Recap of material from notes

A family $a = (a_\hbar)_{0<\hbar\leq\hbar_0}$, with $a_\hbar \in C^\infty(T^*\mathbb{R}^d)$, is a *symbol of order* m, written as $a \in S^m(\mathbb{R}^d)$, if, for any multiindices α, β, there exists $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a_\hbar(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all } (x,\xi) \in T^*\mathbb{R}^d \text{ and for all } 0 < \hbar \leq \hbar_0 \tag{0.1}$$

For $a \in S^m$, we define the *semiclassical quantisation* of a, $\mathrm{Op}_\hbar(a)$ by

$$\big(\mathrm{Op}_\hbar(a)v\big)(x) := (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\exp\big(\mathrm{i}(x-y)\cdot\xi/\hbar\big)\, a(x,\xi)v(y)\,\mathrm{d}y\mathrm{d}\xi \tag{0.2}$$

for $v \in \mathscr{S}(\mathbb{R}^d)$, where the integral is understood as an iterated integral, with the y integration performed first, i.e.,

$$\big(\mathrm{Op}_\hbar(a)v\big)(x) = (2\pi\hbar)^{-d}\int_{\mathbb{R}^d}\exp\big(\mathrm{i}x\cdot\xi/\hbar\big)\, a(x,\xi)\mathcal{F}_\hbar v(\xi)\,\mathrm{d}\xi. \tag{0.3}$$

Lemma 0.1. $\mathrm{Op}_\hbar(a) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$.

Theorem 0.2. (Composition and mapping properties of semiclassical pseudodifferential operators.) *If $A \in \Psi_\hbar^{m_A}$ and $B \in \Psi_\hbar^{m_B}$, then*

(i) $A^* : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ and $A^* \in \Psi_\hbar^{m_A}$,

Lemma 0.3. *If $a \in \hbar^\infty S^{-\infty}$, then $\mathrm{Op}_\hbar(a) = O(\hbar^\infty)_{\Psi^{-\infty}}$.*

Definition 0.4. (Operator wavefront set.) *$(x_0,\xi_0) \in T^*\mathbb{R}^d$ is* not *in the* semiclassical operator wavefront set *of $A = \mathrm{Op}_\hbar(a) \in \Psi_\hbar^m$, denoted by $\mathrm{WF}_\hbar A$, if there exists a neighbourhood U of (x_0,ξ_0) such that for all multiindices α, β and all $N \geq 1$ there exists $C_{\alpha,\beta,N,U} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta,N,U}\hbar^N \quad \text{for all } (x,\xi) \in U \text{ and } 0 < \hbar \leq \hbar_0; \tag{0.4}$$

Definition 0.5. (Symbol class S^m_{phg}.) *$a \in S^m_{\mathrm{phg}}$ if $a \in S^m$ and there exist $a_j \in S^{m-j}$, independent of $\hbar$, such that, for all $N \in \mathbb{Z}^+$,*

$$a - \sum_{j=0}^{N-1}\hbar^j a_j \in \hbar^N S^{m-N}. \tag{0.5}$$

If $A = \mathrm{Op}_\hbar(a)$ for $a \in S^m_{\mathrm{phg}}$, we write $A \in \Psi^m_{\mathrm{phg}}$.

Lemma 0.6. (Definitions of compactly and properly supported in terms of cut-off functions.)
(i) A is compactly supported iff there exist $\chi_1, \chi_2 \in \mathcal{D}$ such that $A = \chi_1 A \chi_2$.
(ii) A is properly supported iff for any $\chi \in \mathcal{D}$ there exist $\chi_1, \chi_2 \in \mathcal{D}$ such that

$$\chi A = \chi A \chi_1, \qquad A\chi = \chi_2 A \chi.$$

Theorem 0.7. (Borel's theorem.) *Given $a_j \in S^{m-j}$, $j = 0, 1, \ldots$, there exists $a \in S^m$ such that $a \sim \sum_{j=0}^\infty \hbar^j a_j$ (in the sense of (0.5)).*

Lemma 0.8. *Suppose $a \in S^m$ and $a_j \in S^{m-j}$ $j = 0, 1, \ldots$ are such that $a \sim \sum_{j=0}^\infty \hbar^j a_j$. If $a_j \in S^{m-j}_{\mathrm{phg}}$, then $a \in S^m_{\mathrm{phg}}$.*

0.1 Exercises for Section 7

1. (i) Show that $a(x,\xi) = \sum_{|\gamma|\leq m} a_\gamma(x)\xi^\gamma$, where $a_\alpha \in C^\infty$ and $\partial^\gamma a_\alpha \in L^\infty$ for all γ and α, is in S^m.

   (ii) Show that $\langle\xi\rangle^{-m} \in S^{-m}$ for $m \in \mathbb{Z}^+$.

   (iii) Show that if $\chi \in C^\infty_{\mathrm{comp}}(T^*\mathbb{R}^d)$, then $\chi \in S^{-N}$ for every $N \geq 1$.

   Solution: (i)

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) = \sum_{|\gamma|\leq m}\partial_x^\alpha a_\gamma(x)\begin{cases}\gamma!((\gamma-\beta)!)^{-1}\xi^{\gamma-\beta} & \text{if } \beta \leq \gamma,\\ 0 & \text{otherwise.}\end{cases}$$

   The bound in (0.1) then follows since, when $\beta \leq \gamma$,

$$|\xi^{\gamma-\beta}| \leq |\xi|^{|\gamma|-|\beta|} = |\xi|^{m-|\beta|} \leq \langle\xi\rangle^{m-|\beta|}.$$

(iii) Since χ has compact support, $\partial_x^\alpha \partial_\xi^\beta \chi(x,\xi)$ vanishes as $|\xi| \to \infty$ faster than $|\xi|^{-N}$ for any $N > 0$, and thus $\chi \in S^{-N}$ for every $N \geq 1$.

(ii) If we can show that $\langle \xi \rangle^{-1} \in S^1$, then the result follows from the composition property ($a \in S^m, b \in S^\ell$, then $ab \in S^{m+\ell}$). We now prove by induction that for all $\beta \in \mathbb{N}^d$,

$$\partial_\xi^\beta \langle \xi \rangle^{-1} = \sum_{|\alpha| \leq |\beta|} P_\alpha(\xi) \langle \xi \rangle^{-1-|\alpha|-|\beta|} \tag{0.6}$$

for some polynomials P_α of degree $|\alpha|$. Indeed, assume that this holds for some β, and consider

$$\begin{aligned}
\partial_{\xi_i} \left[P_\alpha(\xi) \langle \xi \rangle^{-1-|\alpha|-|\beta|} \right] &= (\partial_{\xi_i} P_\alpha(\xi)) \langle \xi \rangle^{-1-|\alpha|-|\beta|} + (-1-|\alpha|-|\beta|) P_\alpha(\xi) \langle \xi \rangle^{-1-|\alpha|-|\beta|-1} \frac{\xi_i}{\langle \xi \rangle} \\
&= \underbrace{(\partial_{\xi_i} P_\alpha(\xi))}_{\text{degree } |\alpha|-1} \langle \xi \rangle^{-1-(|\alpha|-1)-(|\beta|+1)} \\
&\qquad + (-1-|\alpha|-|\beta|) \underbrace{\xi_i P_\alpha(\xi)}_{\text{degree } |\alpha|+1} \langle \xi \rangle^{-1-(|\alpha|+1)-(|\beta|+1)};
\end{aligned}$$

therefore (0.6) holds for multiindices of order $|\beta| + 1$.

Having proved (0.6), we now bound $|\partial_\xi^\beta(\langle \xi \rangle^{-1})|$. To do this, we use the following lemma.

Lemma. *Suppose that $F(X) = P(X)\langle X \rangle^{-m}$, where P is a polynomial of degree ℓ. Then there exists a constant C such that for all $X \in \mathbb{R}^d$,*

$$|F(X)| \leq C \langle X \rangle^{\ell - m}.$$

Proof. It suffices to show that the function

$$R(X) := P(X) \langle X \rangle^{-\ell}$$

is bounded on $\mathbb{R}^d$. We first note that since R is continuous on $\mathbb{R}^d$, it is bounded on the unit ball. Next we consider R outside the unit ball and write

$$R(X) = \frac{\sum_{|\alpha| \leq \ell} a_\alpha X^\alpha}{(1+|X|^2)^{\ell/2}}$$

Hence, for $|X| \geq 1$,

$$|R(X)| \leq \frac{|X|^\ell \sum_{|\alpha| \leq l} |a_\alpha|}{|X|^\ell} \leq \sum_{|\alpha| \leq \ell} |a_\alpha|.$$

This proves that R is also bounded outside the unit ball, and the result follows. $\square$

By (0.6),

$$\left| \partial_\xi^\beta \langle \xi \rangle^{-1} \right| \leq \sum_{|\alpha| \leq |\beta|} \left| P_\alpha(\xi) \right| \langle \xi \rangle^{-1-|\alpha|-|\beta|}. \tag{0.7}$$

By the lemma, there exists C_α such that, for all $\xi \in \mathbb{R}^d$,

$$\| P_\alpha(\xi) \langle \xi \rangle^{-1-|\alpha|-|\beta|} \| \leq C_\alpha \langle \xi \rangle^{-1-|\beta|};$$

thus, for all $\xi \in \mathbb{R}^d$,

$$|\partial_\xi^\beta \langle \xi \rangle| \leq \Big(\sum_\alpha C_\alpha \Big) \langle \xi \rangle^{-1-|\beta|};$$

i.e., $\langle \xi \rangle^{-1} \in S^{-1}$.

2. Prove Lemma 0.1. Hint: perform the y integral in (0.2) and then use the definition of $\mathcal{S}(\mathbb{R}^d)$.

   Solution: by performing the y integral,

$$\left(\mathrm{Op}_\hbar(a) v \right)(x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} \exp\left(\mathrm{i} x \cdot \xi / \hbar \right) a(x,\xi) \mathcal{F}_\hbar v(\xi) \, \mathrm{d}\xi,$$

so that

$$x^\alpha \partial_x^\beta \Big(\big(\mathrm{Op}_\hbar(a)v\big)(x)\Big) = \frac{x^\alpha}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} \partial_x^\beta\Big(\exp\left(\mathrm{i}x\cdot\xi/\hbar\right) a(x,\xi)\Big)\mathcal{F}_\hbar v(\xi)\,\mathrm{d}\xi,$$

$$= \frac{x^\alpha}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} \exp\left(\mathrm{i}x\cdot\xi/\hbar\right)\left(\sum_{\gamma:|\gamma|\leq|\beta|}\left(\begin{array}{c}\beta\\ \gamma\end{array}\right)\left(\frac{\mathrm{i}\xi}{\hbar}\right)^{\beta-\gamma}\partial_x^\gamma a(x,\xi)\right)\mathcal{F}_\hbar v(\xi)\,\mathrm{d}\xi.$$

The idea now is to integrate by parts on the right-hand side, bringing down inverse powers of $|x|$ to show that the right-hand side is bounded. A convenient way to do this is to observe that, with $D := (1/\mathrm{i})\partial$,

$$\left(\frac{1+x\cdot D_\xi}{1+|x|^2/\hbar}\right)\exp(\mathrm{i}x\cdot\xi/\hbar) = \exp(\mathrm{i}x\cdot\xi/\hbar),$$

so that, for any m,

$$x^\alpha \partial_x^\beta \Big(\big(\mathrm{Op}_\hbar(a)v\big)(x)\Big)$$

$$= \frac{x^\alpha}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} \exp\left(\mathrm{i}x\cdot\xi/\hbar\right)\left(\frac{1-x\cdot D_\xi}{1+|x|^2/\hbar}\right)^m\left[\left(\sum_{|\gamma|\leq|\beta|}\left(\begin{array}{c}\beta\\ \gamma\end{array}\right)\left(\frac{\mathrm{i}\xi}{\hbar}\right)^{\beta-\gamma}\partial_x^\gamma a(x,\xi)\right)\mathcal{F}_\hbar v(\xi)\right]\mathrm{d}\xi.$$

Since $\mathcal{F}_\hbar v \in \mathscr{S}(\mathbb{R}^d)$ and the derivatives of $a(x,\xi)$ satisfy (0.1), the integral on the right-hand side of this last equation is finite, and the result follows by choosing $m > |\alpha|$.

3. Prove Part (i) of Theorem 0.2 in the special case when A is a Fourier multiplier. Solution:

$$\big\langle \mathrm{Op}_\hbar(a)u, v\big\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi/\hbar} a(\xi)u(y)\,\mathrm{d}y\,\mathrm{d}\xi\right)\overline{v(x)}\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^d}\overline{\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(y-x)\cdot\xi/\hbar}\overline{a(\xi)}v(x)\,\mathrm{d}x\,\mathrm{d}\xi\right)}u(y)\,\mathrm{d}y = \big\langle u, \mathrm{Op}_\hbar(\overline{a})v\big\rangle_{\mathbb{R}^d}.$$

4. Prove Lemma 0.3. Hint: given $s>0, N\geq 1$, choose an appropriate $M\geq 1$, and use that $a\in\hbar^M S^{-M}$.

   Solution: Given $s>0, N\geq 1$, let $M:=\max\{N,2s\}$. By definition $a\in\hbar^M S^{-M}$, and so, by Part (iv) of Theorem 0.2, given $\hbar_0>0$, there exists $C_{s,M}$ such that

$$\hbar^{-M}\left\|\mathrm{Op}_\hbar(a)\right\|_{H_\hbar^{-s}\to H_\hbar^{-s+M}} \leq C_{s,M} \qquad \text{for all } 0<\hbar\leq\hbar_0.$$

   Then, since $M\geq 2s$, by the definition of $\|\cdot\|_{H_\hbar^s}$,

$$\left\|\mathrm{Op}_\hbar(a)\right\|_{H_\hbar^{-s}\to H_\hbar^{s}} \leq \left\|\mathrm{Op}_\hbar(a)\right\|_{H_\hbar^{-s}\to H_\hbar^{-s+M}} \leq C_{s,M}\hbar^M \lesssim C_{s,M}\hbar^N,$$

   and thus $\mathrm{Op}_\hbar(a) = O(\hbar^\infty)_{\Psi^{-\infty}}$ by the definition of the latter.

5. If $P_\hbar u := -\hbar^2\nabla\cdot(A\nabla u) - nu$, show that

   (i) $P_\hbar$ is the quantisation of a symbol in S^2_{phg}, and

   (ii) $\sigma_\hbar(P_\hbar) = (A\xi)\cdot\xi - n \in S^2$.

   Solution: since $P_\hbar u = -\hbar^2 A_{j\ell}\partial_j\partial_\ell u - \hbar^2(\partial_j A_{j\ell})(\partial_\ell u) - n$, the fact that $P_\hbar = \mathrm{Op}_\hbar\big((A\xi)\cdot\xi - n - \mathrm{i}\hbar\xi_\ell\partial_j A_{j\ell}\big)$ follows from the definition of $\mathrm{Op}_\hbar$. The fact that the symbol is in S^2_{phg} follows from Definition 0.5, and the fact that $\sigma_\hbar(P_\hbar) = (A\xi)\cdot\xi - n\in S^2$ then follows from the fact that $\hbar\xi_\ell\partial_j A_{j\ell}\in\hbar S^1$.

6. Prove that if $a(x,\xi)$ is independent of $\hbar$, then $\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)) = \operatorname{supp} a$. Solution: If $(x_0,\xi_0)\notin\operatorname{supp} a$, then there exists a neighbourhood U of (x_0,ξ_0) such that $a(x,\xi)=0$ for all $(x,\xi)\in U$. Therefore, by (0.4), $(x_0,\xi_0)\in(\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)))^c$; i.e., $(\operatorname{supp} a)^c\subset(\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)))^c$.

   Conversely, if $(x_0,\xi_0)\in(\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)))^c$ and a is independent of $\hbar$, then by (0.4) there exists a neighbourhood U of (x_0,ξ_0) such that $a(x,\xi)=0$ for all $(x,\xi)\in U$. Therefore $(x_0,\xi_0)\in(\operatorname{supp} a)^c$; i.e. $(\mathrm{WF}_\hbar(\mathrm{Op}_\hbar(a)))^c\subset(\operatorname{supp} a)^c$.

7. Prove Lemma 0.6. Solution:

(i) $\Rightarrow$ Since K_A is compactly supported, there exist $\chi_1, \chi_2 \in \mathcal{D}$ such that

$$K_A(x,y) = \chi_1(x) K_A(x,y) \chi_2(y). \tag{0.8}$$

$\Leftarrow$ The assumption implies that there exist $\chi_1, \chi_2 \in \mathcal{D}$ such that (0.8) holds. Since χ_1 and χ_2 both have compact support, so does K_A.

(ii) $\Leftarrow$ Given a compact $X \subset \mathbb{R}^d$, there exists $\chi_X \in \mathcal{D}$ such that $\chi_X = 1$ on X. By assumption there exists $\chi_Y \in \mathcal{D}$ such that $\chi_X A = \chi_X A \chi_Y$. Thus

$$\chi_X(x) K_A(x,y) = \chi_X(x) K_A(x,y) \chi_Y(y) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{0.9}$$

Therefore

$$\{(x,y) \in \operatorname{supp} K_A : x \in \operatorname{supp} X\} \subset \{(x,y) \in \operatorname{supp} K_A : x \in \operatorname{supp} \chi_X\} \subset \operatorname{supp} \chi_Y,$$

which is compact. The proof that, for compact Y, $\{(x,y) \in \operatorname{supp} K_A : y \in \operatorname{supp} Y\}$ is compact is similar.

$\Rightarrow$ Given $\chi_X \in \mathcal{D}$, by assumption $\{(x,y) \in \operatorname{supp} K_A : x \in \operatorname{supp} \chi_X\}$ is compact. Therefore there exists $\chi_Y \in \mathcal{D}$ such that $\chi_Y = 1$ on this last set. Then (0.9) holds so $\chi_X A = \chi_X A \chi_Y$. The proof that, given $\chi_Y \in \mathcal{D}$ there exists $\chi_X \in \mathcal{D}$ such that $A\chi_Y = \chi_X A \chi_Y$ is similar.

8. Prove Theorem 0.7 via the following steps.

   (a) Let $\chi \in C^\infty_{\text{comp}}(\mathbb{R})$ with $\chi \equiv 1$ on $[-1,1]$. Show that if $\{\lambda_j\}_{j=0}^\infty \subset \mathbb{R}$ with $\lambda_j \to \infty$, the sum

$$a(x,\xi) := \sum_{j=0}^\infty \chi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right) \hbar^j a_j(x,\xi)$$

   converges.

   Solution: Given ξ_0 and $\hbar > 0$, since $\lambda_j \to \infty$ and χ has compact support, there exists $J \in \mathbb{Z}^+$ such that $\chi(\lambda_j \hbar \langle \xi \rangle^{-1}) = 0$ for all $j \geq J$. Therefore, for each x_0, ξ_0, and $\hbar > 0$, the sum converges since there are at most finitely-many non-zero terms.

   (b) Show that, given β and $\chi \in C^\infty_{\text{comp}}(\mathbb{R})$, there exists $C_{\beta,\chi}$ such that

$$\partial_\xi^\beta \left(\chi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right)\right) \leq \frac{C_{\beta,\chi}}{\lambda_j \hbar} \langle \xi \rangle^{1-|\beta|}. \tag{0.10}$$

   Solution: We prove the result via induction on $|\beta|$. For $|\beta| = 0$, we write

$$\chi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right) = \frac{\lambda_j \hbar}{\langle \xi \rangle} \chi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right) \frac{\langle \xi \rangle}{\lambda_j \hbar},$$

   and the result for $|\beta| = 0$ holds with $C_{0,\chi} = \sup_{t \in \mathbb{R}} t\chi(t)$.

   Now

$$\partial_{\xi_i} \partial_\xi^\beta \chi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right) = \partial_\xi^\beta \left(-\frac{\lambda_j \hbar \xi_i}{\langle \xi \rangle^3} \chi'\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right)\right). \tag{0.11}$$

   By Exercise 1, $\xi_i \in S^1$ and $\langle \xi \rangle^{-1} \in S^{-1}$.

   Our aim is to apply the Leibniz formula to the right-hand side of the last displayed equation, and use the induction hypothesis. However, a direct application of this to $(\xi_i \langle \xi \rangle^{-3})\chi'$ obtains the bound $C_{\beta,\chi} \langle \xi \rangle^{-1-|\beta|}$, i.e., a better bound in $\langle \xi \rangle$, but missing a factor of $1/(\lambda_j \hbar)$.

   If we apply the Leibniz formula to $b\psi$, with ψ satisfying (0.10) and $b \in S^0$, we get the bound $C_{\beta,\psi} \langle \xi \rangle^{1-|\beta|}$. Motivated by this, we let $b := \xi_i \langle \xi \rangle$ (which is in S^0 by Theorem 0.2 (ii)), and let $\psi(y) = y^2 \chi'(y)$. Observe that $\psi \in C^\infty_{\text{comp}}(\mathbb{R})$, and thus (0.10) holds with χ replaced by ψ.

   Applying the Leibniz formula to $b\psi$, we find

$$\partial_\xi^\beta \left((\lambda_j \hbar)^2 \frac{\xi_i}{\langle \xi \rangle^3} \chi'\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right)\right) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \partial_\xi^{\beta'} \left(\psi\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right)\right) \partial_\xi^{\beta - \beta'} b.$$

   By the triangle inequality, the induction hypothesis applied to ψ, and the fact that $b \in S^0$,

$$\partial_\xi^\beta \left((\lambda_j \hbar)^2 \frac{\xi_i}{\langle \xi \rangle^3} \chi'\left(\frac{\lambda_j \hbar}{\langle \xi \rangle}\right)\right) \leq \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \frac{C_{\beta',\psi}}{\lambda_j \hbar} \langle \xi \rangle^{1-|\beta'|} C_b \langle \xi \rangle^{|\beta'|-|\beta|} \leq \frac{C_{\beta,\chi}}{\lambda_j \hbar} \langle \xi \rangle^{1-|\beta|}.$$

   Combining this last inequality with (0.11) and using that $(\lambda_j \hbar)^{-1} \leq C$ (since $\hbar$ is fixed and $\lambda_j \to \infty$), we obtain the result.

(c) Show that there is an increasing sequence $\{\lambda_j\}_{j=0}^{\infty}$ with $\lambda_j \to \infty$ such that for any multiindices $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \leq j$,

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) a_j \right) \right| \leq 2^{-j} \hbar^{-1} \langle\xi\rangle^{m-j-|\beta|+1}.$$

Solution: by the Leibniz rule

$$\partial_x^\alpha \partial_\xi^\beta \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) a_j \right) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \partial_\xi^{\beta'} \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) \right) \partial_x^\alpha \partial_\xi^{\beta-\beta'} a_j.$$

By the triangle inequality, Part (b), and the fact that $a_j \in S^{m-j}$,

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) a_j \right) \right| \leq \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \frac{C_{\beta',\chi}}{\lambda_j \hbar} \langle\xi\rangle^{1-|\beta'|} C_{\alpha,\beta-\beta'} \langle\xi\rangle^{m-j-|\beta|+|\beta'|} \leq \frac{\widetilde{C}_{\alpha,\beta}}{\lambda_j \hbar} \langle\xi\rangle^{m-|\beta|+1}.$$

Choosing $(\lambda_j)_{j=0}^{\infty}$ such that $\lambda_j \geq \widetilde{C}^{\alpha,\beta} 2^{-j}$, we obtain the result.

(d) With the choice of λ_j from (c), show that for any $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \leq N$,

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \right) \right| \leq C_{\alpha\beta N} \hbar^N \langle\xi\rangle^{m-|\beta|-N}, \tag{0.12}$$

and conclude that $a \sim \sum_j h^j a_j$.

Solution: We now assume that $\hbar \leq 1$ (if instead $\hbar \leq \hbar_0$, we replace 2^{-j} by $(h_0+1)^{-j}$ in Part (c) and in the rest of the argument).

$$\begin{aligned} a(x,\xi) - \sum_{j=0}^{N} \hbar^j a_j(x,\xi) &= \sum_{j=0}^{N} \hbar^j \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) - 1 \right) a(x,\xi) + \sum_{j=N+1}^{\infty} \hbar^j \chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) a_j(x,\xi) \\ &=: T_1(x,\xi,N) + T_2(x,\xi,N). \end{aligned}$$

By the result of Part (c),

$$\begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta T_2(x,\xi,N) \right| \leq \sum_{j=N+1}^{\infty} \frac{1}{2^j} \hbar^{j-1} \langle\xi\rangle^{m-j-|\beta|+1} &= \frac{\langle\xi\rangle^{m-|\beta|+1}}{\hbar} \sum_{j=N+1}^{\infty} \left(\frac{\hbar}{2\langle\xi\rangle} \right)^j \\ &\leq 2^{-N} \hbar^N \langle\xi\rangle^{m-|\beta|-N}, \end{aligned} \tag{0.13}$$

where we have used that $\hbar/(2\langle\xi\rangle) \leq 1/2$.

Since $\chi \equiv 1$ on $[-1,1]$, if $\hbar \leq \lambda_j^{-1}\langle\xi\rangle$ for $j = 1, \ldots, N$, then $\delta_1(x,\xi,N) = 0$. This condition is ensured if $\hbar \leq \lambda_N^{-1}\langle\xi\rangle$ (since λ_j is increasing), so we now assume, without loss of generality, that $\hbar \geq \lambda_N^{-1}\langle\xi\rangle$, i.e.

$$\left(\langle\xi\rangle \hbar^{-1} \lambda_N^{-1} \right)^{-1} \geq 1. \tag{0.14}$$

Now

$$\partial_x^\alpha \partial_\xi^\beta T_1(x,\xi) = \sum_{j=0}^{N} \hbar^j \partial_\xi^\beta \left(\chi\left(\frac{\lambda_j \hbar}{\langle\xi\rangle}\right) \partial_x^\alpha a(x,\xi) \right) - \sum_{j=0}^{N} \hbar^j \partial_x^\alpha \partial_\xi^\beta a(x,\xi) =: T_{11} - T_{12}.$$

Using the fact that $a \in S^m$, the inequality (0.14), and the fact that $\hbar \leq 1$, we have

$$|T_{12}| \leq \sum_{j=0}^{N} \hbar^j C_{\alpha\beta} \langle\xi\rangle^{m-|\beta|} \leq \sum_{j=0}^{N} \hbar^j C_{\alpha\beta} \langle\xi\rangle^{m-|\beta|} \left(\langle\xi\rangle \lambda_N^{-1} \hbar^{-1} \right)^{-N} \leq \left(2\lambda_N^N C_{\alpha,\beta} \right) \hbar^N \langle\xi\rangle^{m-|\beta|-N}. \tag{0.15}$$

By the Leibniz rule, the bound (0.10), and the fact that $a \in S^m$,

$$\begin{aligned} |T_{11}| &\leq \sum_{j=0}^{N} \hbar^j \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \frac{C_{\beta',\chi}}{\lambda_j \hbar} \langle\xi\rangle^{1-|\beta'|} C_{\alpha,\beta-\beta'} \langle\xi\rangle^{m-|\beta|+|\beta'|} \\ &\leq \frac{2}{\lambda_1 \hbar} C_{\alpha,\beta,\chi} \langle\xi\rangle^{m-|\beta|+1} \\ &\leq \frac{2}{\lambda_1 \hbar} C_{\alpha,\beta,\chi} \langle\xi\rangle^{m-|\beta|+1} \left(\langle\xi\rangle \lambda_N^{-1} \hbar^{-1} \right)^{-N-1} \leq 2\lambda_N^N C_{\alpha,\beta,\chi} \hbar^N \langle\xi\rangle^{m-|\beta|-N}. \end{aligned} \tag{0.16}$$

Combining (0.13), (0.15), and (0.16) completes the proof of (0.12).

9. Prove Lemma 0.8.

Solution: Since $a \sim \sum_{j=0}^{\infty} a_j$, given $N \in \mathbb{Z}^+$, there exists $R_N \in S^{m-N}$ such that

$$a = \sum_{j=0}^{N-1} \hbar^j a_j + \hbar^N R_N\,.$$

Since $a_j \in S^{m-j}_{\mathrm{phg}}$, there exist symbols $a_{jk} \in S^{m-j-k}$ independent of $\hbar$, and $Q_{N-j} \in S^{m-N}$ such that

$$a_j = \sum_{k=0}^{N-1-j} \hbar^k a_{jk} + \hbar^{N-j} Q_{N-j}\,.$$

Therefore

$$a = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1-j} \hbar^{j+k} a_{jk} + \hbar^N \left(R_N + \sum_{j=0}^{N-1} Q_{N-j} \right).$$

Now

$$\sum_{j=0}^{N-1} \sum_{k=0}^{N-1-j} \hbar^{j+k} a_{jk} = \sum_{p=0}^{N-1} \sum_{q=0}^{p} \hbar^p a_{q(p-q)}.$$

So

$$a = \sum_{p=0}^{N-1} \hbar^p \tilde{a}_p + \hbar^N \widetilde{R}_N$$

where $\tilde{a}_p := \sum_{q=0}^{p} a_{q,p-q} \in S^{m-p}$ are independent of $\hbar$ and $\widetilde{R}_N := R_N + \sum_{j=0}^{N-1} Q_{N-j} \in S^{m-N}$; i.e., $a \in S^m_{\mathrm{phg}}$.