SINGLE ERROR DETECTION DECODING

THE DECODING PROCESS;

1) COMPUTE THE SYNDROME: $S = R[H]^T$
(FOR ERROR DETECTION IT IS SUFFICIENT TO CHECK JUST FOR A NON-ZERO SYNDROME).

2) LOOK-UP THE ERROR PATTERN ASSOCIATED WITH THE SYNDROME VECTOR, S.

3) OBTAIN THE CODE VECTOR BY
$C = R \oplus E$.

THIS WILL CORRECT THE 2^{n-k} MOST LIKELY ERROR PATTERNS FOR WHICH THE CODE IS DESIGNED.

NOTE: FOR SMALL (SHORT) CODES, A LOOK-UP TABLE CAN BE USED FOR THE SYNDROME TO ERROR PATTERN FUNCTION
e.g $2^7 = 128$.

FOR LONGER CODES eg $2^{15} = 32768$, THIS CAN BE EXPENSIVE, IS THERE AN EASIER WAY?

EXAMPLE

CONSIDER THE (7,4) SYSTEMATIC CODE WE LOOKED AT PREVIOUSLY...

SYNDROME VECTOR, S	ERROR PATTERN
0 0 0	0 0 0 0 0 0 0
0 1 1	1 0 0 0 0 0 0
1 0 1	0 1 0 0 0 0 0
1 1 0	0 0 1 0 0 0 0
1 1 1	0 0 0 1 0 0 0
1 0 0	0 0 0 0 1 0 0
0 1 0	0 0 0 0 0 1 0
0 0 1	0 0 0 0 0 0 1

THE APPLICATION OF THIS LOOK-UP TABLE PROCESS IS EASY; THE CODE IS SHORT, AND WE ARE ONLY CONSIDERING SINGLE ERRORS.

FOR LONGER CODES (WHICH YIELD BETTER CODE EFFICIENCY GENERALLY) AND MULTIPLE ERROR CORRECTING CODES, THE NUMBER OF IDENTIFIABLE ERROR PATTERNS BECOME TOO LARGE FOR CONVENIENT USE.

EXAMPLE

CODEWORD LENGTH, n	15	63	127
No SINGLE ERRORS, nC_1	15	63	127
No DOUBLE ERRORS, $^nC_2$	105	1953	8001
No TRIPLE ERRORS $^nC_3$	455	39711	333375

FOR LARGE CODES MORE STRUCTURE IS REQUIRED TO SIMPLIFY THE PROCESS.

$\Rightarrow$ CYCLIC CODES

CODE PERFORMANCE FOR RANDOM ERRORS

FOR SINGLE ERROR CORRECTION WE REQUIRE;

$$2^r - 1 \geq n.$$

FOR MULTIPLE ERROR CORRECTION WE REQUIRE;

$$2^r - 1 \geq \sum_{i=1}^{t} \binom{n}{i}$$

CODES CAN BE OPERATED IN TWO WAYS;

1) FORWARD ERROR CORRECTION (FEC) WHERE ERRORS ARE CORRECTED AT THE RECEIVER

2) AUTOMATIC REPEAT REQUEST (ARQ) WHERE ERRORS ARE CORRECTED BY REQUESTING THE RE-TRANSMISSION OF THE BLOCK FROM THE TRANSMITTER.

CERTAIN PROBABILITIES ARE COMMON TO BOTH FORMS OF ERROR CORRECTION.

P_{NO} – PROB. THAT NO ERROR OCCURS, OR WORD OK.
P_{WE} – PROB. THAT SOME ERROR OCCURS (WORD)
P_b – PROB. OF A SINGLE BIT ERROR, OR BER – BIT ERROR RATE.

FOR AN 'n' BIT CODEWORD;

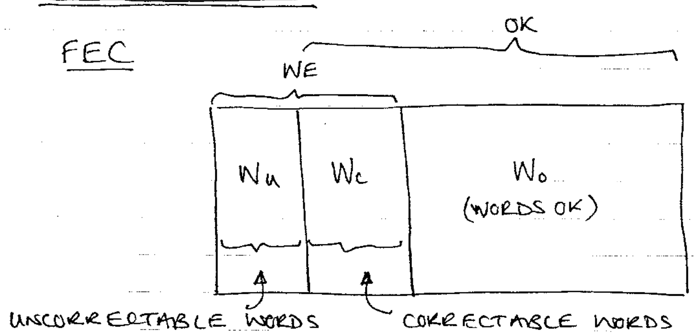
(5)

$$P_{WO} = (1 - P_b)^n \quad , \quad P_{WE} = 1 - P_{WO}$$
$$= 1 - (1 - P_b)^n$$
$$\simeq nP_b \text{ WHEN } P_b \ll 1$$

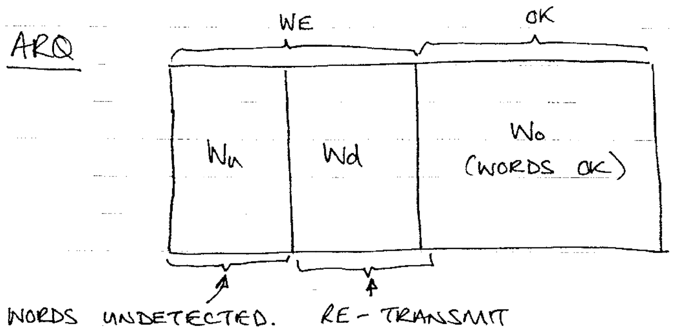
BINOMIAL THEOREM APPROX.

VENN DIAGRAMS

FEC



ARQ



(6)

FEC

THE PROBABILITY OF A CORRECTABLE ERROR, P_{WC} IS THE PROBABILITY THAT NO MORE ERRORS OCCUR THAN THE CODE CAN CORRECT.

THE PROBABILITY THAT 'e' ERRORS OCCUR IN A WORD OF 'n' BITS LONG IS GIVEN BY;

$$P_n(e) = \binom{n}{e} P_b^{\,e} (1 - P_b)^{n-e}$$

WHERE $\binom{n}{e} = \frac{n!}{e!(n-e)!}$ ← THE BINOMIAL COEFFICIENT

IF THE CODE CAN CORRECT UP TO 't' ERRORS THEN

$$P_{WC} = P_n(1) + P_n(2) + \ldots + P_n(t)$$
$$= \sum_{e=1}^{e=t} P_n(e)$$

$$\boxed{P_{WC} = \sum_{e=1}^{e=t} \binom{n}{e} P_b^{\,e} (1 - P_b)^{n-e}}$$

IF MORE THAN 't' ERRORS OCCUR THEN THE RESULT IS AN <u>UNCORRECTABLE</u> ERROR WITH A PROBABILITY OF P_{WU}

$$P_{WU} = P_n(t+1) + P_n(t+2) + \ldots + P_n(n)$$

$$\Rightarrow \boxed{P_{WU} = \sum_{e=t+1}^{e=n} \binom{n}{e} P_b^{\,e} (1-P_b)^{n-e}}$$

FOR $P_b \ll 1$, THIS WILL BE DOMINATED BY THE tTH CASE (THIS CAN BE SEEN BY LOOKING AT THE BINOMIAL EXPANSION)

HENCE; P_{WU} CAN BE APPROXIMATED

$$\boxed{P_{WU} \simeq \binom{n}{e} P_b^{\,t+1} (1-P_b)^{[n-(t+1)]} \quad \text{IF } P_b \ll 1}$$

<u>FOR EXAMPLE;</u>

CONSIDER A (31, 21, 2) SYSTEMATIC BLOCK CODE, CAPABLE OF CORRECTING UP TO <u>2</u> ERRORS, IN A CODEWORD OF <u>31</u> BITS, OF WHICH <u>21</u> BITS ARE INFORMATION BITS.

ASSUME THAT THE PROBABILITY OF A BIT ERROR IS 10^{-3}, ie $P_b = 10^{-3} = 0.001$

PROB. WORDS OK;

$$P_{WO} = (1-P_b)^n = (1-0.001)^{31} = 0.96946$$

PROB. WORD IN ERROR;

$$P_{WE} = 1 - P_{WO} = 1 - 0.96946 = 0.03054$$

(ie ~3%)

PROB OF ERRORS OCCURING;

$$P_n(e) = \binom{n}{e} P_b^{\,e} (1-P_b)^{n-e} \qquad \binom{n}{e} = \frac{n!}{e!(n-e)!}$$

$$P_n(1) = {}^{31}C_1 (10^{-3})^1 (1-10^{-3})^{(31-1)} = 3.008 \times 10^{-2}$$

$$P_n(2) = 4.517 \times 10^{-4}$$
$$P_n(3) \simeq 4.375 \times 10^{-6}$$
$$P_n(4) = 3.060 \times 10^{-8}$$
$$P_n(5) = 1.655 \times 10^{-10}$$

THE CODE IS CAPABLE OF DETECTING <u>TWO</u> ERRORS SO THE PROBABILITY OF AN <u>UNCORRECTABLE ERROR</u> IS <u>APPROX</u>;

$$P_{WU} \simeq P_n(3) \simeq 4.4 \times 10^{-6}$$

[TO BE STRICTLY CORRECT WE SHOULD PERFORM THE SUM, BUT IT CAN SEEN THAT P_{WU} IS DOMINATED BY THE $P_n(3)$ TERM]

$$P_{WU} = \sum_{i=3}^{31} P_n(i)$$

FOR A SINGLE WORD ERROR

$$E = N_{WORDS} \times P_{WU}$$
$$1 = N_{WORDS} \times 4.4 \times 10^{-6}$$

$$\Rightarrow N_{WORDS} = \frac{10^6}{4.4} = 272,272.\dot{7}\dot{2}$$

THAT IS 272, 272 WORDS ARE TRANSMITTED ON AVERAGE BEFORE AN UNDETECTABLE ERROR OCCURS

SUPPOSE WE HAVE A DATA RATE OF 2.4 kbit sec^{-1} $= \frac{2400}{31}$ words sec^{-1}

THAT IS, AN UNDETECTED ERROR OCCURS ON AVERAGE EVERY ...

$$T = \frac{N_{WORDS}}{2400/31} \text{ seconds}$$

$$= \frac{31 \times 10^6}{4.4 \times 2400} \simeq 2935.6 \text{ sec.}$$

$$\simeq 49 \text{ minutes}$$

ARQ

OUR (31, 21, 2) CAN CORRECT 2 ERRORS,

HENCE $d_{min} = 5$ $\qquad t = \left\lfloor \frac{d_{min}-1}{2} \right\rfloor$

THAT IS 5 ERRORS MUST OCCUR BEFORE AN UNDETECTABLE ERROR OCCURS

HENCE;

$$P_{WU} \simeq P_n(5) \simeq 1.655 \times 10^{-10}$$

AS BEFORE, FOR A SINGLE WORD ERROR;

$$E = N_{WORDS} \times P_{WU}$$
$$1 = N_{WORDS} \times 1.655 \times 10^{-10}$$

$$\Rightarrow N_{WORDS} = \frac{10^{10}}{1.655} \simeq 6042296073$$

AT THE SAME 2.4 kbit sec^{-1} DATA RATE ONE UNDETECTED WORD ERROR OCCURS EVERY ...

$$T = \frac{10^{10} \times 31}{1.655 \times 2400} \text{ seconds}$$

$$\sim 2.5 \text{ years!}$$

THE NUMBER OF REPEAT TRANSMISSIONS REQUESTED IS APPROXIMATELY THE NUMBER OF WORD ERRORS – SINCE THE VAST MAJORITY OF ERRORS ARE CORRECTABLE BY RE-TRANS-MISSION AND THE NUMBER THAT FAIL TO BE DETECTED WILL BE VERY SMALL;

HENCE;

PROB. OF REPEAT REQUEST IS $P_{WE} = 0.03054$

FOR THE SAME DATA RATE, ONE BLOCK IS REPEATED EVERY...

$$E = N_{WORDS} \times P_{WE}$$

$$\Rightarrow N_{WORDS} = \frac{10^{-2}}{3.05} \approx 32.744$$

... 32 WORDS

$$T = \frac{N_{WORDS}}{2400/31} = \frac{32744}{2400/31} = 0.43 \text{ seconds}$$

HENCE, RE-TRANSMISSION IS A FAIRLY FREQUENT OCCURRENCE, BUT THIS ONLY DIMINISHES THE DATA RATE THROUGHPUT BY ONLY ~3%

EXAMPLE 2

WHEN COMPARING UN-CODED AND CODED SYSTEMS THE MESSAGE OR INFORMATION TRANSMISSION RATE IS ASSUMED TO BE THE SAME FOR BOTH SYSTEMS, AND BOTH SYSTEMS ARE OPERATING WITH THE SAME AVERAGE POWER.

BECAUSE MORE BITS ARE TRANSMITTED IN THE CODED CASE, THE REDUCTION IN BIT ENERGY WILL INCREASE THE BER (THE SIGNAL TO NOISE RATIO HAS BEEN LOWERED

CONSIDER A SINGLE-ERROR CORRECTING (7,4) CODE OPERATING WITH AN E_b/N_o OF 9.6dB FOR THE UNCODED CASE. ASSUME PSK MOD.

UNCODED CASE

THE PROBABILITY OF A BIT ERROR IS (FOR PSK)

$$P_b = Q\left[\sqrt{\frac{2E_b}{N_o}}\right]$$

$$\frac{E_b}{N_o} = 9.6\,dB \quad \Rightarrow \quad \frac{E_b}{N_o} = 10^{[9.6/10]} = 9.12$$

SINCE $\sqrt{\frac{2E_b}{N_o}} > 3$ WE CAN APPROXIMATE $Q(x)$

FOR $x > 3$, $Q(x) \simeq \frac{1}{x\sqrt{2\pi}} \exp(-x^2/2)$

$$P_b = \frac{1}{\sqrt{2E_b/N_o}\sqrt{2\pi}} \exp\left(-\frac{\left(\sqrt{2E_b/N_o}\right)^2}{2}\right)$$

$$= \frac{1}{2\sqrt{\pi E_b/N_o}} \exp(-E_b/N_o)$$

$$= \frac{1}{2\sqrt{\pi \times 9.12}} \exp(-9.12)$$

$$= \underline{1.02 \times 10^{-5}}$$

THE PROBABILITY OF A WORD ERROR P_{WE} IN THE UNCODED CASE (ie ONLY 4 MESSAGE BITS) IS

$$P_{WE} = 1 - (1 - P_b)^n$$

$$= 1 - (1 - 1.02 \times 10^{-5})^4$$

$$= \underline{4.08 \times 10^{-5}}$$

CODED CASE

IN THIS CASE WE TRANSMIT 7 INSTEAD OF 4 BITS AND THUS FOR THE SAME POWER WE MUST DEGRADE OUR E_b/N_o

$$\left[\frac{E_b}{N_o}\right]_{CODED} = \frac{4}{7}\left[\frac{E_b}{N_o}\right]_{UNCODED}$$

ie A LOWER ENERGY PER BIT PER POWER SPECTRI DENSITY.

PROB. OF ERROR;

$$P_b = Q\left[\sqrt{\frac{2E_b}{N_o} \times \frac{4}{7}}\right]$$

$$P_b = \frac{1}{2\sqrt{\pi \times 9.1 \times 4/7}} \exp(-5.2)$$

$$P_b = 6.82 \times 10^{-4}.$$

SINCE THE CODE CAN CORRECT A SINGLE ERROR, IT REQUIRES TWO OR MORE BIT ERRORS TO GENERATE A WORD ERROR

HENCE

$$P_{WE} = {}^7C_2 \, P_b^2 (1 - P_b)^5$$

$$^7C_2 = \frac{7!}{2!\,(7-2)!} = \frac{7\times 6\times \cancel{5!}}{2\times \cancel{5!}} = \frac{42}{2} = 21$$

$$P_{WE} = 21 \times (6.82\times10^{-4})^2 (1-6.82\times10^{-4})^5$$

$$P_{WE} \simeq 21 \times (6.82\times10^{-4})^2$$

$$\simeq 9.767\times10^{-6}$$

SO; CODED CASE HAS LOWER E_b/N_o (SNR)
HENCE A HIGHER P_b.
__BUT__ CODED CASE HAS A LOWER P_{WE}

FOR A GIVEN BIT ERROR PROBABILITY THE IMPROVEMENT (REDUCTION) IN E_b/N_o THAT CODING GIVES IS CALLED __CODING GAIN__

ERROR FUNCTIONS

THE "Q" FUNCTION IS DEFINED AS;

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp(-u^2/2)\,du.$$

NO TRACTABLE ANALYTIC SOLUTION:

$x < 3$ EVALUATE $Q(x)$ FROM TABLES

$x > 3$ $Q(x) \simeq \frac{1}{x\sqrt{2\pi}} \exp(-x^2/2)$

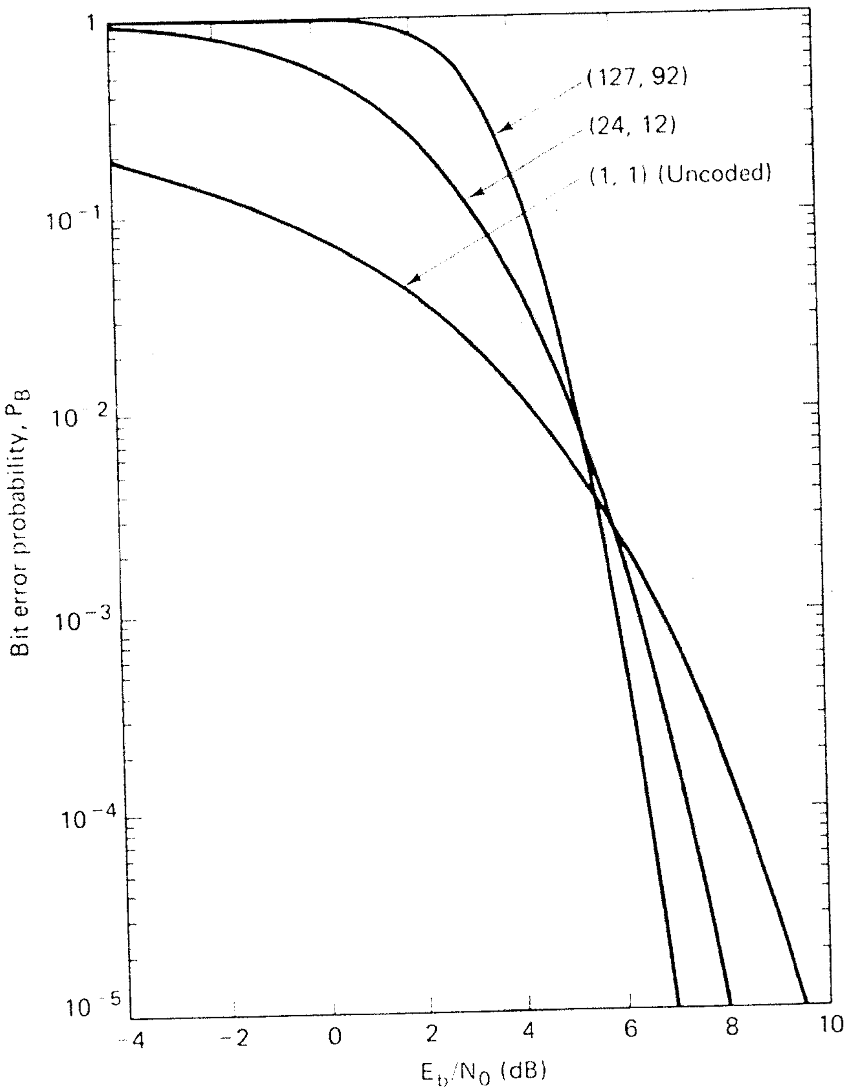
A RELATED FUNCTION:

$$\mathrm{erfc} = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-u^2)\,du$$

$$\mathrm{erfc} = 2Q(x\sqrt{2})$$

$$\text{OR } Q(x) = \frac{1}{2}\,\mathrm{erfc}(x/\sqrt{2})$$

CHECK THE DEFINITION!



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