	$+2$	$+1$	$+1$					
										$+5$	$+5$	$+5$	$+1$	$+1$
$\sum f_i^2$	2	6	20	4	20	10	14	10	70	70	84	180	28	252
λ_i	1	3	2	1	$\frac{10}{3}$	1	1	$\frac{5}{6}$	$\frac{35}{12}$	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{7}{12}$	$\frac{21}{10}$

7					8					9				
f_1	f_2	f_3	f_4	f_5	f_1	f_2	f_3	f_4	f_5	f_1	f_2	f_3	f_4	f_5
0	-4	0	$+6$	0						0	-20	0	$+18$	0
					$+1$	-5	-3	$+9$	$+15$					
$+1$	-3	-1	$+1$	$+5$						$+1$	-17	-9	$+9$	$+9$
					$+3$	-3	-7	-3	$+17$					
$+2$	0	-1	-7	-4						$+2$	-8	-13	-11	$+4$
					$+5$	$+1$	-5	-13	-23					
$+3$	$+5$	$+1$	$+3$	$+1$						$+3$	$+7$	-7	-21	-11
					$+7$	$+7$	$+7$	$+7$	$+7$					
										$+4$	$+28$	$+14$	$+14$	$+4$
28	84	6	154	84	168	168	264	616	2184	60	2772	990	2002	464
1	1	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{7}{20}$	2	1	$\frac{2}{3}$	$\frac{7}{12}$	$\frac{7}{10}$	1	3	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{3}{20}$

$$
\begin{aligned}
f_1(x) &= \lambda_1(x) \\
f_2(x) &= \lambda_2\left\{x^2 - \frac{1}{12}(n^2-1)\right\} \\
f_3(x) &= \lambda_3\left\{x^3 - \frac{1}{20}(3n^2-7)x\right\} \\
f_4(x) &= \lambda_4\left\{x^4 - \frac{1}{14}(3n^2-13)x^2 + \frac{3}{560}(n^2-1)(n^2-9)\right\} \\
f_5(x) &= \lambda_5\left\{x^5 - \frac{5}{18}(n^2-7)x^3 + \frac{1}{1008}(15n^4-230n^2+407)x\right\}
\end{aligned}
$$

14 Physical constants

Note: Numerical values have been rounded to four significant figures.

Quantity	Symbol	Value	Unit	Dimensions
Atomic mass unit	u	1.661×10^{-27}	kg	M
Avogadro constant	$\mathrm{N_A}$	6.022×10^{23}	$\mathrm{mol^{-1}}$	
Bohr magneton $(\mathrm{e}\hbar/2\mathrm{m_e})$	μ_B	9.274×10^{-24}	$\mathrm{J\,T^{-1}}$	$\mathrm{I\,L^2}$
Bohr radius $(4\pi\hbar^2/\mu_\mathrm{o}\mathrm{c^2e^2m_e})$	$\mathrm{a_o}$	5.292×10^{-11}	m	L
Boltzmann constant	k	1.381×10^{-23}	$\mathrm{J\,K^{-1}}$	$\mathrm{ML^2T^{-2}}\theta^{-1}$
Charge of electron (magnitude)	e	1.602×10^{-19}	C	$\mathrm{I\,T}$
Charge (magnitude)/rest mass ratio (electron)	$\mathrm{e/m_e}$	1.759×10^{11}	$\mathrm{C\,kg^{-1}}$	$\mathrm{I\,M^{-1}T}$
Fine-structure constant $(\mu_\mathrm{o}\mathrm{ce^2/2h})$	α	7.292×10^{-3}		
	$1/\alpha$	137.0		
Gravitational constant	G	6.672×10^{-11}	$\mathrm{Nm^2kg^{-2}}$	$\mathrm{M^{-1}L^3T^{-2}}$
Mass ratio, $\mathrm{m_p/m_e}$	$\mathrm{m_p/m_e}$	1836		
Molar gas constant	R	8.314	$\mathrm{J\,mol^{-1}K^{-1}}$	$\mathrm{ML^2T^{-2}}\theta^{-1}$
Molar volume (ideal gas, STP)	$\mathrm{V_m}$	2.241×10^{-2}	$\mathrm{m^3}$	$\mathrm{L^3}$
Permeability of vacuum	μ_o	$4\pi \times 10^{-7}$	$\mathrm{Hm^{-1}}$	$\mathrm{I^{-2}MLT^{-2}}$
Permittivity of vacuum $(1/\mu_\mathrm{o}\mathrm{c^2})$	ε_o	8.854×10^{-12}	$\mathrm{Fm^{-1}}$	$\mathrm{I^2M^{-1}L^{-3}T^4}$
	$4\pi\varepsilon_\mathrm{o}$	1.113×10^{-10}	$\mathrm{Fm^{-1}}$	$\mathrm{I^2M^{-1}L^{-3}T^4}$
Planck constant	h	6.626×10^{-34}	$\mathrm{J\,s}$	$\mathrm{ML^2T^{-1}}$
	$\hbar$	1.055×10^{-34}	$\mathrm{J\,s}$	$\mathrm{ML^2T^{-1}}$
Rest mass of electron	$\mathrm{m_e}$	9.110×10^{-31}	kg	M
Rest mass of proton	$\mathrm{m_p}$	1.673×10^{-27}	kg	M
Speed of light in vacuum	c	2.998×10^{8}	$\mathrm{m\,s^{-1}}$	$\mathrm{LT^{-1}}$
Stefan-Boltzmann constant $(2\pi^5\mathrm{k^4/15h^3c^2})$	σ	5.670×10^{-8}	$\mathrm{Wm^{-2}K^{-4}}$	$\mathrm{MT^{-3}}\theta^{-4}$

15 Astrophysical constants

Symbol	Quantity	Value
$\mathrm{M}_\odot$	Mass of Sun	$1.989 \times 10^{30}\,\mathrm{kg}$
$\mathrm{R}_\odot$	Radius of Sun	$6.957 \times 10^{8}\,\mathrm{m}$
$\mathrm{L}_\odot$	Bolometric luminosity of Sun	$3.828 \times 10^{26}\,\mathrm{W}$
		$3.828 \times 10^{33}\,\mathrm{erg\,s^{-1}}$
$M_{\mathrm{bol}}^{\odot}$	Absolute bolometric magnitude of Sun	$+4.74$
$M_{\mathrm{vis}}^{\odot}$	Absolute visual magnitude of Sun	$+4.83$
$T_{\mathrm{eff}}^{\odot}$	Effective temperature of Sun	$5770\,\mathrm{K}$
	Spectral type of Sun	G2 V
$\mathrm{M_J}$	Mass of Jupiter	$1.898 \times 10^{27}\,\mathrm{kg}$
$\mathrm{R_J}$	Equatorial radius of Jupiter	$71\,492\,\mathrm{km}$
$\mathrm{M}_\oplus$	Mass of Earth	$5.972 \times 10^{24}\,\mathrm{kg}$
$\mathrm{R}_\oplus$	Equatorial radius of Earth	$6378\,\mathrm{km}$
$\mathrm{M}_{☾}$	Mass of Moon	$7.348 \times 10^{22}\,\mathrm{kg}$
$\mathrm{R}_{☾}$	Equatorial radius of Moon	$1738\,\mathrm{km}$
	Sidereal year	$3.156 \times 10^{7}\,\mathrm{s}$
au	Astronomical Unit	$1.496 \times 10^{11}\,\mathrm{m}$
ly	Light year	$9.461 \times 10^{15}\,\mathrm{m}$
pc	Parsec	$3.086 \times 10^{16}\,\mathrm{m}$
Jy	Jansky	$10^{-26}\,\mathrm{W\,m^{-2}Hz^{-1}}$
H_0	Hubble constant (Planck 2016)	$67.8 \pm 0.9\,\mathrm{km\,s^{-1}Mpc^{-1}}$
G	Gravitational constant	$6.672 \times 10^{-11}\,\mathrm{N\,m^{2}kg^{-2}}$
		$4.301 \times 10^{-9}\,\mathrm{km^{2}MpcM_\odot^{-1}s^{-2}}$
$\mathrm{D_{GC}}$	Distance of Galactic Centre	$8.3 \pm 0.4\,\mathrm{kpc}$

15.1 Rest-wavelengths of Hydrogen lines

Line	Wavelength (nm)
Lyα	121.57
Lyβ	102.57
Lyγ	97.254
Hα	656.3
Hβ	486.1
Hγ	434.0
Paα	1875
Paβ	1282
Paγ	1094

Book B

Statistical tables

1 Some Common Families of Distributions

1.1 Discrete Distributions

Distribution	Point probability	Mean	Variance	Probability generating function
Binomial (n, p)	$\binom{n}{r} p^r (1-p)^{n-r}$, $r = 0, 1, 2, \cdots n$	np	$np(1-p)$	$(1-p+pz)^n$
Poisson (λ)	$\frac{e^{-\lambda}\lambda^r}{r!}$, $r = 0, 1, 2, \cdots$	λ	λ	$e^{\lambda(z-1)}$
Negative Binomial (k, p)	$\binom{k+r-1}{r} p^k (1-p)^r$, $r = 0, 1, 2, \cdots$	$\frac{k(1-p)}{p}$	$\frac{k(1-p)}{p^2}$	$\left(\frac{p}{1-z+pz}\right)^k$
Geometric (p)	is the same as Negative Binomial $(1, p)$			
Hyper-geometric (N_1, N_2, n)	$\frac{\binom{N_1}{r}\binom{N_2}{n-r}}{\binom{N_1+N_2}{n}}$, $r = 0, 1, 2, \cdots$ $\cdots, \min(n, N_1)$, $N_1 < N_2$.	$\frac{nN_1}{N_1+N_2}$	$\frac{nN_1N_2(N_1+N_2-n)}{(N_1+N_2)^2(N_1+N_2-1)}$	

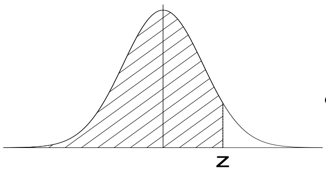
1.2 Continuous Distributions

Distribution	Density Function	Mean	Variance	Moment generating function
Uniform (a,b)	$\frac{1}{b-a}$, $(a<x<b)$	$\frac{1}{2}(a+b)$	$\frac{1}{12}(b-a)^2$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Beta (r,s)	$\frac{\Gamma(r+s)x^{r-1}(1-x)^{s-1}}{\Gamma(r)\Gamma(s)}$	$\frac{r}{r+s}$	$\frac{rs}{(r+s)^2(r+s+1)}$	
Gamma (s,α)	$\frac{\alpha^s x^{s-1}e^{-\alpha x}}{\Gamma(s)}$, $(x>0)$	$\frac{s}{\alpha}$	$\frac{s}{\alpha^2}$	$\left(\frac{\alpha}{\alpha-t}\right)^s$
Exponential (α)	is the same as Gamma $(1,\alpha)$			
χ^2 Chi-squared (n)	is the same as Gamma $\left(\frac{n}{2},\frac{1}{2}\right)$			
Normal (μ,σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2	$e^{\mu t+\frac{1}{2}\sigma^2t^2}$

1.3 p-variate normal distribution (μ,Σ)

- Density function: $(2\pi)^{-\frac{1}{2}p}\,|\Sigma|^{-\frac{1}{2}}\,e^{-\frac{1}{2}\{(x-\mu)^T\Sigma^{-1}(x-\mu)\}}$ if Σ^{-1} exists.
- Mean: μ
- Variance: Σ
- Moment Generating Function: $e^{(t^T\mu+\frac{1}{2}t^T\Sigma t)}$.

2 The Normal Distribution Function $\Phi(z)$



$$\Phi(z) \;=\; P(Z < z) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt$$

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993

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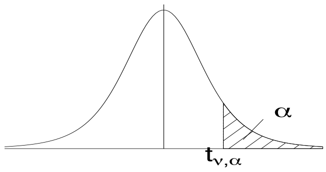
z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

2.1 Percentage Points of the Normal Distribution

The value is that at which the upper tail probability equals the product of the row and column labels, rounded up in the 3rd D.P.

	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
5.0	0.000	1.645	2.576	3.291	3.891	4.417	4.892	5.327	5.731	6.109
2.5	0.674	1.960	2.807	3.481	4.056	4.565	5.026	5.451	5.847	6.219
1.0	1.282	2.326	3.090	3.719	4.265	4.753	5.199	5.612	5.998	6.361

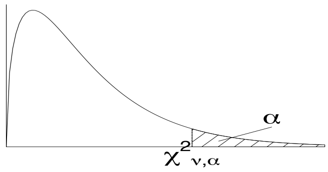
3 Percentage Points of Student's t-Distribution



The value given is $t_{\nu,\alpha}$ where $P(t_\nu > t_{\nu,\alpha}) = \alpha$ for Student's t-distribution on ν degrees of freedom. Note that $P\left(|t_\nu| > t_{\nu,\alpha/2}\right) = \alpha$.

	Two-tailed probabilities				
$\alpha/2$	0.5	0.1	0.05	0.02	0.01
	One-tailed probabilities				
α	0.25	0.05	0.025	0.01	0.005
ν					
1	1.000	6.314	12.706	31.821	63.657
2	0.816	2.920	4.303	6.965	9.925
3	0.765	2.353	3.182	4.541	5.841
4	0.741	2.132	2.776	3.747	4.604
5	0.727	2.015	2.571	3.365	4.032
6	0.718	1.943	2.447	3.143	3.707
7	0.711	1.895	2.365	2.998	3.499
8	0.706	1.860	2.306	2.896	3.355
9	0.703	1.833	2.262	2.821	3.250
10	0.700	1.812	2.228	2.764	3.169
11	0.697	1.796	2.201	2.718	3.106
12	0.695	1.782	2.179	2.681	3.055
13	0.694	1.771	2.160	2.650	3.012
14	0.692	1.761	2.145	2.624	2.977
15	0.691	1.753	2.131	2.602	2.947
16	0.690	1.746	2.120	2.583	2.921
17	0.689	1.740	2.110	2.567	2.898
18	0.688	1.734	2.101	2.552	2.878
19	0.688	1.729	2.093	2.539	2.861
20	0.687	1.725	2.086	2.528	2.845
21	0.686	1.721	2.080	2.518	2.831
22	0.686	1.717	2.074	2.508	2.819
23	0.685	1.714	2.069	2.500	2.807
24	0.685	1.711	2.064	2.492	2.797
25	0.684	1.708	2.060	2.485	2.787
26	0.684	1.706	2.056	2.479	2.779
27	0.684	1.703	2.052	2.473	2.771
28	0.683	1.701	2.048	2.467	2.763
29	0.683	1.699	2.045	2.462	2.756
30	0.683	1.697	2.042	2.457	2.750
35	0.682	1.690	2.030	2.438	2.724
40	0.681	1.684	2.021	2.423	2.704
45	0.680	1.679	2.014	2.412	2.690
50	0.679	1.676	2.009	2.403	2.678
60	0.679	1.671	2.000	2.390	2.660
∞	0.674	1.645	1.960	2.326	2.576

4 Percentage Points of the χ^2-distribution

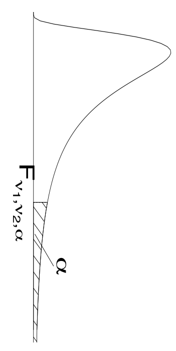


The value given is $\chi^2_{\nu,\alpha}$ where $P(\chi^2_\nu > \chi^2_{\nu,\alpha}) = \alpha$ for the χ^2 distribution on ν degrees of freedom.

α	0.99	0.975	0.95	0.5	0.1	0.05	0.025	0.01
ν								
1	0.000157	0.000982	0.00393	0.455	2.706	3.841	5.024	6.635
2	0.0201	0.0506	0.103	1.386	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	2.366	6.251	7.815	9.348	11.345
4	0.297	0.484	0.711	3.357	7.779	9.488	11.143	13.277
5	0.554	0.831	1.145	4.351	9.236	11.070	12.833	15.086
6	0.872	1.237	1.635	5.348	10.645	12.592	14.449	16.812
7	1.239	1.690	2.167	6.346	12.017	14.067	16.013	18.475
8	1.646	2.180	2.733	7.344	13.362	15.507	17.535	20.090
9	2.088	2.700	3.325	8.343	14.684	16.919	19.023	21.666
10	2.558	3.247	3.940	9.342	15.987	18.307	20.483	23.209
11	3.053	3.816	4.575	10.341	17.275	19.675	21.920	24.725
12	3.571	4.404	5.226	11.340	18.549	21.026	23.337	26.217
13	4.107	5.009	5.892	12.340	19.812	22.362	24.736	27.688
14	4.660	5.629	6.571	13.339	21.064	23.685	26.119	29.141
15	5.229	6.262	7.261	14.339	22.307	24.996	27.488	30.578
16	5.812	6.908	7.962	15.338	23.542	26.296	28.845	32.000
17	6.408	7.564	8.672	16.338	24.769	27.587	30.191	33.409
18	7.015	8.231	9.390	17.338	25.989	28.869	31.526	34.805
19	7.633	8.907	10.117	18.338	27.204	30.144	32.852	36.191
20	8.260	9.591	10.851	19.337	28.412	31.410	34.170	37.566
21	8.897	10.283	11.591	20.337	29.615	32.671	35.479	38.932
22	9.542	10.982	12.338	21.337	30.813	33.924	36.781	40.289
23	10.196	11.689	13.091	22.337	32.007	35.172	38.076	41.638
24	10.856	12.401	13.848	23.337	33.196	36.415	39.364	42.980
25	11.524	13.120	14.611	24.337	34.382	37.652	40.646	44.314
26	12.198	13.844	15.379	25.336	35.563	38.885	41.923	45.642
27	12.879	14.573	16.151	26.336	36.741	40.113	43.195	46.963
28	13.565	15.308	16.928	27.336	37.916	41.337	44.461	48.278
29	14.256	16.047	17.708	28.336	39.087	42.557	45.722	49.588
30	14.953	16.791	18.493	29.336	40.256	43.773	46.979	50.892
40	22.164	24.433	26.509	39.335	51.805	55.758	59.342	63.691
50	29.707	32.357	34.764	49.335	63.167	67.505	71.420	76.154
60	37.485	40.482	43.188	59.335	74.397	79.082	83.298	88.379
80	53.540	57.153	60.391	79.334	96.578	101.879	106.629	112.329
100	70.065	74.222	77.929	99.334	118.498	124.342	129.561	135.807

For $\nu > 100$, $\sqrt{2\chi^2_\nu} - \sqrt{2\nu - 1}$ is approximately distributed as a standard normal.

5 Percentage Points of the F-Distribution



The value given is $F_{\nu_1,\nu_2,\alpha}$ where $P(F_{\nu_1,\nu_2} > F_{\nu_1,\nu_2,\alpha}) = \alpha$ for the F-Distribution with degrees of freedom ν_1 (numerator) and ν_2 (denominator).

Upper 5% points

ν_1 / ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Upper 2.5% points

ν_1 / ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001.4	1005.6	1009.8	1014.0	1018.3
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00

Upper 1% points

ν_1 / ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	4052	4999	5403	5625	5764	5859	5928	5981	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Upper 0.5% points

ν_1 / ν_2	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	16211	19999	21615	22500	23056	23437	23715	23925	24091	24224	24426	24630	24836	24940	25044	25148	25253	25359	25464
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.45	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01	3.90
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	3.10	2.98
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73	2.61
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60	2.48
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50	2.38
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45	2.33
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41	2.29
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37	2.25
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33	2.21
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61	1.43
∞	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.36	2.19	2.00	1.90	1.79	1.67	1.53	1.36	1.00

6 Poisson Tables

Values of $P(r) = \dfrac{\mu^r e^{-\mu}}{r!}$

μ \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	…	21
0.02	0.980	0.020																	
0.04	0.961	0.038	0.001																
0.06	0.942	0.057	0.002																
0.08	0.923	0.074	0.003																
0.10	0.905	0.090	0.005																
0.15	0.861	0.129	0.010																
0.20	0.819	0.164	0.016	0.001															
0.25	0.779	0.195	0.024	0.002															
0.30	0.741	0.222	0.033	0.003															
0.35	0.705	0.247	0.043	0.005															
0.40	0.670	0.268	0.054	0.007	0.001														
0.45	0.638	0.287	0.065	0.010	0.001														
0.50	0.607	0.303	0.076	0.013	0.002														
0.55	0.577	0.317	0.087	0.016	0.002														
0.60	0.549	0.329	0.099	0.020	0.003														
0.65	0.522	0.339	0.110	0.024	0.004	0.001													
0.70	0.497	0.348	0.122	0.028	0.005	0.001													
0.75	0.472	0.354	0.133	0.033	0.006	0.001													
0.80	0.449	0.359	0.144	0.038	0.008	0.001													
0.85	0.427	0.363	0.154	0.044	0.009	0.002													
0.90	0.407	0.366	0.165	0.049	0.011	0.002													
0.95	0.387	0.367	0.175	0.055	0.013	0.002													
1.00	0.368	0.368	0.184	0.061	0.015	0.003	0.001												
1.10	0.333	0.366	0.201	0.074	0.020	0.004	0.001												
1.20	0.301	0.361	0.217	0.087	0.026	0.006	0.001												
1.30	0.273	0.354	0.230	0.100	0.032	0.008	0.002												
1.40	0.247	0.345	0.242	0.113	0.039	0.011	0.003	0.001											
1.50	0.223	0.335	0.251	0.126	0.047	0.014	0.004	0.001											
1.60	0.202	0.323	0.258	0.138	0.055	0.018	0.005	0.001											

μ	r																		
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...	21
1.70	0.183	0.311	0.264	0.150	0.064	0.022	0.006	0.001											
1.80	0.165	0.298	0.268	0.161	0.072	0.026	0.008	0.002											
1.90	0.150	0.284	0.270	0.171	0.081	0.031	0.010	0.003	0.001										
2.00	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.001										
2.10	0.122	0.257	0.270	0.189	0.099	0.042	0.015	0.004	0.001										
2.20	0.111	0.244	0.268	0.197	0.108	0.048	0.017	0.005	0.002										
2.30	0.100	0.231	0.265	0.203	0.117	0.054	0.021	0.007	0.002										
2.40	0.091	0.218	0.261	0.209	0.125	0.060	0.024	0.008	0.002	0.001									
2.50	0.082	0.205	0.257	0.214	0.134	0.067	0.028	0.010	0.003	0.001									
2.60	0.074	0.193	0.251	0.218	0.141	0.074	0.032	0.012	0.004	0.001									
2.70	0.067	0.181	0.245	0.220	0.149	0.080	0.036	0.014	0.005	0.001									
2.80	0.061	0.170	0.238	0.222	0.156	0.087	0.041	0.016	0.006	0.002									
2.90	0.055	0.160	0.231	0.224	0.162	0.094	0.045	0.019	0.007	0.002	0.001								
3.00	0.050	0.149	0.224	0.224	0.168	0.101	0.050	0.022	0.008	0.003	0.001								
3.10	0.045	0.140	0.216	0.224	0.173	0.107	0.056	0.025	0.010	0.003	0.001								
3.20	0.041	0.130	0.209	0.223	0.178	0.114	0.061	0.028	0.011	0.004	0.001								
3.30	0.037	0.122	0.201	0.221	0.182	0.120	0.066	0.031	0.013	0.005	0.002								
3.40	0.033	0.113	0.193	0.219	0.186	0.126	0.072	0.035	0.015	0.006	0.002	0.001							
3.50	0.030	0.106	0.185	0.216	0.189	0.132	0.077	0.039	0.017	0.007	0.002	0.001							
3.60	0.027	0.098	0.177	0.212	0.191	0.138	0.083	0.042	0.019	0.008	0.003	0.001							
3.70	0.025	0.091	0.169	0.209	0.193	0.143	0.088	0.047	0.022	0.009	0.003	0.001							
3.80	0.022	0.085	0.162	0.205	0.194	0.148	0.094	0.051	0.024	0.010	0.004	0.001							
3.90	0.020	0.079	0.154	0.200	0.195	0.152	0.099	0.055	0.027	0.012	0.005	0.002	0.001						

μ	r																			
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	· · ·		21
4.00	0.018	0.073	0.147	0.195	0.195	0.156	0.104	0.060	0.030	0.013	0.005	0.002	0.001							
4.10	0.017	0.068	0.139	0.190	0.195	0.160	0.109	0.064	0.033	0.015	0.006	0.002	0.001							
4.20	0.015	0.063	0.132	0.185	0.194	0.163	0.114	0.069	0.036	0.017	0.007	0.003	0.001							
4.30	0.014	0.058	0.125	0.180	0.193	0.166	0.119	0.073	0.039	0.019	0.008	0.003	0.001							
4.40	0.012	0.054	0.119	0.174	0.192	0.169	0.124	0.078	0.043	0.021	0.009	0.004	0.001							
4.50	0.011	0.050	0.112	0.169	0.190	0.171	0.128	0.082	0.046	0.023	0.010	0.004	0.002	0.001						
4.60	0.010	0.046	0.106	0.163	0.188	0.173	0.132	0.087	0.050	0.026	0.012	0.005	0.002	0.001						
4.70	0.009	0.043	0.100	0.157	0.185	0.174	0.136	0.091	0.054	0.028	0.013	0.006	0.002	0.001						
4.80	0.008	0.040	0.095	0.152	0.182	0.175	0.140	0.096	0.058	0.031	0.015	0.006	0.003	0.001						
4.90	0.007	0.036	0.089	0.146	0.179	0.175	0.143	0.100	0.061	0.033	0.016	0.007	0.003	0.001						
5.00	0.007	0.034	0.084	0.140	0.175	0.175	0.146	0.104	0.065	0.036	0.018	0.008	0.003	0.001						
5.10	0.006	0.031	0.079	0.135	0.172	0.175	0.149	0.109	0.069	0.039	0.020	0.009	0.004	0.002	0.001					
5.20	0.006	0.029	0.075	0.129	0.168	0.175	0.151	0.113	0.073	0.042	0.022	0.010	0.005	0.002	0.001					
5.30	0.005	0.026	0.070	0.124	0.164	0.174	0.154	0.116	0.077	0.045	0.024	0.012	0.005	0.002	0.001					
5.40	0.005	0.024	0.066	0.119	0.160	0.173	0.156	0.120	0.081	0.049	0.026	0.013	0.006	0.002	0.001					
5.50	0.004	0.022	0.062	0.113	0.156	0.171	0.157	0.123	0.085	0.052	0.029	0.014	0.007	0.003	0.001					
5.60	0.004	0.021	0.058	0.108	0.152	0.170	0.158	0.127	0.089	0.055	0.031	0.016	0.007	0.003	0.001					
5.70	0.003	0.019	0.054	0.103	0.147	0.168	0.159	0.130	0.092	0.059	0.033	0.017	0.008	0.004	0.001					
5.80	0.003	0.018	0.051	0.098	0.143	0.166	0.160	0.133	0.096	0.062	0.036	0.019	0.009	0.004	0.002	0.001				
5.90	0.003	0.016	0.048	0.094	0.138	0.163	0.160	0.135	0.100	0.065	0.039	0.021	0.010	0.005	0.002	0.001				

μ	r																					
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
6.00	0.002	0.015	0.045	0.089	0.134	0.161	0.161	0.138	0.103	0.069	0.041	0.023	0.011	0.005	0.002	0.001						
6.20	0.002	0.013	0.039	0.081	0.125	0.155	0.160	0.142	0.110	0.076	0.047	0.026	0.014	0.007	0.003	0.001						
6.40	0.002	0.011	0.034	0.073	0.116	0.149	0.159	0.145	0.116	0.082	0.053	0.031	0.016	0.008	0.004	0.002	0.001					
6.60	0.001	0.009	0.030	0.065	0.108	0.142	0.156	0.147	0.121	0.089	0.059	0.035	0.019	0.010	0.005	0.002	0.001					
6.80	0.001	0.008	0.026	0.058	0.099	0.135	0.153	0.149	0.126	0.095	0.065	0.040	0.023	0.012	0.006	0.003	0.001					
7.00	0.001	0.006	0.022	0.052	0.091	0.128	0.149	0.149	0.130	0.101	0.071	0.045	0.026	0.014	0.007	0.003	0.001	0.001				
7.20	0.001	0.005	0.019	0.046	0.084	0.120	0.144	0.149	0.134	0.107	0.077	0.050	0.030	0.017	0.009	0.004	0.002	0.001				
7.40	0.001	0.005	0.017	0.041	0.076	0.113	0.139	0.147	0.136	0.112	0.083	0.056	0.034	0.020	0.010	0.005	0.002	0.001				
7.60	0.001	0.004	0.014	0.037	0.070	0.106	0.134	0.145	0.138	0.117	0.089	0.061	0.039	0.023	0.012	0.006	0.003	0.001	0.001			
7.80	0.000	0.003	0.012	0.032	0.063	0.099	0.128	0.143	0.139	0.121	0.094	0.067	0.043	0.026	0.015	0.008	0.004	0.002	0.001			
8.00	0.000	0.003	0.011	0.029	0.057	0.092	0.122	0.140	0.140	0.124	0.099	0.072	0.048	0.030	0.017	0.009	0.005	0.002	0.001			
9.00	0.000	0.001	0.005	0.015	0.034	0.061	0.091	0.117	0.132	0.132	0.119	0.097	0.073	0.050	0.032	0.019	0.011	0.006	0.003	0.001	0.001	
10.00	0.000	0.000	0.002	0.008	0.019	0.038	0.063	0.090	0.113	0.125	0.125	0.114	0.095	0.073	0.052	0.035	0.022	0.013	0.007	0.004	0.002	0.001

7 Random Numbers

The below table presents a typical series of random numbers for the convenience of class exercises. For practical work, reference should be made to a more extensive series such as that in the Fisher and Yates statistical tables.

99050	30876	80821	14955	11495
08090	84688	36332	86858	73763
67619	00352	32735	59654	97851
63779	66008	02516	93874	67930
03259	72119	04769	95593	02754
92914	02066	97320	00328	51685
80001	70542	01530	63033	64384
37815	09824	86504	14817	74434
15897	74758	12779	69608	76893
06193	94893	24598	02714	69670
40134	12803	33942	46600	05681
88480	27598	48458	65639	08810
49989	94369	80429	97152	67613
62089	52111	92190	85413	95362
01675	12741	94334	86069	71353
04259	19768	47711	63262	06316
63859	63087	91886	43467	55595
17709	21642	56384	85699	24310
11727	83872	22553	17012	02949
02838	03160	92864	23985	63585

8 Wilcoxon Matched-Pairs Test

Critical values of T at Various Levels of Probability

	Level of significance for two-tailed test			
N	0.10	0.05	0.02	0.01
5	0	−−	−−	−−
6	2	0	−−	−−
7	3	2	0	−−
8	5	3	1	0
9	8	5	3	1
10	10	8	5	3
11	13	10	7	5
12	17	13	9	7
13	21	17	12	9
14	25	21	15	12
15	30	25	19	15
16	35	29	23	19
17	41	34	27	23
18	47	40	32	27
19	53	46	37	32
20	60	52	43	37
21	67	58	49	42
22	75	65	55	48
23	83	73	62	54
24	91	81	69	61
25	100	89	76	68
26	110	98	84	75
27	119	107	92	83
28	130	116	101	91
29	140	126	110	100
30	151	137	120	109
31	163	147	130	118
32	175	159	140	128
33	187	170	151	138
34	200	182	162	148
35	213	195	173	159

9 Mann-Whitney Test

1. Critical values of U for a **One-tailed** Test at $\alpha = 0.05$ or a **Two-tailed** Test at $\alpha = 0.10$

	n_1																			
n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1																				
2																				
3			0																	
4			0	1																
5		0	1	2	4															
6		0	2	3	5	7														
7		0	2	4	6	8	11													
8		1	3	5	8	10	13	15												
9		1	3	6	9	12	15	18	21											
10		1	4	7	11	14	17	20	24	27										
11		1	5	8	12	16	19	23	27	31	34									
12		2	5	9	13	17	21	26	30	34	38	42								
13		2	6	10	15	19	24	28	33	37	42	47	51							
14		2	7	11	16	21	26	31	36	41	46	51	56	61						
15		3	7	12	18	23	28	33	39	44	50	55	61	66	72					
16		3	8	14	19	25	30	36	42	48	54	60	65	71	77	83				
17		3	9	15	20	26	33	39	45	51	57	64	70	77	83	89	96			
18		4	9	16	22	28	35	41	48	55	61	68	75	82	88	95	102	109		
19	0	4	10	17	23	30	37	44	51	58	65	72	80	87	94	101	109	116	123	
20	0	4	11	18	25	32	39	47	54	62	69	77	84	92	100	107	115	123	130	138

2. Critical values of U for a **One-tailed** Test at $\alpha = 0.025$ or a **Two-tailed** Test at $\alpha = 0.05$

	n_1																			
n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1																				
2																				
3																				
4				0																
5			0	1	2															
6			1	2	3	5														
7			1	3	5	6	8													
8		0	2	4	6	8	10	13												
9		0	2	4	7	10	12	15	17											
10		0	3	5	8	11	14	17	20	23										
11		0	3	6	9	13	16	19	23	26	30									
12		1	4	7	11	14	18	22	26	29	33	37								
13		1	4	8	12	16	20	24	28	33	37	41	45							
14		1	5	9	13	17	22	26	31	36	40	45	50	55						
15		1	5	10	14	19	24	29	34	39	44	49	54	59	64					
16		1	6	11	15	21	26	31	37	42	47	53	59	64	70	75				
17		2	6	11	17	22	28	34	39	45	51	57	63	67	75	81	87			
18		2	7	12	18	24	30	36	42	48	55	61	67	74	80	86	93	99		
19		2	7	13	19	25	32	38	45	52	58	65	72	78	85	92	99	106	113	
20		2	8	13	20	27	34	41	48	55	62	69	76	83	90	98	105	112	119	127

Mann-Whitney Test (continued)

3. Critical values of U for a **One-tailed** Test at $\alpha = 0.01$ or a **Two-tailed** Test at $\alpha = 0.02$

	n_1																			
n_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1																				
2																				
3																				
4																				
5				0	1															
6				1	2	3														
7			0	1	3	4	6													
8			0	2	4	6	8	10												
9			1	3	5	7	9	11	14											
10			1	3	6	8	11	13	16	19										
11			1	4	7	9	12	15	18	22	25									
12			2	5	8	11	14	17	21	24	28	31								
13		0	2	5	9	12	16	20	23	27	31	35	39							
14		0	2	6	10	13	17	22	26	30	34	38	43	47						
15		0	3	7	11	15	19	24	28	33	37	42	47	51	56					
16		0	3	7	12	16	21	26	31	36	41	46	51	56	61	66				
17		0	4	8	13	18	23	28	33	38	44	49	55	60	66	71	77			
18		0	4	9	14	19	24	30	36	41	47	53	59	65	70	76	82	88		
19		1	4	9	15	20	26	32	38	44	50	56	63	69	75	82	88	94	101	
20		1	5	10	16	22	28	34	40	47	53	60	67	73	80	87	93	100	107	114

10 Rank Correlation Coefficients (Spearman's)

Critical Values of r

	Level of significance for two-tailed test			
n	0.10	0.05	0.02	0.01
5	0.900	1.000	1.000	−−
6	0.829	0.886	0.943	1.000
7	0.714	0.786	0.893	0.929
8	0.643	0.738	0.833	0.881
9	0.600	0.683	0.783	0.833
10	0.564	0.648	0.746	0.794
12	0.506	0.591	0.712	0.777
14	0.456	0.544	0.645	0.715
16	0.425	0.506	0.601	0.665
18	0.399	0.475	0.564	0.625
20	0.377	0.450	0.534	0.591
22	0.359	0.428	0.508	0.562
24	0.343	0.409	0.485	0.537
26	0.329	0.392	0.465	0.515
28	0.317	0.377	0.448	0.496
30	0.306	0.364	0.432	0.478

11 Correlation Coefficients

Critical Values of r

	Level of significance for two-tailed test			
n	0.10	0.05	0.02	0.01
4	0.900	0.950	0.980	0.990
5	0.805	0.878	0.934	0.959
6	0.729	0.811	0.882	0.917
7	0.669	0.754	0.833	0.874
8	0.621	0.707	0.789	0.834
9	0.582	0.666	0.750	0.798
10	0.549	0.632	0.716	0.765
12	0.497	0.576	0.658	0.708
14	0.457	0.532	0.612	0.661
16	0.426	0.497	0.574	0.623
18	0.400	0.468	0.543	0.590
20	0.378	0.444	0.516	0.561
25	0.337	0.397	0.463	0.507
30	0.308	0.361	0.423	0.464
35	0.283	0.335	0.392	0.430
40	0.264	0.312	0.367	0.403
50	0.235	0.279	0.328	0.361

12 Constants for Use in Constructing Quality Control Charts

$A_{0.025} = 1.96a_n/\sqrt{n}\,. \qquad A_{0.001} = 3.1a_n/\sqrt{n}\,.$

Control limits at $\overline{x} \pm A_\alpha \overline{w}$ where $\overline{w}$ is the average sample range when system is under control.

$$\text{Prob (range} < D_\alpha \overline{w}) = \alpha$$

No. in Sample	**Chart for means** Factors for control limits		$\sigma = a_n\overline{\omega}$	**Chart for ranges** Factors for control limits			
n	$A_{0.025}$	$A_{0.001}$	a_n	$D_{0.95}$	$D_{0.995}$	$D_{0.999}$	$F_{0.95}$
2	1.23	1.94	0.8862	2.45	3.52	4.12	0.08
3	0.67	1.05	0.5908	1.96	2.58	2.98	0.25
4	0.48	0.75	0.4857	1.76	2.26	2.57	0.37
5	0.37	0.59	0.4299	1.66	2.08	2.34	0.44
6	0.32	0.50	0.3946	1.59	1.97	2.21	0.49
7	0.27	0.43	0.3698	1.54	1.90	2.11	0.53
8	0.24	0.38	0.3512	1.51	1.84	2.04	0.56
9	0.22	0.35	0.3367	1.48	1.79	1.99	0.59
10	0.20	0.32	0.3249	1.45	1.75	1.93	0.60