Recap of material from notes

We say that $a \in C^\infty(\mathbb{R}^d)$ is a *Fourier symbol* if there exists $m \in \mathbb{R}$ such that for any multiindex β there exists C_β such that

$$|\partial_\xi^\beta a(\xi)| \leq C_\beta \langle\xi\rangle^{m-|\beta|} \quad \text{for all } \xi \in \mathbb{R}^d. \tag{0.1}$$

We say that m is the *order* of the Fourier symbol and use the (non-standard) notation that $a \in (FS)^m$.

Given a Fourier symbol a, the *Fourier multiplier* defined by a is given by

$$\big(a(\hbar D)v\big)(x) = \mathcal{F}_\hbar^{-1}\big(a(\cdot)(\mathcal{F}_\hbar v)(\cdot)\big)(x). \tag{0.2}$$

Lemma 0.1 *$a(\hbar D) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}^*(\mathbb{R}^d) \to \mathscr{S}^*(\mathbb{R}^d)$.*

$$\|u\|^2_{H^s_\hbar(\mathbb{R}^d)} := (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |\mathcal{F}_\hbar u(\xi)|^2 \, d\xi; \tag{0.3}$$

Theorem 0.2 (Composition and mapping properties of Fourier multipliers.) *If $a \in (FS)^{m_1}$ and $b \in (FS)^{m_2}$ then the following hold.*

(i) $ab \in (FS)^{m_1+m_2}$.

(ii) $a(\hbar D)b(\hbar D) = (ab)(\hbar D) = b(\hbar D)a(\hbar D)$.

(iii) $a(\hbar D) : H^s_\hbar \to H^{s-m_1}_\hbar$ and there exists $C > 0$ such that, for all $s \in \mathbb{R}$ and $\hbar > 0m$

$$\|a(\hbar D)\|_{H^s_\hbar \to H^{s-m_1}_\hbar} \leq C.$$

i.e., $a(\hbar D)$ is bounded uniformly in both $\hbar$ and s as an operator from $H^s_\hbar$ to $H^{s-m_1}_\hbar$.

Exercises for Section 5

1. Prove Lemma 0.1.

   Solution: if $v \in \mathscr{S}$, then $\mathcal{F}_\hbar v \in \mathscr{S}$. By the derivative bounds on a in (0.1), $a(\cdot)(\mathcal{F}_\hbar v)(\cdot) \in \mathscr{S}$, and then so is $\mathcal{F}_\hbar^{-1}(a\mathcal{F}_\hbar v) =: a(\hbar D)v$.

2. Prove Theorem 0.2. Solution:

   (i) By Leibnitz's rule,

   $$\partial^\beta(ab) = \sum_{|\alpha|\leq|\beta|} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} (\partial^\alpha a)(\partial^{\beta-\alpha} b),$$

   so that

   $$\big|\partial^\beta(ab)(\xi)\big| \lesssim \sum_{|\alpha|\leq|\beta|} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} |\xi|^{m_A-|\alpha|} |\xi|^{m_B-(|\beta|-|\alpha|)} = \sum_{|\alpha|\leq|\beta|} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} |\xi|^{m_A+m_B-|\beta|},$$

   where we have used that $|\beta - \alpha| = |\beta| - |\alpha|$ since $|\alpha| \leq |\beta|$.

   (ii)

   $$\begin{aligned} a(\hbar D)\big(b(\hbar D)v\big)(x) &= \mathcal{F}_\hbar^{-1}\big(a(\cdot)(\mathcal{F}_\hbar(b(\hbar D)v))(\cdot)\big)(x) \\ &= \mathcal{F}_\hbar^{-1}\big(a(\cdot)b(\cdot)(\mathcal{F}_\hbar v)(\cdot)\big)(x) \\ &= (ab)(\hbar D)v(x) \\ &= (ba)(\hbar D)v(x) = b(\hbar D)\big(a(\hbar D)\big)v(x). \end{aligned}$$

   (iii) By the definitions of $\|\cdot\|_{H^s_\hbar}$ (0.3) and $a(\hbar D)v$ (0.2),

   $$\begin{aligned} \|a(\hbar D)v\|^2_{H^{s-m_A}_\hbar} &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \langle\xi\rangle^{2(s-m_A)} \big|\mathcal{F}_\hbar\big(a(\hbar D)v\big)(\xi)\big|^2 \, d\xi \\ &= \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \langle\xi\rangle^{2(s-m_A)} \big|a(\xi)\big(\mathcal{F}_\hbar v\big)(\xi)\big|^2 \, d\xi \\ &\lesssim \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \langle\xi\rangle^{2(s-m_A)} \langle\xi\rangle^{2m_A} \big|\big(\mathcal{F}_\hbar v\big)(\xi)\big|^2 \, d\xi = \|v\|^2_{H^s_\hbar}. \end{aligned}$$