L#9

CONVOLUTIONAL CODING ①

* THERE ARE TWO MAIN CLASSES OF CHANNEL CODE
  - BLOCK CODES
  - CONVOLUTIONAL CODES

* SO FAR, WE HAVE ONLY LOOKED AT BLOCK CODES, TODAY WE WILL BEGIN OUR LOOK AT CONVOLUTION CODES.

CHANNEL CODES

* BLOCK CODES: DEFINED BY TWO INTEGERS n AND k.

  $\frac{k}{n}$ – GIVES THE CODE RATE

* CONVOLUTION CODES: DEFINED BY THREE INTEGERS; n, k AND K.

  $\frac{k}{n}$ - GIVES CODE RATE; BUT n DOES NOT HAVE THE SAME SIGNIFICANCE AS IT DOES FOR BLOCK CODES

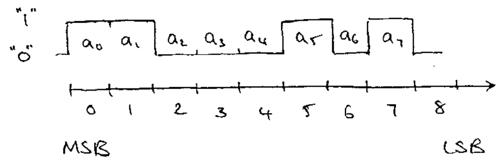
  K - CONSTRAINT LENGTH

DELAY OPERATOR ②

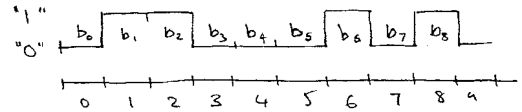
A BINARY SEQUENCE "A", CONTAINS A NUMBER OF SEQUENTIAL ELEMENTS a_r WHICH OCCUR AT REGULAR TIME INTERVALS

$$A = a_0\, a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7$$
$$(1 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 1)$$

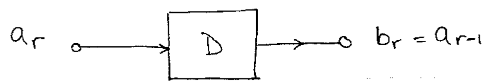
n-TUPLE OR n-VECTOR

Ⓐ


MSB - TRANSMITTED FIRST

Ⓑ


"B" IS SUBJECT TO A DIGITAL STORAGE ELEMENT, IMPLICITLY ASSUMED TO BE CLOCKED. THIS CAUSES THE INPUT SEQUENCE "A" TO SUFFER A UNIT DELAY. SO;



THAT IS; OPERATING ON A SEQUENCE BY D (THE DELAY OPERATOR) CAUSES A SINGLE SHIFT TO THE RIGHT;

$$B(D) = D\,A(D),$$

OR JUST

$$B = DA.$$

IN POLYNOMIAL FORM $g(x)$ FOR EXAMPLE, WHEN MULTIPLIED BY x SHIFTED TO THE LEFT.

MULTIPLYING BY x^{-1} SHIFTS SYMBOLS TO THE RIGHT (DELAY).

HENCE;

$$D \equiv x^{-1}, \quad \text{OR} \quad x = D^{-1}$$

OR $\times x$ - SHIFT LEFT
$\times D$ - SHIFT RIGHT.

FOR EXAMPLE IF WE APPLY "A" TO AN M-STAGE SHIFT REGISTER THEN

$$B(D) = D^{m} A(D)$$

SO, FOR $m = 4$.

$$B(D) = D^{m} A(D) = D^{4} A(D)$$
$$= D^{4} \oplus D^{5} \oplus D^{9} \oplus D^{11}$$

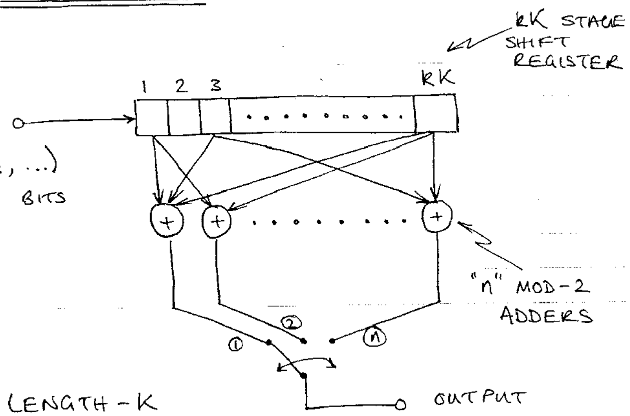
CONVOLUTIONAL CODING

WITH CONVOLUTIONAL CODING A SLIDING SEQUENCE OF MESSAGE BITS IS USED TO PRODUCE A CODED STREAM OF BITS

AS WITH BLOCK CODING WE CAN USE SYMBOLS INSTEAD OF JUST BITS, BUT, FOR SIMPLICITY WE WILL ONLY CONSIDER BIT STREAMS IN THIS COURSE.

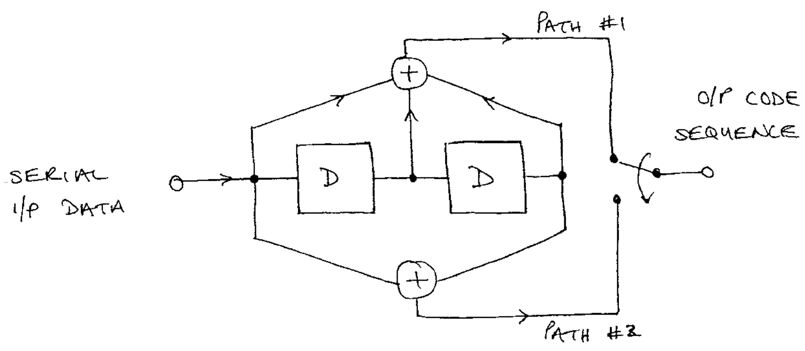
CONVOLUTIONAL CODER

M
($M = m_1, m_2, m_3, \ldots$)
SHIFTED IN "k" BITS AT A TIME

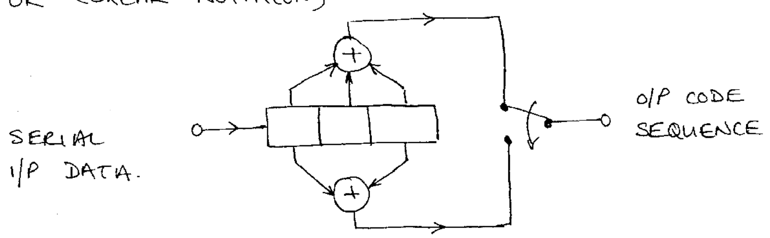


CONSTRAINT LENGTH - K
CODE RATE - $\frac{k}{n}$

TYPICAL CODER, K=3, $\frac{k}{n} = \frac{1}{2}$



OR (SKLAR NOTATION)



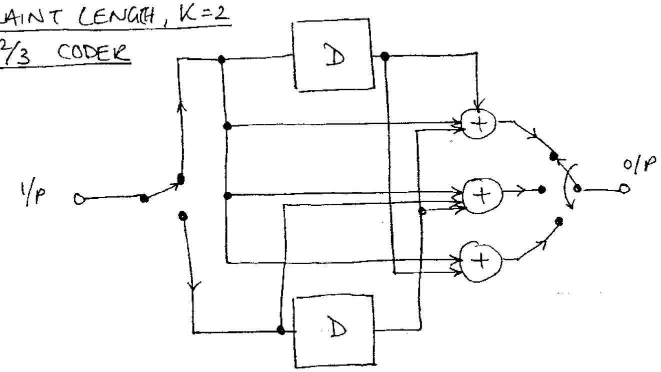
THE INPUT BITS ARE CLOCKED IN, AFTER EACH INPUT BIT IS RECEIVED THE CODER OUTPUT IS GENERATED BY SAMPLING AND MULTIPLEXING THE $n=2$ PATH OUTPUTS FROM THE MOD-2 ADDERS.

NOTE CONSTRAINT LENGTH K=3 IS ONE PLUS THE NUMBER OF PAST BITS AFFECTING THE CURRENT OUTPUT

THAT IS, ONE PARTICULAR BIT INFLUENCES THE OUTPUT DURING ITS OWN INTERVAL, AS WELL AS THE NEXT TWO INTERVALS.

WE WILL ONLY CONSIDER THE CASE $k=1$ IN THIS COURSE.

CONSTRAINT LENGTH, K=2
RATE = 2/3 CODER



SYSTEMATIC AND NON-SYSTEMATIC CODES

LIKE BLOCK CODES, CONVOLUTIONAL CODES CAN BE SYSTEMATIC OR NON-SYSTEMATIC DEPENDING ON WHETHER OR NOT THE INFORMATION SEQUENCE APPEARS DIRECTLY WITHIN THE CODE SEQUENCE i.e THE INPUT CONNECTED DIRECTLY TO ONE OF THE "n" OUTPUTS

(THAT IS $g_1(D) = 1$, - SEE LATER)

FOR THE SAME CODE PERFORMANCE, A SYSTEMATIC ENCODER WILL HAVE A MORE COMPLEX STRUCTURE THAN A NON-SYSTEMATIC ONE

$\boxed{\text{THEREFORE, MOST CONVOLUTIONAL CODES USED IN PRACTICE ARE NON SYSTEMATIC}}$

K=3, ½ RATE ENCODER

RETURNING TO OUR ½ RATE K=3 ENCODER SHOWN ON PAGE 5;

USING THE DELAY OPERATOR D, FOR THE TOP ADDER OF THE ENCODER THE IMPULSE RESPONSE (THE RESPONSE TO A SINGLE ONE, WITH THE REGISTERS RESET INITIALLY TO ZERO) FOR PATH #1 CAN BE EXPRESSED AS;

$$g^1(D) = g_0^1 \oplus g_1^1 D \oplus g_2^1 D^2 \oplus \dots\ g_N^1 D^N$$

SIMILARLY FOR PATH #2;

$$g^2(D) = g_0^2 \oplus g_1^2 D \oplus g_2^2 D^2 \oplus \dots\ g_N^2 D^N$$

WHERE

$g_i^{1,2} = 0$ OR 1 IF THE CONNECTION IS OPEN OR MADE.

THE TWO POLYNOMIALS $g^1(D)$ AND $g^2(D)$ ARE THE GENERATOR POLYNOMIALS OF THE CODE;

WITH A (SEMI-INFINITE) INPUT MESSAGE SEQUENCE OF L BITS;

$$M = (M_0, M_1, M_2, \dots\ M_{L-1})$$

THEREFORE THE OUTPUTS ARE;

PATH #1 : $x^1(D) = g^1(D)M(D)$

PATH #2 : $x^2(D) = g^2(D)M(D)$

SO, FROM OUR EXAMPLE, THE IMPULSE RESPONSE OF PATH #1 IS 111, SO;

$$g^1(D) = 1 \oplus D \oplus D^2$$

AND FOR PATH #2

$$g^2(D) = 1 \oplus D^2$$

FOR AN INPUT SEQUENCE OF 10011

$\Rightarrow$ $M(D) = 1 \oplus D^3 \oplus D^4$

$$x^1(D) = g^1(D)M(D)$$
$$= (1 \oplus D \oplus D^2)(1 \oplus D^3 \oplus D^4)$$
$$= 1 \oplus D \oplus D^2 \oplus D^3 \oplus D^6$$
$$= (1111001)$$

AND

$$x^2(D) = g^2(D)\,M(D)$$
$$= (1 \oplus D^2)(1 \oplus D^3 \oplus D^4)$$
$$= 1 \oplus D^2 \oplus D^3 \oplus D^4 \oplus D^5 \oplus D^6$$
$$(1011111)$$

HENCE, THE OUTPUT CODEWORD IS;
(BY MULTIPLEXING x^1 AND x^2)

$x^1(D)$: 1 1 1 1 0 0 1

$x^2(D)$: 1 0 1 1 1 1 1

OUTPUT : 1 1 1 0 1 1 1 1 0 1 0 1 1 1

MESSAGE LENGTH : $L = 5$ PRODUCES AN OUTPUT CODED SEQUENCE OF; $n(L+K-1)$ BITS
$= 2(5+2) = 14$ BITS.

NOTE: SINCE A CONVOLUTIONAL CODE, UNLIKE A BLOCK CODE, HAS NO PARTICULAR BLOCK SIZE, THEY ARE OFTEN (FOR CONVENIENCE) PERIODICALLY TRUNCATED. WHEN THIS IS DONE, A TAIL OF $(K-1)$ ZEROS ARE APPENDED TO THE END OF THE MESSAGE SEQUENCE TO FLUSH THE ENCODER. THIS GIVES RISE TO A TAIL IN THE CODE SEQUENCE

ENCODER STATE REPRESENTATION

THE STATE OF THE ENCODER (SHIFT REGISTER CONTENTS) CAN ONE OF 2^{K-1} STATES:

KNOWLEDGE OF THE PRESENT STATE PLUS THE NEXT INPUT IS SUFFICIENT INFORMATION TO DETERMINE THE NEXT STATE.

THE ENCODER STATE IS SAID TO BE MARKOV IN THAT THE PROBABILITY OF BEING IN ONE STATE DEPENDS ONLY ON THE MOST RECENT STATE.

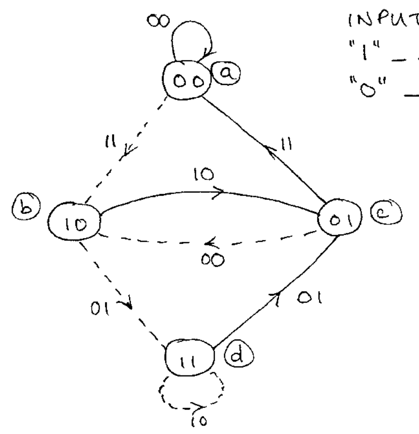
STATE TRANSITION DIAGRAM

FROM THE STATE DIAGRAM, TO DETERMINE THE OUTPUT SEQUENCE FOR SOME INPUT MESSAGE SEQUENCE, START AT STATE (a) AND WALK THROUGH THE STATE DIAGRAM IN ACCORDANCE WITH THE MESSAGE SEQUENCE OUTPUTTING THE APPROPRIATE BITS FOR EACH BRANCH.

CONSIDER OUR EXAMPLE $K=3$ 1/2 RATE CODE

STATE TRANSITION DIAGRAM K=3 1/2 RATE

(11)



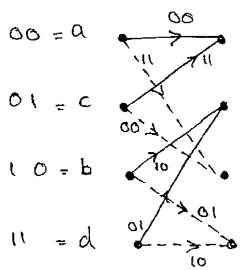
TRANSITION FOR:
INPUT
"1" - - - -> - - -
"0" ———>———
01
RESULTANT OUTPUT

FOR EXAMPLE IF M = 10011

$$a \xrightarrow{} b \xrightarrow{} c \xrightarrow{} a \xrightarrow{} b \xrightarrow{} d \xrightarrow{} c \xrightarrow{} a$$

11 10 11 11 01 01 11

TRELLIS SEGMENT



NOTE: WE CAN GET TO ANY STATE BY ONE OF TWO PATHS

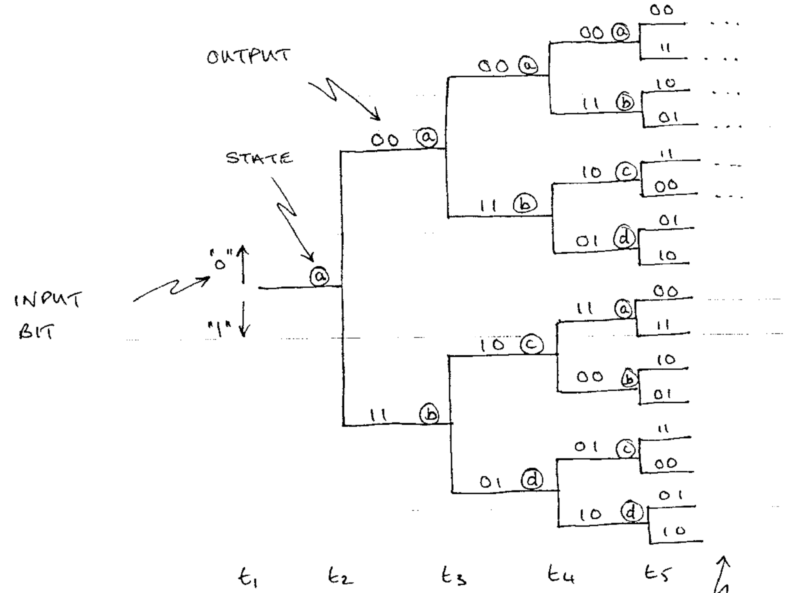
WE CAN LEAVE ANY STATE BY ONE OF TWO PATHS.

TREE DIAGRAM K=3 1/2 RATE

(12)

ALTHOUGH THE STATE DIAGRAM CAN COMPLETELY CHARACTERIZE THE ENCODER, STATE, WE CANNOT TRACK THE OUTPUT AS A FUNCTION OF TIME SINCE IT HAS NO TEMPORAL DIMENSION.

WITH THE TREE DIAGRAM, EACH BRANCH OF THE TREE REPRESENTS AN INPUT BIT WITH AN UPPER BIFURCATION FOR A "0" INPUT AND A LOWER BIFURCATION FOR A "1" INPUT



REPEATS AFTER K=3 3rd BRANCH.

TRELLIS DIAGRAM

* IT CAN BE SEEN FROM THE TREE DIAGRAM THAT THE STRUCTUR REPEATS ITSELF AFTER THE K^{th} BRANCH

* AFTER THE K^{th} BRANCH THERE ARE EIGHT NODES : TWO LABELLED ⓐ , TWO LABELLED ⓑ , TWO LABELLED ⓒ AND TWO LABELLED ⓓ.

* FROM THIS POINT IT CAN BE SEEN THAT THE UPPER AND LOWER PARTS OF THE TREE ARE IDENTICAL.

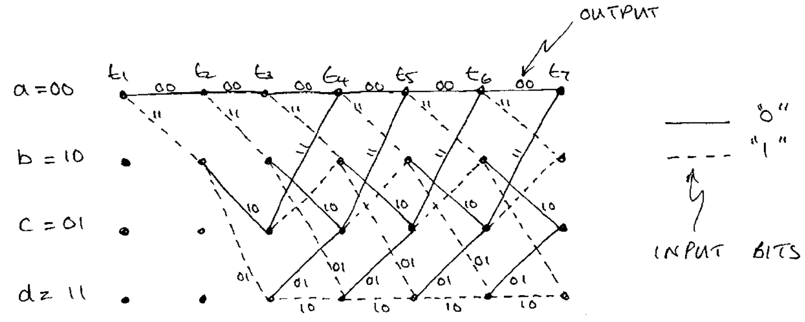
* THIS MEANS THAT ANY TWO NODES HAVING THE SAME STATE LABEL, AT THE SAME TIME t_i CAN BE MERGED SINCE ALL SUBSEQUENT PATHS WILL BE INDISTINGUISHABLE.

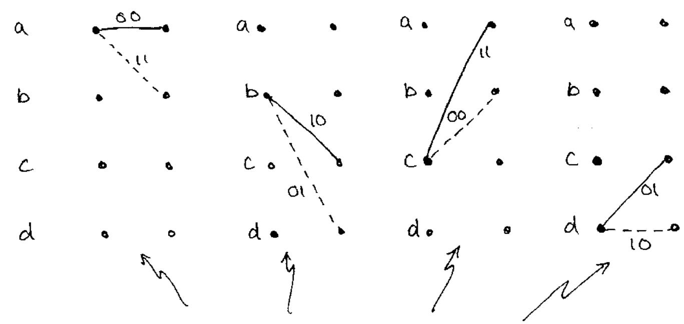
* IF WE MERGE THESE PATHS WE OBTAIN THE TRELLIS DIAGRAM.

* AT EACH UNIT OF TIME WE NEED 2^{K-1} NODES TO REPRESENT THE 2^{K-1} POSSIBLE ENCODER STATES

* TRELLIS DIAGRAMS ARE MOSTLY USED IN PRACTICE TO REPRESENT CONVOLUTIONAL CODES – WE SOON RUN OUT OF PAPER WITH THE TREE DIAGRAM!!

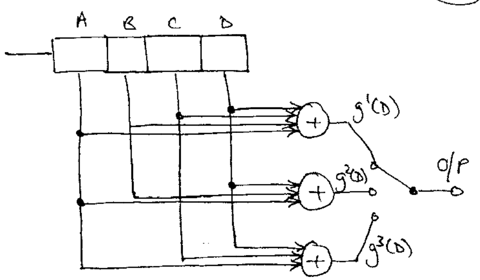
TRELLIS DIAGRAM K=3 1/2 RATE





TRELLIS FRAGMENTS

FOR EXAMPLE



$K=4$ 1/3 RATE

$g^1(D) = 1 \oplus D \oplus D^2 \oplus D^3$
$g^2(D) = 1 \oplus D \oplus D^3$
$g^3(D) = 1 \oplus D^2 \oplus D^3$

$K=4 \quad 2^{K-1} = 2^3 = 8$ STATES

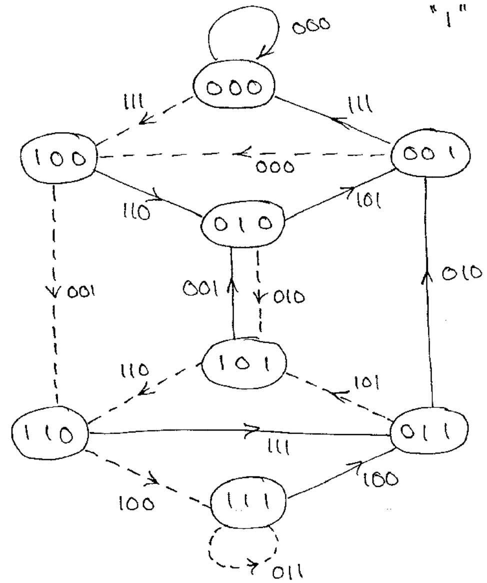
INPUT BIT	ABCD	STATE t_i	STATE t_{i+1}	g^1	g^2	g^3
0	0000	000	000	0	0	0
1	1000	000	100	1	1	1
0	0001	001	000	1	1	1
1	1001	001	100	0	0	0
0	0010	010	001	1	0	1
1	1010	010	101	0	1	0
0	0011	011	001	0	1	0
1	1011	011	101	1	0	1
0	0100	100	010	1	1	0
1	1100	100	110	0	0	1
0	0101	101	010	0	0	1
1	1101	101	110	1	1	0
0	0110	110	011	1	1	1
1	1110	110	111	1	0	0
0	0111	111	011	1	0	0
1	1111	111	111	0	1	1

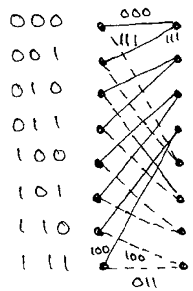
t_{i+1} t_i

A⊕B⊕C⊕D A⊕C⊕D A⊕C⊕D

STATE DIAGRAM

INPUT BIT
"0" ————
"1" - - - -





TRELLIS SEGMENT.

DECODING OF CONVOLUTIONAL CODES

* DECODER HAS KNOWLEDGE OF THE CODE STRUCTURE (e.g TRELLIS STRUCTURE) AND THE RECEIVED SIGNAL (e.g THE STATISTICAL CHARACTERISTICS OF THE CHANNEL).

* THE TRANSMITTED SIGNAL CORRESPONDS TO A SPECIFIC PATH THROUGH THE TRELLIS. THE DECODER USES THE RECEIVED SEQUENCE (WHICH MAY CONTAIN ERRORS) TO FIND THE MOST LIKELY PATH THROUGH THE TRELLIS - THAT CORRESPONDS TO THE RECEIVED SEQUENCE

* THE MOST LIKELY PATH IS THEN USED TO SPECIFY THE DECODED DATA SEQUENCE

  - THIS IS CALLED **MAXIMUM LIKELIHOOD DECODING**

MAXIMUM LIKELIHOOD DECODING

DENOTE THE ENCODER SEMI-INFINITE OUTPUT SEQUENCE CORRESPONDING TO A MESSAGE SEQUENCE (OR PATH) BY;

$$C_M = C_{m1}, C_{m2}, C_{m3}, \ldots$$

THE RECEIVED SIGNAL FROM THE DISCRETE MEMORYLESS CHANNEL IS;

$$R = r_1, r_2, r_3, \ldots$$

THE RECEIVED SEQUENCE R, MAY DIFFER FROM C_m BECAUSE OF CHANNEL ERRORS.

THE PROBABILITY OF RECEIVING R GIVEN THAT THE CHANNEL INPUT WAS C_m IS GIVEN BY;

$$P(R|C_m) = \prod_{i=1}^{N} P(r_i|C_{mi})$$

N ← semi-infinite length

GIVEN R, THE MOST LIKEY PATH THROUGH THE TRELLIS IS ONE THAT MAXIMIZES $P(R|C_m)$ - THIS IS A **METRIC**

THIS IN NOT VERY CONVENIENT, BUT IF WE USE THE LOGARITHM OF THE PROBABILITY WE CAN USE SUMMATION RATHER THAT MULTIPLICATION

THE USE OF THE LOG-LIKELIHOOD FUNCTION IS STILL PERMISSIBLE SINCE THE LOG FUNCTION IS A MONOTONICALLY INCREASING FUNCTION.

- THAT IS, THE OPTIMUM PATH THROUGH THE TRELLIS IS STILL ONE WHICH MAXIMIZES $\log[P(R|C_m)]$

HENCE, WE CAN USE THE LOG-LIKELIHOOD FUNCTION

$$\log[P(R|C_m)] = \sum_{i=1}^{N} \log[P(r_i|C_{mi})]$$

FINDING THE MOST LIKELY PATH THROUGH EXHAUSTIVE SEARCHING OF THE TREE DIAGRAM, REQUIRES A "BRUTE FORCE" TECHNIQUE. FOR AN "L" BIT RECEIVED SEQUENCE, 2^L ACCUMULATED LOG-LIKELIHOOD METRICS HAVE TO BE COMPUTED AND COMPARED – COMPLEX AND SLOW

BY TAKING INTO ACCOUNT THE SPECIAL STRUCTURE OF THE TRELLIS DIAGRAM CAN IMPROVE THIS CONSIDERABLY, BY DISCARDING IMPOSSIBLE PATHS.

<u>VITERBI</u> DECODING GOES ONE STAGE FURTHER ...

CONVOLUTIONAL CODE QUIZ

A CONVOLUTIONAL CODE IS DESCRIBED BY THE FOLLOWING GENERATOR POLYNOMIALS

$g_1(D) = 1$; $g_2(D) = 1 \oplus D^2$; $g_3(D) = 1 \oplus D \oplus D^2$

a) DRAW THE ENCODER (USING THE SHIFT REGISTER NOTATION OF SKLAR & PROAKIS)

b) DRAW THE STATE TABLE

c) DRAW THE STATE TRANSITION DIAGRAM

d) DRAW THE TRELLIS DIAGRAM

e) FIND THE OUTPUT CODE FOR THE MESSAGE STREAM 1011000

a) ENCODER:

b) STATE TABLE:

c) STATE DIAGRAM: