How to write maths (well)

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These are the slides from a talk I gave to the new first-year students at Bath, annotated with some of the things I said (which appear in boxes like this one).

Some of you may be thinking "What's writing got to do with maths? Is this some kind of joke? What's next week – how to dance maths?!?"

Well, although this talk was given on a Friday, it was not a joke.

WHAT does writing maths well involve?

WHY write maths well?

HOW do you write maths well?

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At this point I asked the class to look at the two calculations on the handout (`example1.pdf`) and decide which one they find easier to understand.

A show of hands revealed that the majority of people found Calculation 2 easier to understand. This is the one that I find easier to understand, and I claim that the vast majority of mathematicians would also find this one easier to understand. The reason is that it keeps the reader informed about what's going on at each step of the calculation.

The point of this exercise was to demonstrate that the majority of you already have some sort of idea of what constitutes good mathematical writing (i.e. it's not a completely alien concept).

Guiding principle of writing maths well

When you read the maths you've written it should make sense as a piece of English.

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At this point you might say to me "Ok, I can see that Calculation 2 makes sense when you read it aloud whereas Calculation 1 doesn't, and I can see that having these words in Calculation 2 makes it easier for the reader to follow the argument, but if the chips were down the reader would still be able to understand Calculation 1."

I can see where this (hypothetical) person is coming from. Indeed, the vast majority of the mathematical arguments that you've met so far (i.e. at school) are a bit like the one in Example 1 (calculating $\int \sin^4 x \, dx$): you can pretty much get away without writing any words.

However, the majority of mathematical arguments that you'll meet at university don't make sense without any words.

Why?

- The majority of mathematical arguments don't make sense without any words.
- In the future you will be in situations where you need to communicate mathematical arguments to other people:
  - short term: exams
  - longer term: in whatever you do after you graduate

Regarding the short term (i.e. exams): I marked $\sim$300 exam scripts last summer and, when looking at a new script, I could quickly make a good guess how well the student had done based on how many words there were on the page. Of course, this doesn't mean that writing down words just for the sake of it will get you a better mark, but it does mean that writing words is an inherent part of constructing correct mathematical arguments.

Regarding the longer term: even if you end up doing a job that doesn't involve any maths (as some of you might), the ability to construct valid, well-structured arguments will still be very useful.

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Example 2:

Prove that the sum of any two odd numbers is even.

When I was thinking about why the sum of any two odd numbers is even, I wrote down the following on the piece of paper in front of me:

$$2n + 1$$

$$\begin{aligned}(2n+1)+(2m+1) &= 2n+1+2m+1 \\ &= 2n+2m+2 \\ &= 2(n+m+1)\end{aligned}$$

These few lines of maths are the essence of why the sum of any two odd numbers is even, but they're not an argument – when read aloud they don't make sense since they're just a collection of symbols and numbers. To make these lines of maths into a proper argument, I need to write down (at least some of) the thoughts/commentary that was going on in my head at the time.

Here is a proper argument that proves the result.

Proof.
Any odd number equals $2n+1$ for some integer n.
Therefore, given any two odd numbers p and q there exist integers n and m such that $p = 2n+1$ and $q = 2m+1$. Hence

$$p + q = (2n+1) + (2m+1).$$

Now

$$\begin{aligned}(2n+1)+(2m+1) &= 2n+1+2m+1\\ &= 2n+2m+2\\ &= 2(n+m+1).\end{aligned}$$

Since $2(n+m+1)$ is divisible by 2 it is even.
Therefore, $p+q$ is even.

□

Example 3:

Prove that the product of any two odd numbers is odd.

I gave the students in the class some time to have a go at this example themselves.

When I gave this talk last year, five people left the pieces of paper with their arguments on behind. Two of these five arguments were correct; here is one of the incorrect ones (copied verbatim by me to protect the identity of the guilty party!).

An attempt at Example 3 from last year.....

let $n, m \in \mathbb{Z}$

odd numbers : $2n+1$, $2m+1$

product $(2n+1)(2m+1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

let $2mn + m + n = p$

$\Rightarrow 2p+1$ is odd and $p \in \mathbb{Z}$

as $m, n \in \mathbb{Z}$

The logical error in the previous argument is that the author starts with integers n and m, states (correctly) that $2n+1$ and $2m+1$ are odd numbers, and then goes on to show that the product of $2n+1$ and $2m+1$ is odd. This is not quite the same as showing that the product of any two odd numbers is odd, because here one has to start with two arbitrary odd numbers, state that they can be written as $2n+1$ and $2m+1$ for some $n, m \in \mathbb{Z}$, and then show that the product of $2n+1$ and $2m+1$ is odd.

Here is a correct argument that proves the result.

Proof.
Any odd number equals $2n+1$ for some integer n.
Therefore, given any two odd numbers p and q there exist integers n and m such that $p = 2n+1$ and $q = 2m+1$. Hence

$$pq = (2n+1)(2m+1).$$

Now,

$$\begin{aligned}(2n+1)(2m+1) &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1.\end{aligned}$$

Therefore, $(2n+1)(2m+1) = 2r+1$ for $r = 2nm + n + m$ (which is an integer if n and m are integers).
Therefore, pq is odd. □

A note on symbols:

During the next few weeks you will learn symbols that stand for phrases that are commonly used when writing maths (e.g. “there exists”, “for all”, “such that”, “therefore”).

These symbols can be very useful, but when students first encounter them, they are sometimes tempted to use them as much as possible.

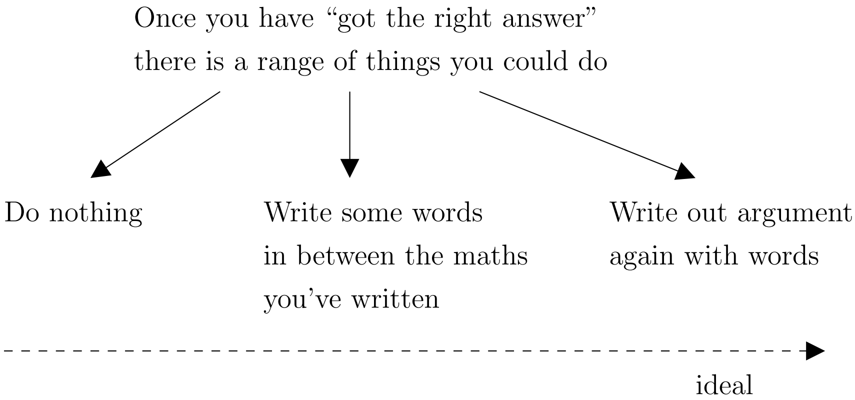
Bear in mind that the goal is not to write everything only in terms of symbols (with no words), but to use the symbols to *aid* communication.

My advice is: if you wouldn’t say it that way out loud, don’t write it that way (even if this means using more words and fewer symbols).

Once you have “got the right answer” there is a range of things you could do

Do nothing

Write out argument again with words


Once you have "got the right answer"
there is a range of things you could do
Do nothing
Write some words
in between the maths
you've written
Write out argument
again with words
ideal

Although writing out the argument again with words is what you should be aiming for, if you're short on time (e.g. in an exam) you could add some words in between the maths that you've written.

This is illustrated in `example1annotated.pdf`.

If you only remember one thing from this talk....

When you read the maths you've written it should make sense as a piece of English.

This is the beginning....

Learning to write maths well (i.e. learning to construct arguments that make sense when read aloud) will take some practice, but you will have lots of opportunities to do this through the problem sheets. You will also see many examples of mathematical arguments in lectures which (I hope!) will be written well.

Resources

- Pages 154–158 of "Discrete mathematics with applications" by Susanna S. Epp (in library as 510 EPP).
- Chapter 8 of "How to study for a mathematics degree" by Lara Alcock (physical copies in library as 510.711 ALC, online access available via library website).

And finally, for those disappointed that there will be no talk on how to dance maths....

Beautiful Dance Moves