## Exercises for Section 4

**Lemma 0.1 (Analyticity from derivative bounds)** *If $u \in C^\infty(D)$ and there exist $C_1, C_2 > 0$ such that*

$$\|\partial^\alpha u\|_{L^2(D)} \leq C_1 (C_2)^{|\alpha|} |\alpha|! \quad \text{for all } \alpha, \tag{0.1}$$

*then $u$ is real analytic in $D$.*

1. Prove Lemma 0.1 via the following steps.

   (a) Show that the result follows if there exists $n_0 \in \mathbb{Z}^+$ such that

   $$\|\partial^\alpha u\|_{L^\infty(D)} \leq \widetilde{C}_1 (\widetilde{C}_2)^{|\alpha|} (|\alpha| + n_0)!. \tag{0.2}$$

   Hint: bound the Lagrange form of the remainder in the Taylor-series up to $n-1$ terms, i.e.,

   $$\sum_{|\alpha|=n} \frac{(x-x')^\alpha}{\alpha!} \big(\partial^\alpha u\big(x' + c(x-x')\big)\big),$$

   for some $c \in (0,1)$, and use the consequence of the binomial theorem that

   $$\sum_{|\alpha|=n} \frac{n!}{\alpha!} = d^n. \tag{0.3}$$

   Solution: By (0.2) and (0.3),

   $$\left| \sum_{|\alpha|=n} \frac{(x-x')^\alpha}{\alpha!} \big(\partial^\alpha u\big(x' + c(x-x')\big)\big) \right| \leq \sum_{|\alpha|=n} \frac{n!}{\alpha!} (n+1)\ldots(n+n_0) \widetilde{C}_1 (\widetilde{C}_2)^n |x-x'|^n,$$
   $$= \widetilde{C}_1 (n+1)\ldots(n+n_0) \big(d\widetilde{C}_2 |x-x'|\big)^n,$$

   which $\to 0$ as $n \to \infty$ if $|x-x'| < (d\widetilde{C}_2)^{-1}$.

   (b) Prove (0.2) using the Sobolev embedding theorem (see, e.g., [1, Theorem 3.26]).
   Solution: Let $n_0 := \lceil (d+1)/2 \rceil$. Then, by the Sobolev embedding theorem (see, e.g., [1, Theorem 3.26]), (0.1), and the fact that $|\alpha+\beta| = |\alpha| + |\alpha|$, there exists $C > 0$ such that

   $$\|\partial^\alpha u\|_{L^\infty(D)} \leq C \sum_{|\alpha| \leq n_0} \big\|\partial^{\alpha+\beta} u\big\|_{L^2(D)} \leq C\, C_1\, C_2^{|\alpha|} \bigg( \sum_{|\alpha| \leq n_0} C_2^{|\alpha|} (|\alpha| + |\alpha|)! \bigg),$$
   $$\leq C\, C_1\, C_2^{|\alpha|} (|\alpha| + n_0)! \bigg( \sum_{|\alpha| \leq n_0} C_2^{|\alpha|} \bigg),$$

   so that (0.2) holds with $\widetilde{C}_2 := C_2$ and $\widetilde{C}_1 := C\, C_1 \big( \sum_{|\alpha| \leq n_0} C_2^{|\alpha|} \big)$.

2. (Proof of the bound on the solution of the "modified Helmholtz equation".) Given $f \in L^2(\mathbb{R}^d)$, and $A$ and $n$ satisfying Assumption 1.1 with $\Omega_- = \emptyset$, let $u \in H^1(\mathbb{R}^d)$ be the solution of $-k^{-2}\nabla \cdot (A\nabla u) + nu = f$ in $\mathbb{R}^d$. Prove that $u$ exists, is unique, and satisfies the bound

   $$\|u\|_{H^1_k(\mathbb{R}^d)} \leq \frac{1}{\min\{A_{\min}, n_{\min}\}} \|f\|_{L^2(\mathbb{R}^d)}$$

   for all $k > 0$. Hint: consider the variational problem satisfied by $u$. Solution: $u \in H^1(\mathbb{R}^d)$ is the solution of the variational problem

   $$\int_{\mathbb{R}^d} k^{-2} (A\nabla u) \cdot \overline{\nabla v} + nu\overline{v} = \int_{\mathbb{R}^d} f\overline{v} \quad \text{for all } v \in H^1(\mathbb{R}^d).$$

   The sesquilinear form on the left-hand side is continuous and coercive on $H^1(\mathbb{R}^d)$ (compare to the sesquilinear form in Question 1 from the exercises in §2), with coercivity constant $\min\{A_{\min}, n_{\min}\}$ in the $H^1_k(\mathbb{R}^d)$ norm. The result then follows from the Lax–Milgram theorem.

# References


[1] W. C. H. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.