

Synthesis, as Opposed to Separation, of Variables*

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Abstract. Every applied mathematician has used separation of variables. For a given boundary value problem (BVP) in two dimensions, the starting point of this powerful method is the separation of the given PDE into two ODEs. If the spectral analysis of either of these ODEs yields an appropriate transform pair, i.e., a transform consistent with the given boundary conditions, then the given BVP can be reduced to a BVP for an ODE. For simple BVPs it is straightforward to choose an appropriate transform and hence the spectral analysis can be avoided. In spite of its enormous applicability, this method has certain limitations. In particular, it requires the given domain, PDE, and boundary conditions to be separable, and also may not be applicable if the BVP is non-self-adjoint. Furthermore, it expresses the solution as either an integral or a series, neither of which are uniformly convergent on the boundary of the domain (for nonvanishing boundary conditions), which renders such expressions unsuitable for numerical computations. This paper describes a recently introduced transform method that *can* be applied to certain nonseparable and non-self-adjoint problems. Furthermore, this method expresses the solution as an integral in the complex plane that is uniformly convergent on the boundary of the domain. The starting point of the method is to write the PDE as a one-parameter family of equations formulated in a divergence form, and this allows one to consider the variables together. In this sense, the method is based on the “synthesis” as opposed to the “separation” of variables. The new method has already been applied to a plethora of BVPs and furthermore has led to the development of certain novel numerical techniques. However, a large number of related analytical and numerical questions remain open. This paper illustrates the method by applying it to two particular non-self-adjoint BVPs: one for the linearized KdV equation formulated on the half-line, and the other for the Helmholtz equation in the exterior of the disc (the latter is non-self-adjoint due to the radiation condition). The former problem played a crucial role in the development of the new method, whereas the latter problem was instrumental in the full development of the classical transform method. Although the new method can now be presented using only classical techniques, it actually originated in the theory of certain nonlinear PDEs called *integrable*, whose crucial feature is the existence of a *Lax pair* formulation. It is shown here that Lax pairs provide the generalization of the divergence formulation from a separable linear to an integrable nonlinear PDE.

Key words. separation of variables, transform method, boundary value problem, KdV equation, heat equation, diffusion equation, Helmholtz equation, Watson transformation, Lax pair, integrability

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I. Introduction. Separation of variables is the main tool used for finding explicit (or “exact”) solutions to boundary value problems (BVPs). Although introduced in the 1750s, this technique continued to be developed until the 1960s, particularly regarding the question of how the different expressions of the solution of a given BVP obtained via different transforms are related to each another. Indeed, in 1964 D. S. Cohen proclaimed, “The technique of separation of variables has not yet been fully exploited” [19]. The purpose of this paper is to add the further remark, “and one can do even better by using synthesis, as opposed to separation, of variables.”

This new development has arisen out of ideas and techniques originally introduced in [38] in the context of certain nonlinear PDEs and applied to linear PDEs in [39], [41]; see also [23]. The connection of this new transform method to classical approaches for solving linear BVPs, such as classical transforms and Green’s functions, although partially established soon afterwards [14], [49], has only recently been fully elucidated [102], [103], [41].

The goal of this paper is to make clear the connection of the new method to the classical approaches based on separation of variables, and to justify the statement that the application of the new method to linear PDEs can be understood as synthesis, as opposed to separation, of variables. We emphasize that a large number of analytical and numerical questions related to the new method remain open; an additional goal of this paper is to introduce the method to a wide audience with the expectation that this will lead to the solution of at least some of these problems. We begin by introducing the two examples considered in this paper. Although these examples involve BVPs in two dimensions, we note that the new method, like the classical transform method, can also be applied to BVPs in three dimensions.

I.1. Example I: The Linearized KdV Equation. Our first example is the linearized KdV equation

$$(1.1) \quad u_t + u_x + u_{xxx} = 0.$$

It will be useful to juxtapose this equation with the heat, or diffusion, equation

$$(1.2) \quad u_t - u_{xx} = 0.$$

The heat equation is one of the classical equations of linear mathematical physics. Indeed, transform methods were discovered through Fourier’s investigation of this equation posed on a finite interval in x . Equation (1.1) is the linearized version of the celebrated KdV equation

$$(1.3) \quad u_t + u_x + uu_x + u_{xxx} = 0,$$

which has played a key role in the understanding of certain nonlinear phenomena. The KdV equation, which is the simplest PDE combining nonlinearity (the uu_x term) with dispersion (the u_{xxx} term), describes several physical processes, including long, small amplitude water waves in a shallow channel [1, section 1.2], [27, section 1.2] (see [69] for a short but inspiring discussion of the KdV equation and its significance by the late Martin Kruskal). The KdV equation usually appears without the u_x term because it is usually studied on the line $-\infty < x < \infty$, and so u_x can be eliminated via a Galilean transformation. However, if the KdV is formulated on the half-line, the Galilean transformation changes the domain into a wedge. The transformation $u \mapsto u + 1$, which also eliminates the u_x term, is not suitable since the resulting dependent variable, $u + 1$, does not decay as $x \rightarrow \infty$.

The simplest problem one can pose for an evolution PDE is the following initial value problem (IVP): $u(x, t)$ satisfies the given PDE on the line $-\infty < x < \infty$, with a given initial condition $u(x, 0) = u_0(x)$ (where $u_0(x)$ decays as $|x| \rightarrow \infty$), and $u(x, t)$ vanishes as $|x| \rightarrow \infty$ for all $t > 0$. The Fourier transform immediately yields the solution of the IVP of *any* evolution PDE. For example, for the linearized KdV (1.1),

$$(1.4) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu x + i(\nu^3 - \nu)t} \hat{u}_0(\nu) d\nu, \quad -\infty < x < \infty, \quad t > 0,$$

where $\hat{u}_0(\nu)$ denotes the Fourier transform of $u_0(x)$. Indeed, the exponential dependence on (x, t) of the integrand in (1.4) immediately implies that $u(x, t)$ satisfies (1.1). Also, evaluating (1.4) at $t = 0$ and using the inverse Fourier transform we find $u(x, 0) = u_0(x)$.

The simplest initial-boundary value problem (IBVP) for an evolution PDE is the problem on the half-line $0 < x < \infty$, where the initial condition $u(x, 0) = u_0(x)$ is supplemented with an appropriate number of boundary conditions at $x = 0$ and $u(x, t)$ decays as $x \rightarrow \infty$ for all $t > 0$. Both the heat equation and the linearized KdV equation are well-posed by specifying one boundary condition at $x = 0$.

The IBVP for the heat equation on the half-line can be solved by either a transform in t , namely the Laplace transform, or a transform in x . For the Dirichlet problem, i.e., for the case that $u(0, t)$ is given,

$$u(0, t) = g_0(t), \quad 0 < t < T,$$

where T is a positive constant, it is straightforward to verify that the appropriate x -transform is the sine transform. Similarly, for the Neumann problem, i.e., for the case that $u_x(0, t)$ is given, the appropriate transform is the cosine transform. As will be discussed below, the Laplace transform involves certain complications, thus we employ the sine transform to obtain the following expression for the solution:

$$(1.5) \quad u(x, t) = \frac{2}{\pi} \int_0^{\infty} e^{-\nu^2 t} \sin(\nu x) \left[\int_0^{\infty} u_0(\xi) \sin(\nu \xi) d\xi + \nu \int_0^t e^{\nu^2 \tau} g_0(\tau) d\tau \right] d\nu.$$

The ν -integral on the right-hand side of (1.5) is *not* uniformly convergent with respect to the parameter x . Indeed, since the integrand vanishes at $x = 0$, if the integral were a continuous function of x , then $u(0, t)$ would be zero, contradicting the boundary condition $u(0, t) = g_0(t)$. This lack of uniform convergence, in addition to rendering such expressions unsuitable for numerical computations, also makes it difficult to verify that the solution to an IBVP obtained using an appropriate transform is indeed a solution. In this respect we note that the construction of the solution via any transform method *assumes* that a solution exists. Thus, unless one can appeal to PDE existence results, one must verify that the final formula obtained via any transform method *does* satisfy the PDE and the given initial and boundary conditions. For IVPs this is straightforward, as already shown for the linearized KdV equation (1.1), but for IBVPs, in order to verify that the given boundary conditions hold, one must overcome the difficulty of nonuniform convergence.

Using classical transforms to derive an expression for the solution of the linearized KdV equation (1.1) on the half-line is rather problematic. Indeed, as will be discussed in section 2, there does *not* exist a classical x -transform for this IBVP (and this is true even if the u_x term is omitted). In other words, for an evolution PDE containing a third order derivative, there do *not* exist appropriate analogues of the sine and

cosine transforms! Thus, as far as classical transforms are concerned one can only use the Laplace transform in t . However, this transform is, in a sense, inappropriate, since it involves integrating from zero to infinity in time, and hence the final formula involves boundary conditions at times greater than t . By causality, one expects that these contributions vanish, but establishing this fact involves both formal and rigorous complications. Furthermore, even bypassing this difficulty, one encounters the problem that if a PDE involves a derivative of order n , one has to analyze the n th root of a polynomial. In particular, for the linearized KdV (1.1) on the half-line, after taking the Laplace transform one finds

$$(1.6) \quad \tilde{u}_{xxx}(x, s) + \tilde{u}_x(x, s) + s\tilde{u}(x, s) = u_0(x), \quad 0 < x < \infty,$$

where $\tilde{u}(x, s)$ denotes the Laplace transform of $u(x, t)$. In order to construct an appropriate Green's function for this ODE, one seeks a solution of the homogeneous version of (1.6) in the form $\exp(\lambda(s)x)$, which yields the *cubic* equation $\lambda(s)^3 + \lambda(s) + s = 0$. This cube root dependence on the transform parameter in the Laplace transform solution should be contrasted with the polynomial dependence in the Fourier transform solution to the IVP (1.4); thus, for an evolution PDE a transform in x is preferable to a transform in t .

The new method yields uniformly convergent integral expressions of the solution of IBVPs for both (1.2) and (1.1), and indeed for evolution PDEs of arbitrary order in x . These expressions provide the proper generalization of the Fourier transform solution of the IVP. Indeed, (a) the (x, t) -dependence of these novel expressions is *identical* to the exponential (x, t) -dependence of the Fourier transform solution (which immediately implies that these expressions satisfy the given PDE); (b) the contour of integration along the real axis is now supplemented with a contour of integration in the complex ν -plane (the complex Fourier plane); (c) the function $\hat{u}_0(\nu)$ is now supplemented by the functions $\{\tilde{g}_j(\nu)\}_{j=0}^{n-1}$, which denote appropriate t -transforms of $\{\partial_x^j u(0, t)\}_{j=0}^{n-1}$ (where n is the order of the highest x -derivative in the PDE).

In particular, for the Dirichlet problem of the heat equation on the half-line, instead of (1.5), we now find

$$(1.7) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu x - \nu^2 t} \hat{u}_0(\nu) d\nu - \frac{1}{2\pi} \int_{\partial D^+} e^{i\nu x - \nu^2 t} [\hat{u}_0(-\nu) + 2i\nu \tilde{g}_0(\nu^2)] d\nu,$$

where ∂D^+ is the boundary of the region D^+ shown in Figure 1.1, $\hat{u}_0(\nu)$ is the Fourier transform of the initial condition, and $\tilde{g}_0(\nu^2)$ is a t -transform of the boundary condition,

$$(1.8) \quad \hat{u}_0(\nu) = \int_0^{\infty} e^{-i\nu \xi} u_0(\xi) d\xi, \quad \Im \nu \leq 0; \quad \tilde{g}_0(\nu) = \int_0^T e^{\nu \tau} g_0(\tau) d\tau, \quad \nu \in \mathbb{C}.$$

In contrast to (1.5), it is now straightforward to verify that (1.7) satisfies the boundary condition at $x = 0$: evaluating (1.7) at $x = 0$ we find

$$(1.9) \quad u(0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\nu^2 t} \hat{u}_0(\nu) d\nu - \frac{1}{2\pi} \int_{\partial D^+} e^{-\nu^2 t} \hat{u}_0(-\nu) d\nu - \frac{1}{2\pi} \int_{\partial D^+} 2i\nu e^{-\nu^2 t} \tilde{g}_0(\nu^2) d\nu.$$

By deforming the contour of integration in the second integral on the right-hand side of (1.9) from ∂D^+ to the real axis and then using the change of variables $\nu \mapsto -\nu$ in the resulting integral, it follows that the first two integrals on the right-hand side of

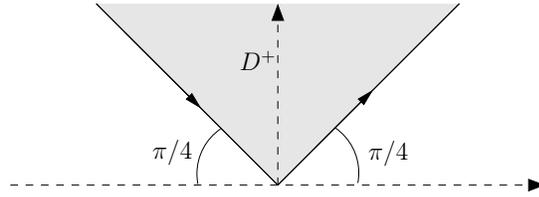


Fig. 1.1 The region D^+ for the heat equation.

(1.9) cancel. Then, making the change of variables $\nu^2 = il$ in the third integral on the right-hand side of (1.9) and using the inversion formula of the classical Fourier transform, we find that $u(0, t) = g_0(t)$.

As noted earlier, the only (x, t) -dependence of the right-hand side of (1.7) is in the form $\exp(i\nu x - \nu^2 t)$, which immediately implies that $u(x, t)$ solves the heat equation. Also, using the fact that the exponential $\exp(i\nu x - \nu^2(t - \tau))$ is bounded for $\tau > t$ in D^+ , and applying Cauchy's theorem in D^+ , it follows that $\tilde{g}_0(\nu^2)$ can be replaced with a similar expression involving an integral up to t instead of up to T , thus making the expression of the solution (1.7) consistent with causality (see section 4 for details).

It is of course straightforward, using Cauchy's theorem, to deform the contours in (1.7) to obtain (1.5). However, in general it is *not* possible to deform the contours in the expression obtained by the new method to obtain an expression involving integrals on the real axis. For example, such a deformation does *not* exist for the analogous expression for the linearized KdV equation obtained in section 4, which is consistent with the fact that in this case there does *not* exist a classical x -transform. Furthermore, even when such a deformation is possible, it appears that the expression formulated in the complex ν -plane has both analytical and numerical advantages. In order to illustrate the latter advantage, we consider the following example for the heat equation: let $u_0(x) = x \exp(-a^2 x)$ and $g_0(t) = \sin bt$, with a and b real constants. In this case, evaluating the integrals in the definitions of \hat{u}_0 and \tilde{g}_0 and then deforming the contours of integration in (1.7), we find

$$u(x, t) = \frac{1}{2\pi} \int_L e^{i\nu x - \nu^2 t} \left[\frac{1}{(i\nu + a)^2} - \frac{1}{(-i\nu + a)^2} - \nu \left(\frac{e^{(\nu+ib)t} - 1}{\nu + ib} - \frac{e^{(\nu-ib)t} - 1}{\nu - ib} \right) \right] d\nu,$$

where L is a curve above the real axis and below ∂D^+ . On this curve the term $\exp(i\nu x - \nu^2 t)$ decays exponentially for large $|\nu|$, thus $u(x, t)$ can be computed numerically very efficiently for any x and t [35].

1.2. Example 2: The Helmholtz Equation. The simplest possible model of wave propagation is the Helmholtz equation. Indeed, assuming a time dependence of the form $e^{-i\omega t}$, the wave equation reduces to the Helmholtz equation

$$(1.10) \quad \Delta u + k^2 u = 0,$$

where $k = \omega/c$, with ω the frequency and c the wave speed.

A classical problem in the theory of wave scattering is the Helmholtz equation posed in the exterior of a disc,

$$(1.11) \quad \Omega = \{a < r < \infty, 0 \leq \theta < 2\pi\},$$

with a Dirichlet boundary condition at $r = a$,

$$(1.12) \quad u(a, \theta) = d(\theta), \quad 0 \leq \theta < 2\pi.$$

If u describes the field scattered when an incident wave hits the disc, then $d(\theta)$ is given in terms of the restriction of the incident wave to the boundary. A radiation condition must be prescribed at infinity, which ensures that waves propagate away from the scatterer,

$$(1.13) \quad \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

The two-dimensional BVP (1.10)–(1.13) also describes the scattering of certain waves from a three-dimensional circular cylinder. Furthermore, it has certain similarities with the three-dimensional BVP of the Helmholtz equation in the exterior of a sphere. Scattering by a disc and a sphere are two of the most studied problems in the theory of wave scattering, for several reasons. First, they involve the only bounded obstacles for which explicit solutions can be obtained. Second, Keller’s celebrated geometrical theory of diffraction [67] allows one to understand the scattering of high frequency waves by an obstacle by splitting the obstacle into component parts (e.g., flat parts, curved parts, corners, edges) and analyzing each component separately. Scattering by a disc and a sphere are the appropriate “canonical problems” for understanding scattering by surfaces with positive curvature. Hence the solutions of these problems have formed the basis of several investigations of scattering from general convex obstacles [75], [7, Chapter 13], [57], [108], [109].

The appropriate transform in the θ -variable for solving the above BVP is the standard Fourier series, which yields

$$(1.14) \quad u(r, \theta) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} e^{in\theta} D(-in),$$

where $H_n^{(1)}$ denotes the Hankel function of the first kind and

$$(1.15) \quad D(-in) = \int_0^{2\pi} e^{-in\phi} d(\phi) d\phi, \quad n \in \mathbb{Z}^+.$$

However, it is well known that this series solution is essentially useless for computing the solution for high frequencies, $ka \gg 1$. This difficulty motivated the search for an alternative expression for the solution, and these attempts were instrumental in the further development of the classical transform method (leading Cohen to make the remark mentioned at the beginning of the introduction). Surprisingly, as will be shown in section 5, the new method provides further insight into this classical problem.

Given that this particular BVP played a key role in the development of the classical transform method, we continue its discussion in section 2, where we give a brief overview of the standard transform method based on separation of variables. Here we note only that finding an effective expression for the solution of this BVP involves going into the complex plane (in common with the IBVP for the linear KdV equation in Example 1 given in section 1.1).

Outline of Paper. Section 2 presents a brief overview of the classical transform method, and also includes some historical remarks about its development, emphasizing the important role played by the BVP of Example 2 of section 1.2. Section 3

summarizes the algorithmic steps needed for the implementation of the new method. These steps are illustrated in section 4 for Example 1 and in section 5 for Example 2. The new method is discussed further and compared to both the standard transform method and other methods in section 6. This section also discusses how the new method can be applied to certain BVPs that apparently cannot be solved by any other method. Finally, in section 7 the new method is placed in a wider context by demonstrating that the concept of a Lax pair, which is a fundamental concept in the theory of integrable nonlinear PDEs, arises naturally from the divergence form of separable linear PDEs.

2. Separation of Variables and the Resulting Classical Approaches. In this section we briefly recall the classical transform method based on separation of variables and discuss its application to the BVPs of Examples 1 and 2.

2.1. The Classical Transform Method. For simplicity we consider BVPs in two dimensions. Starting with a separable BVP, the classical transform method consists of the following four steps (see, e.g., [65, section 8.1.3], [105, p. 154], [51, p. 259], [85, sections 4.4, 5.7, 5.8]).

1. Separate the PDE into two ODEs.
2. Choose one of the ODEs and derive the associated transform pair (which depends on the ODE and boundary conditions). This is constructed through the formula

$$(2.1) \quad \delta(x - \xi) = - \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|\lambda|=R} g(x, \xi; \lambda) d\lambda,$$

where $g(x, \xi, \lambda)$ is the one-dimensional Green's function of the ODE with eigenvalue λ (the separation constant). Evaluating the above integral by collapsing the contour onto singularities of g in the complex λ -plane, one finds the appropriate transform pair (see, e.g., [65, Chapter 7], [104, Chapter 4], [106, Chapter 7]).

3. Apply the transform to the PDE and use integration by parts to derive the ODE associated with this transform (thus one differential operator in the PDE is replaced by multiplication by a transform variable).
4. Solve the ODE of step 3 using, for example, an appropriate one-dimensional Green's function, and then apply the appropriate inverse transform.

The solution to the given BVP is expressed as a superposition of eigenfunctions of the ODE chosen in step 2 involving either an integral or a series depending on whether this ODE has a continuous or a discrete spectrum. Thus, for each BVP there exist two different expressions for the solution depending on which ODE is chosen.

A necessary condition for the application of the above steps is that the domain, PDE, and boundary conditions are separable (for an introduction to the various coordinate systems in which common differential operators such as the Laplacian are separable, see, for example, [79, section 5.1], [78]).

Perhaps some readers will not be familiar with step 2, since usually when one is taught how to solve BVPs via transform methods one is told which transform to use. For example, for the second order operator d^2/dx^2 on $0 < x < 2\pi$ with periodic boundary conditions, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$, the formula (2.1) yields

$$(2.2) \quad \delta(x - \xi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-\xi)}.$$

This expansion in the eigenfunctions $\exp(ix)$ is just the usual Fourier series expansion. For the same second order operator posed on $-\infty < x < \infty$, (2.1) yields the usual Fourier transform,

$$(2.3) \quad \delta(x - \xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu(x-\xi)} d\nu.$$

Formulae of the form (2.2) and (2.3) are often called *completeness relations*. This is due to the fact that when the ODE has a discrete spectrum, the eigenfunctions may form a complete orthonormal set in an appropriate Hilbert space; i.e., they may form an orthonormal basis and a formula such as (2.2) gives the coefficients of this expansion. We note that for non-self-adjoint problems one must abandon the Hilbert space setting, and then a complete sequence need not be a basis [25, section 3.3]. This is relevant, for example, for the linear KdV equation on a finite interval $0 < x < L$ [95, Theorem 5.1]. For simplicity, we ignore this fact and refer to a transform as being “complete” when an arbitrary function can be expanded in a convergent series or integral of the relevant eigenfunctions.

We conclude the discussion of the classical transform method algorithm by making a comment on terminology. Many authors refer to the different expressions for the solution obtained by the method as different “representations” of the solution, e.g., in [19], [68], [51], [65]. This is partly motivated by the terminology that a transform pair is the *spectral representation* associated with the relevant ODE. However, the term “representation” is also used to describe the general form of the solution in terms of its boundary values, not all of which are specified as boundary conditions. For example, Green’s integral representation gives the solution of a BVP in terms of its boundary values, some of which are *not* specified as boundary conditions. To avoid confusion we shall use the term “representation” for the latter case, and “expression” to describe a formula that gives the solution in terms of *only* the given boundary conditions.

2.2. The Watson Transformation and Its Consequences. Although the idea of expanding an arbitrary function in terms of eigenfunctions of an ODE dates back to the 18th century, the fact that the appropriate transforms for solving a given BVP actually arise from the spectral analysis of the associated separated ODEs was only realized in the 1950s in connection with the BVP for the Helmholtz equation of Example 2.

As noted in the introduction, the BVPs of scattering by a disc and a sphere have been extensively studied. The Fourier series solution (1.14) was essentially obtained by Lord Rayleigh in 1881 [91], [92, section 343] (the analogous solution for the sphere was obtained in 1872 [90], [92, section 334]). For the case of a disc, the ODE in the angular variable θ involves $d^2/d\theta^2$ and under periodic boundary conditions the appropriate completeness relation is (2.2). However, when $ka \gg 1$ this Fourier series solution converges extremely slowly (this is because, for certain values of θ , the series is much smaller than the magnitude of the individual terms). As a consequence of the slow convergence, for a relative error of 1% one requires N terms, where $N = \mathcal{O}(ka)$ [13, Chapter 2], [84, section II]. In the case when the sphere is the Earth and one considers radio waves, Love [74] estimated that ka is approximately 8000!

In 1918, Watson overcame this difficulty via the so-called *Watson transformation* [112]. This transformation converts the slowly converging series solution into a different series which converges rapidly. For the problem of radio waves propagating around the Earth, instead of 8000 terms the first couple of terms of the transformed series now gave the desired accuracy. The Watson transformation consists of two steps:

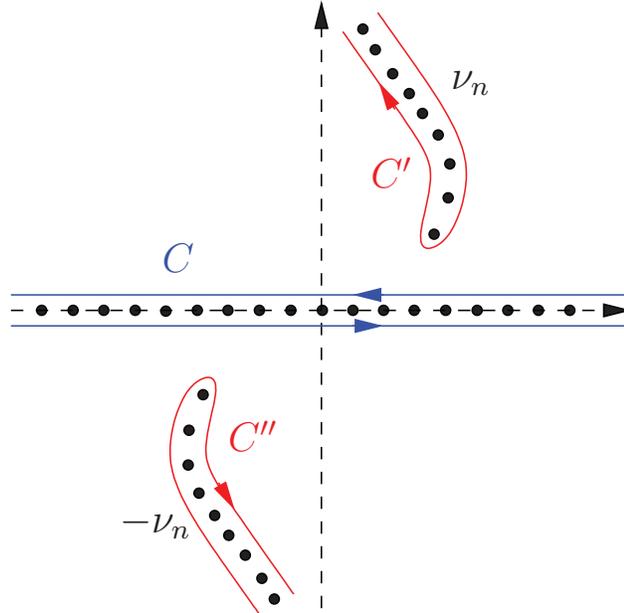


Fig. 2.1 The contours and poles involved in the Watson transformation for the BVP (1.10)–(1.13). The contours C' and C'' enclose the zeros ν_n defined by $H_{\nu_n}^{(1)}(ka) = 0$.

- (a) Convert the series solution into an integral using Cauchy's residue theorem. For example,

$$(2.4) \quad \sum_{n=-\infty}^{\infty} f(n) = - \int_C \frac{f(\nu)}{1 - e^{2\pi i \nu}} d\nu,$$

where C is a contour that encloses the real ν axis (in the positive sense) but not any of the singularities of $f(\nu)$ (see Figure 2.1).

- (b) Deform the contour C to enclose the poles of $f(\nu)$ and then evaluate the integral in terms of the residues at these poles.

Around 1950, Sommerfeld showed that the series obtained by Watson can actually be obtained *directly* by considering the *radial* ODE associated with the Helmholtz equation [98, Appendix II of Chapter 5, Appendix to Chapter 6]. To illustrate this fact, consider the BVP (1.10)–(1.13) but with a homogeneous boundary condition. The PDE (1.10) in polar coordinates has the separable solution $e^{i\nu\theta} H_{\nu}^{(1)}(kr)$, $\nu \in \mathbb{C}$. The eigenfunctions of the angular ODE are $e^{in\theta}$, $n \in \mathbb{Z}$. This yields the Fourier series solution (1.14) consisting of a sum over $n \in \mathbb{Z}$ of terms involving $e^{in\theta} H_n^{(1)}(kr)$. The eigenfunctions of the radial ODE satisfying the boundary condition $u = 0$ on $r = a$ are $H_{\nu_n}^{(1)}(kr)$, where the ν_n are the zeros of the Hankel function with respect to its order, i.e., ν_n are solutions of the equation $H_{\nu_n}^{(1)}(ka) = 0$. These zeros are in the first and third quadrants of the complex plane, as shown in Figure 2.1. The radial solution is a sum over these zeros of terms involving $e^{i\nu_n\theta} H_{\nu_n}^{(1)}(kr)$ (note that these radial solutions are *not* 2π -periodic in θ , hence the need to restrict θ as in (1.11)). Thus, by applying the Watson transformation (2.4) to the Fourier series solution (1.14) and

by deforming the contour C in Figure 2.1 onto the contours C' and C'' which enclose the poles of $f(\nu)$ (i.e., the zeros of $H_\nu^{(1)}(ka)$), one obtains the radial solution (this is given as (5.19) below). However, this deformation procedure is *not* valid, since the integral at infinity is unbounded—we will return to this crucial point in section 2.3.

2.3. Establishing Completeness of a Transform. Following the new understanding provided by Watson and Sommerfeld, the different expressions available for the solution of a separable BVP were systematically derived for many important BVPs [76], [51, Chapters 4 and 5], [33], [34, Chapters 3 and 6], [19], [18]. However, the important question about the completeness of the associated transforms arose, namely the basic question of whether an arbitrary function (from a suitable class) can be expanded in the appropriate eigenfunctions and whether the relevant series or integral converges.

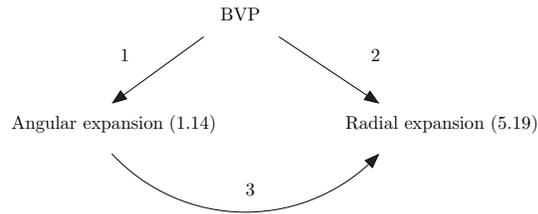
Before discussing Examples 1 and 2, we present a brief outline of the main methods for establishing completeness of eigenfunction expansions. In this respect, we recall the crucial concept of self-adjointness. It turns out that the transforms associated with self-adjoint problems are always complete, but the transforms associated with non-self-adjoint problems may or may not be complete, and actually in many generic cases are not.

Recall that for a differential operator \mathcal{L} , its *formal adjoint*, denoted by \mathcal{L}^* , is such that the quantity $\bar{v}\mathcal{L}u - u\overline{\mathcal{L}^*v}$ is the divergence of an expression involving lower order derivatives of u and \bar{v} ; see, e.g., [105, section 5.7], [65, section 4.3.2], [80, section 1.5], [106, p. 197]. For example, for equations with constant real coefficients one can construct the formal adjoint by reversing the sign of all odd derivatives. An operator \mathcal{L} is *self-adjoint* if both $\mathcal{L} = \mathcal{L}^*$ and the boundary conditions imposed on u satisfy certain constraints (see the references cited above).

Broadly speaking, there are two different approaches for proving rigorously that a transform derived from the spectral analysis of an ODE (step 2 of section 2.1) is complete (actually, more recent investigations have merged the two approaches [77]):

- (i) Use integration in the complex plane [11], [12], [107], [80], [17, Chapter 12]. This method is based on the rigorous justification of step 2 of section 2.1 and consists of two parts: (a) prove directly that (2.1) holds for a given Green's function by obtaining the asymptotics of the Green's function g as $\lambda \rightarrow \infty$; (b) obtain the location and types of singularities of $g(x, \xi; \lambda)$ as a function of λ . This method proves completeness both for self-adjoint problems and for some particular non-self-adjoint problems (these latter problems satisfy certain subtle constraints on the boundary conditions; see, for example, [80, section 5]).
- (ii) Use techniques of functional analysis to provide an indirect proof that the eigenfunctions of the relevant differential operator are complete. Results of this form for self-adjoint operators can be found in [81], [93, section VII], [30, Chapter XIII] (the references [30, section XIII.10A], [31, section XIX.6], and [113, section 2-3] provide good overviews of the development of this theory). The theory for non-self-adjoint operators, which was developed later (see, e.g., [31, Chapter 19]) is more challenging and is still far from complete [72], [73], [77].

For the Helmholtz equation, the differential operator is formally self-adjoint. However, because of the presence of a complex coefficient in the radiation condition (1.13), the appropriate constraints on the boundary conditions for self-adjointness are not satisfied, and thus the Helmholtz equation in unbounded domains is non-self-adjoint.



1. Using the angular transform
2. Using the radial transform (invalid due to lack of completeness)
3. Using the Watson transformation and a residue calculation
(invalid due to the contribution at infinity)

Fig. 2.2 *The relationship between the different expressions for the solution obtained by classical transforms and the Watson transformation for the BVP (1.10)–(1.13).*

At least one of the separated ODEs associated with the Helmholtz equation in an unbounded domain will include the radiation condition (in polar coordinates this will be the radial ODE). Thus, the one-dimensional BVP involving this ODE is non-self-adjoint. A large class of BVPs for the Helmholtz equation in unbounded domains was investigated by Cohen [19], [18], [20], who established, via the method (i) above, that for many particular BVPs the associated transforms are indeed complete [19]. However, the radial transform for the BVP (1.10)–(1.13) turns out *not* to be complete [18], and thus the solution to the BVP obtained via this transform is *not* valid. This is consistent with the *failure* of the Watson transformation procedure for this BVP. Indeed, when C is deformed to C' and C'' there is an *unbounded contribution from the integrals at infinity* for certain θ and therefore the deformation to the radial series expansion is *not* always valid. The situation is shown schematically in Figure 2.2. This deformation procedure is rather subtle and was actually overlooked by Watson himself (it was later investigated by many authors [84, section III], [55, section 3], [88], [75, section 5] [82, pp. 115, 120]; see [101] for a more accessible account).

In summary, for the high frequency limit of the BVP (1.10)–(1.13), $ka \gg 1$, the angular series expansion obtained by Lord Rayleigh is correct but useless, and the radial series expansion obtained by Watson and Sommerfeld is useful but incorrect! (The usefulness of the radial series expansion stems from the fact that although it is not a convergent series, it can be shown that it is an asymptotic series as $ka \rightarrow \infty$ [110].)

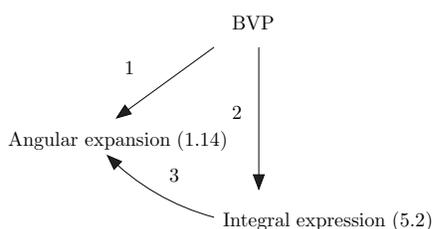
We now discuss the IBVPs of Example 1 in the context of this classical theory of transforms. The separated ODE in the x -variable for the heat equation involves d^2/dx^2 . Following step 2 of section 2.1, the transform associated with this operator on $-\infty < x < \infty$ is the usual Fourier transform. Actually this is also true for the n th order operator d^n/dx^n . Following step 3 of section 2.1 it is straightforward to show that the transform associated with d^2/dx^2 on $0 < x < \infty$ with the boundary condition $u(0) = 0$ is the sine transform (leading to the solution (1.5) of the IBVP).

However, the third order operator d^3/dx^3 on $0 < x < \infty$ with either one or two homogeneous boundary conditions at $x = 0$ does *not* have a classical transform pair associated with it [36, Appendix A]. After multiplying by i , this third order operator is formally self-adjoint in a complex inner product; however, the boundary conditions do not satisfy the constraints for self-adjointness.

The separated ODE in the t -variable for both the heat and linearized KdV equations involves the operator d/dt . Like the third order operator, after multiplying by i this first order operator is formally self-adjoint in a complex inner product. Strictly speaking, for the IBVPs in Example 1 one should consider this operator on $0 < t < T$. However, in this case the relevant transform yields an expression involving $u(x, T)$, which should not appear due to causality (the solution at time t should only depend on earlier times). Thus, one usually considers the first order operator on $0 < t < \infty$, and following step 2 of section 2.1 one obtains the Laplace transform [106, p. 432] (Stakgold actually obtains the appropriate Green's function [106, equation (1.65)] as a limit of Green's functions on $0 < t < T$ with self-adjoint boundary conditions; however, one can obtain it directly by considering the BVP on $0 < t < \infty$).

2.4. The Use of the Complex Plane by the Classical Approaches. We conclude our discussion of the classical transform method and related approaches with the following observations regarding the use of complex analysis. (a) The derivation of an appropriate transform takes place in the complex plane (step 2 of section 2.1). However, in order to construct the solution of a BVP (steps 3 and 4 of section 2.1) one employs transforms involving *real* variables. (b) In hindsight, the “moral” of the Watson transformation is that the best expression for the solution is given as an *integral in the complex plane*; this integral can then be deformed (and evaluated via residues if necessary) to yield either of the two expressions obtained by transforms. In other words, the classical approach begins in the complex plane (in order to derive the transform), abandons the complex plane (in order to solve a given BVP), and sometimes returns to the complex plane via the Watson transformation (in order to find the best expression for the solution). This motivates the following two questions: (i) Is there a way to *remain in the complex plane* whilst solving a given BVP? (ii) Is this advantageous? The answer to both questions is yes, and this is precisely the achievement of the new method: the solution of a given BVP is obtained directly as an *integral in the complex plane* (through which the classical expressions, if they exist, can be obtained by appropriate contour deformations).

The relationship between the new method and the classical transform method, applied to the BVP of Example 2, i.e., (1.10)–(1.13), is illustrated in Figure 2.3.



1. Using the angular transform
2. Using the new method
3. Via a residue calculation (i.e., a suitable “reverse” Watson transformation)

Fig. 2.3 The relationship between the classical expressions for the solution of the BVP (1.10)–(1.13) and the expression obtained by the new method.

3. Description of the New Method. The new method is based on the following three steps:

1. Rewrite the PDE as a one-parameter family of equations in divergence form. Integrating this divergence form over the boundary of the domain gives the *global relation*, which is an algebraic equation coupling the transforms of all boundary values. The terminology emphasizes that this relation contains global, as opposed to local, information about the boundary values.
2. Derive an *integral representation* of the solution involving *transforms* of all boundary values. This equation provides the analogue of the classical Green's integral representation, but rather than being formulated in the physical space, it is formulated in the spectral (Fourier) space.
3. Eliminate from the integral representation the transforms of the unknown boundary values and thus obtain an expression for the solution in terms of the known boundary data. This step involves the algebraic manipulation of both the global relation and the equations obtained from the global relation via certain transformations in the complex Fourier plane, as well as appropriate deformations of the contours of the integral representation and the use of Cauchy's theorem.

4. Solution of Example I. In this section we apply the new method described in section 3 to solve the linear version of the KdV equation (1.1) posed on the half-line $0 < x < \infty$. Actually, in order to keep the technicalities to a minimum and present the elements of the new method in their simplest possible form, we present the details for the linearized KdV equation without the u_x term, i.e.,

$$(4.1) \quad u_t + u_{xxx} = 0.$$

As with the proper linearized KdV (1.1), there does *not* exist an x -transform for solving (4.1) on the half-line. The solution of (1.1) is conceptually identical and differs only in the fact that some of the expressions are slightly more complicated; the details can be found in [41, Examples 1.5 and 1.12], [47].

PROPOSITION 4.1. *The solution of the PDE (4.1) posed on $0 < x < \infty$, $0 < t < T$, with the initial and boundary conditions*

$$(4.2) \quad u(x, 0) = u_0(x), \quad 0 < x < \infty; \quad u(0, t) = g_0(t), \quad 0 < t < T,$$

is given by

$$(4.3) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu x + i\nu^3 t} \hat{u}_0(\nu) d\nu \\ - \frac{1}{2\pi} \int_{\partial D^+} e^{i\nu x + i\nu^3 t} [3\nu^2 \tilde{g}_0(-i\nu^3) - \alpha \hat{u}_0(\alpha\nu) - \alpha^2 \hat{u}_0(\alpha^2\nu)] d\nu,$$

where the contour ∂D^+ is shown in Figure 4.1(a), $\hat{u}_0(\nu)$ and $\tilde{g}_0(\nu)$ are given by (1.8), and $\alpha = \exp(2\pi i/3)$.

The solution of the analogous IBVP for the linearized KdV (1.1) is given by a similar expression to (4.3), where $-i\nu^3$ is replaced by $\omega(\nu) := -i(\nu^3 - \nu)$, D^+ is the shaded domain in Figure 4.1(b), and the expression inside the square brackets in the second integral of (4.3) is replaced by a slightly more complicated, but explicit, expression involving $\tilde{g}_0(\omega(\nu))$ and $\hat{u}_0(\nu_j)$, $j = 1, 2$, where ν_1 and ν_2 are the solutions of

$$(4.4) \quad \nu_j^2 + \nu\nu_j + \nu^2 - 1 = 0, \quad \nu_j \sim e^{2\pi ij/3} \nu \text{ as } |\nu| \rightarrow \infty, \quad j = 1, 2.$$

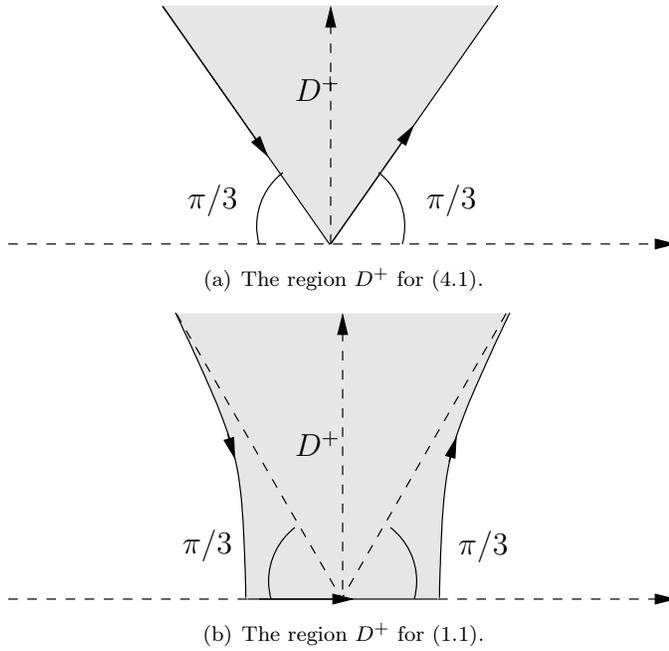


Fig. 4.1 The regions D^+ for (4.1) and (1.1).

4.1. Step I. Equation (4.1) can be written as the following one-parameter family of equations in divergence form:

$$(4.5) \quad \left[e^{-i\nu x - i\nu^3 t} u \right]_t - \left[e^{-i\nu x - i\nu^3 t} (\nu^2 u - i\nu u_x - u_{xx}) \right]_x = 0, \quad \nu \in \mathbb{C}.$$

Indeed, the formal adjoint (defined in section 2.3) of the PDE (4.1) is

$$(4.6) \quad -v_t - v_{xxx} = 0$$

and the divergence form is

$$(4.7) \quad (vu)_t + (v_{xx}u - v_x u_x + v u_{xx})_x = 0.$$

Separation of variables gives the one-parameter family of solutions to (4.6)

$$(4.8) \quad v = e^{-i\nu x - i\nu^3 t},$$

and thus we obtain (4.5). Note that we find v without specifying the domain or boundary conditions. Furthermore, in contrast to the classical transform method, we have *not* favored either the x - or the t -variable (hence the “synthesis” as opposed to “separation”).

Suppose that the PDE (4.1) is valid in the domain $\{0 < \xi < \infty, 0 < \tau < t\}$. Then, applying Green’s theorem in this domain, (4.5) implies the following global relation:

$$(4.9) \quad e^{-i\nu^3 t} \hat{u}(\nu, t) = \hat{u}_0(\nu) - \tilde{g}(\nu, t), \quad \Im \nu \leq 0, \quad 0 < t < T,$$

where $\hat{u}_0(\nu)$ denotes the Fourier transform of $u_0(x)$ defined by (1.8), $\hat{u}(\nu, t)$ denotes the Fourier transform of $u(x, t)$, and $\tilde{g}(\nu, t)$ involves the t -transforms of the relevant boundary values:

$$\tilde{g}(\nu, t) = \nu^2 \tilde{g}_0(-i\nu^3, t) - i\nu \tilde{g}_1(-i\nu^3, t) - \tilde{g}_2(-i\nu^3, t), \quad \nu \in \mathbb{C}, \quad 0 < t < T,$$

where

$$\tilde{g}_j(\nu, t) = \int_0^t e^{\nu\tau} \partial_x^j u(0, \tau) d\tau, \quad j = 0, 1, 2, \quad \nu \in \mathbb{C}, \quad 0 < t < T.$$

The function \tilde{g} is an entire function of ν , whereas both $\hat{u}_0(\nu)$ and $\hat{u}(\nu, t)$ are well-defined for $\Im\nu \leq 0$, thus the global relation (4.9) is valid for all complex ν in the lower half-plane.

4.2. Step 2. Multiplying (4.9) by $\exp(i\nu^3 t)$, taking ν to be real, using the inverse Fourier transform, and then employing Jordan's lemma (from complex analysis), we find the integral representation

$$(4.10) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu x + i\nu^3 t} \hat{u}_0(\nu) d\nu - \frac{1}{2\pi} \int_{\partial D^+} e^{i\nu x + i\nu^3 t} \tilde{g}(\nu, t) d\nu, \quad 0 < x < \infty, \quad 0 < t < T,$$

where $D := \{\nu \in \mathbb{C}, \Re(i\nu^3) > 0\}$ and D^+ is the part of the region D that is in the upper half-plane. Thus ∂D^+ is the union of the rays in the complex ν -plane defined by $\arg \nu = \pi/3$ and $\arg \nu = 2\pi/3$; see Figure 4.1(a).

Indeed, the inverse Fourier transform yields an equation almost identical to (4.10) except that the second term on the right-hand side is an integral along the real axis. This integral can be deformed to the contour ∂D^+ using the fact that the function $\tilde{g}(\nu, t) \exp(i\nu x + i\nu^3 t)$ is bounded and analytic in the region bounded by the real axis and ∂D^+ . To see this, we note that by definition

$$(4.11) \quad e^{i\nu x + i\nu^3 t} \tilde{g}(\nu, t) = e^{i\nu x} \left[\nu^2 \int_0^t e^{i\nu^3(t-\tau)} u(0, \tau) d\tau - i\nu \int_0^t e^{i\nu^3(t-\tau)} u_x(0, \tau) d\tau + \int_0^t e^{i\nu^3(t-\tau)} u_{xx}(0, \tau) d\tau \right].$$

The function $\exp(i\nu x)$, $x \geq 0$, is bounded in the upper half complex ν -plane, and the function $\exp(i\nu^3(t-\tau))$, with $t-\tau \geq 0$, is bounded in the region $\Re(i\nu^3) \leq 0$. Furthermore, integration by parts implies that the term in square brackets in (4.11) is $\mathcal{O}(1/\nu)$ as $\nu \rightarrow \infty$ in $\mathbb{C}^+ \setminus D^+$. Thus, Jordan's lemma implies that the contour can be deformed from the real axis to ∂D^+ .

4.3. Step 3. The global relation (4.9) was already used for the derivation of the integral representation (4.10). However, this only required that ν was real, whereas (4.9) is valid in a much larger region, namely in the lower half of the complex ν -plane. Using this fact it is possible to eliminate the unknown functions \tilde{g}_1 and \tilde{g}_2 from (4.10). Indeed, we note that $\{\tilde{g}_j(-i\nu^3)\}_{j=0}^2$ depend on ν only through ν^3 , thus they remain invariant under those transformations in the complex ν -plane that leave ν^3 invariant, i.e., $\nu \mapsto \alpha\nu$ and $\nu \mapsto \alpha^2\nu$, for $\alpha = \exp(2\pi i/3)$. Let D^- denote the part of the region D that is in the lower half complex plane, and let $D_j^-, j = 1, 2$, denote the two components of D^- ; see Figure 4.2. If $\nu \in D^+$, then $\alpha\nu \in D_1^-$ and $\alpha^2\nu \in D_2^-$. Thus,

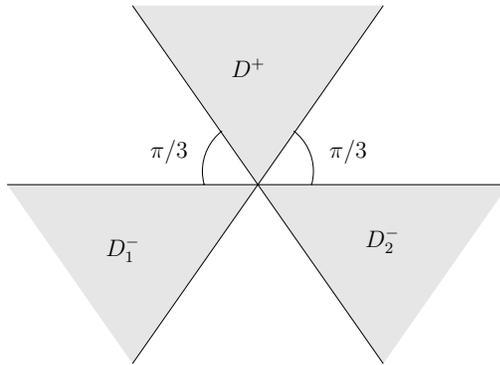


Fig. 4.2 The regions D^+ and D^- in the complex ν -plane.

replacing ν by $\alpha\nu$ and $\alpha^2\nu$ in the global relation we find two equations valid in D^+ :

$$(4.12a) \quad \hat{u}(\alpha\nu, t)e^{-i\nu^3 t} = \hat{u}_0(\alpha\nu) - \alpha^2\nu^2\tilde{g}_0(-i\nu^3, t) + i\alpha\nu\tilde{g}_1(-i\nu^3, t) + \tilde{g}_2(-i\nu^3, t),$$

$$(4.12b) \quad \hat{u}(\alpha^2\nu, t)e^{-i\nu^3 t} = \hat{u}_0(\alpha^2\nu) - \alpha\nu^2\tilde{g}_0(-i\nu^3, t) + i\alpha^2\nu\tilde{g}_1(-i\nu^3, t) + \tilde{g}_2(-i\nu^3, t),$$

$$\nu \in D^+, \quad 0 < t < T.$$

Given that \tilde{g}_0 is known, (4.12) can be thought of as two equations for the two unknown functions \tilde{g}_1, \tilde{g}_2 (neglecting for a moment the fact that $\hat{u}(\alpha\nu, t)$ and $\hat{u}(\alpha^2\nu, t)$ are also unknown). Solving these two equations for \tilde{g}_1 and \tilde{g}_2 and then substituting the resulting expressions into \tilde{g} , we find

$$\tilde{g}(\nu, t) = 3\nu^2\tilde{g}_0(-i\nu^3, t) - \alpha\hat{u}_0(\alpha\nu) - \alpha^2\hat{u}_0(\alpha^2\nu) + e^{-i\nu^3 t} [\alpha\hat{u}(\alpha\nu, t) + \alpha^2\hat{u}(\alpha^2\nu, t)], \nu \in D^+.$$

Substituting the above expression into the right-hand side of (4.10) and using Jordan's lemma, we find that the contribution of the terms $\hat{u}(\alpha\nu, t)$ and $\hat{u}(\alpha^2\nu, t)$ vanishes by analyticity.

Thus we have obtained the solution (4.3) but with $\tilde{g}_0(-i\nu^3, t)$ instead of $\tilde{g}_0(-i\nu^3)$. To show that these two expressions for the solution are equivalent, note that their difference involves

$$\frac{1}{2\pi} \int_{\partial D^+} 3\nu^2 e^{i\nu x} \left[\int_t^T e^{i\nu^3(t-\tau)} g_0(\tau) d\tau \right] d\nu;$$

the integrand is an entire function of ν that tends to zero exponentially as $\nu \rightarrow \infty$ in the region D^+ . Thus, closing the contour at infinity in D^+ , the integral equals zero by Cauchy's theorem.

In the case of the proper linearized KdV equation (1.1), the transformations that leave $\omega(\nu) = -i(\nu^3 - \nu)$ invariant are ν_1 and ν_2 defined by (4.4).

REMARK 4.2. The expression involving the t -transform of $g_0(t)$ with the integral up to t , namely $\tilde{g}_0(-i\nu^3, t)$, is (a) useful for numerical computations; (b) necessary for obtaining the expression constructed via the classical x -transform, when such a transform exists; and (c) consistent with causality (only the boundary values at times less than t determine the solution at time t).

The expression with the integral up to T , i.e., involving $\tilde{g}_0(-i\nu^3)$, is (a) more convenient for verifying that the expression satisfies both the PDE and the boundary

condition at $t = 0$; and (b) more convenient for obtaining the long-time asymptotics of the solution; see [37, sections 2–3].

REMARK 4.3 (return to the real axis). *In order to derive the integral representation in the complex ν -plane, the integral involving \tilde{g} is deformed from the real axis to the curve ∂D^+ . However, after using the global relation to eliminate the transforms of the unknown boundary values, \tilde{g} involves \hat{u}_0 and now, in general, it is not possible to “return to the real axis.” In particular, $\hat{u}_0(\alpha\nu)$ is bounded in the half-plane that includes D^+ and D_1^- , and $\hat{u}_0(\alpha^2\nu)$ is bounded in the half-plane that includes D^+ and D_2^- . Thus, although the contour for the term involving \tilde{g}_0 in (4.3) can be deformed from ∂D^+ back to the real axis, the contours for the terms involving $\hat{u}_0(\alpha\nu)$ and $\hat{u}_0(\alpha^2\nu)$ cannot. In the example of the heat equation, \tilde{g} involves $\hat{u}_0(-\nu)$, which is analytic in the upper half-plane, thus it is possible to deform back to the real axis.*

REMARK 4.4 (evolution PDEs of arbitrary order on the half-line). *A well-posed IBVP for an evolution PDE with an arbitrary number of derivatives in x can be solved in a similar way to the IBVP of Example 1. As discussed in the introduction, this is in stark contrast to the classical transform method, which depends crucially on the order of the x -derivatives. Consider, for example, the evolution PDE*

$$(4.13) \quad u_t + \omega(-i\partial_x)u = 0,$$

where $\omega(\nu)$ is a polynomial of degree n and $\Re\omega(\nu) \geq 0$ for ν real (this restriction on $\omega(\nu)$ ensures that the IVP of (4.13) on the full line is well-posed). Equation (4.13) possesses the particular solution $\exp(i\nu x - \omega(\nu)t)$, whereas the formal adjoint to (4.13) possesses the solution $\exp(-i\nu x + \omega(\nu)t)$. This immediately implies that u is given by the right-hand side of (4.10) with $\exp(i\nu x + i\nu^3 t)$ replaced by $\exp(i\nu x - \omega(\nu)t)$, where the contour ∂D^+ is now the boundary of the region D^+ , where $D = \{\nu \in \mathbb{C}, \Re\omega(\nu) < 0\}$ and D^+ is the part of the region D that is in the upper half-plane. We note that if the highest derivative of (4.13) is of order n , then for n even the global relation involves $n/2$ unknown functions, and for n odd either $(n+1)/2$ or $(n-1)/2$ unknown functions (depending on the sign of the highest derivative). However, there exist $n-1$ transformations in the complex ν -plane that leave $\omega(\nu)$ invariant, and by using these transformations it is possible to obtain a linear system of n or $(n \pm 1)/2$ equations for the transforms of the n or $(n \pm 1)/2$ unknown boundary values [46], [41, Chapter 2].

Another example is that of the evolution PDE

$$(4.14) \quad \beta(-i\partial_x)u_t + \omega(-i\partial_x)u = 0,$$

where both $\beta(\nu)$ and $\omega(\nu)$ are polynomials [111]. A particular case of (4.14) is the PDE

$$(4.15) \quad u_t - u_{xxt} + u_x = 0,$$

which is the linearized Benjamin–Bona–Mahony equation, a model for long waves in shallow water [10].

5. Solution of Example 2. Our goal in this section is to prove the following proposition.

PROPOSITION 5.1. *Let Ω be the domain exterior to a disc of radius a centered at the origin with a cut along the line $\theta = 0$, i.e., Ω is given by (1.11). Let $u(r, \theta)$ satisfy the PDE (1.10), the radiation condition (1.13), and the Dirichlet boundary condition (1.12). Furthermore let u satisfy the following boundary conditions on the cut (which*

ensure that the solution is 2π -periodic in θ):

$$(5.1) \quad u(r, 2\pi) = u(r, 0), \quad u_\theta(r, 2\pi) = u_\theta(r, 0).$$

Then u is given by

$$(5.2) \quad u(r, \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} e^{i\nu\theta} D(-i\nu) d\nu \\ - \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} \frac{e^{i\nu\theta} D(-i\nu) + e^{-i\nu\theta} D(i\nu)}{1 - e^{-2\pi i\nu}} d\nu,$$

where the known functions $D(\pm i\nu)$ are defined by

$$(5.3) \quad D(\pm i\nu) = \int_0^{2\pi} e^{\pm i\nu\phi} d(\phi) d\phi, \quad \nu \in \mathbb{C},$$

and c in the second integral is such that $0 < c < \Im\nu_1$, where ν_1 is the zero of $H_\nu^{(1)}(ka)$ in the first quadrant of the complex ν -plane with the smallest imaginary part.

Note that in our definition of the domain Ω we consider space to be nonperiodic in the θ variable. Thus $\theta = 0$ and $\theta = 2\pi$ are not automatically identified as the same line, and we have to impose periodicity via the boundary conditions (5.1). From the discussion of the classical radial series solution in section 2.2, we know that it is necessary to consider nonperiodic θ ; it turns out that the new method imposes this requirement independently; see Step 2 in section 5.2.

5.1. Step 1. The Helmholtz equation is formally self-adjoint, and the divergence form of (1.10) is

$$(5.4) \quad \nabla \cdot (v\nabla u - u\nabla v) = 0.$$

Integrating (5.4) over Ω and using the divergence theorem yields the global relation

$$(5.5) \quad 0 = \int_{\partial\Omega} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) dS,$$

where v is a one-parameter family of solutions to (1.10) such that the integral at infinity vanishes; this is the case if v also satisfies the radiation condition (1.13).

We can obtain a one-parameter family of adjoint solutions v via separation of variables. Since the BVP is separable in polar coordinates, a one-parameter family is given by

$$(5.6) \quad v = e^{i\nu\theta} H_\nu^{(1)}(kr), \quad \nu \in \mathbb{C},$$

where the solution in the radial variable, $H_\nu^{(1)}(kr)$, is chosen by the requirement that it satisfies the radiation condition. Note that, although we have chosen to work in polar coordinates, the adjoint solutions are *independent* of both the domain in which the PDE is posed and the type of boundary conditions prescribed. Furthermore, in contrast to the classical transform method, we have given equal importance to the r - and θ -variables (again, “synthesis,” as opposed to “separation”).

Substituting into (5.5) v given by (5.6), writing the resulting equation in polar coordinates, and recalling that we have a cut along $\theta = 0$ where across the cut $u(r, \theta)$ satisfies (5.1), we find the following global relation:

$$(5.7) \quad -aH_\nu^{(1)}(ka)N(i\nu) + akH_\nu^{(1)'}(ka)D(i\nu) + (1 - e^{2\pi i\nu}) \left(i\nu D_0(\nu) - N_0(\nu) \right) = 0,$$

where \prime denotes differentiation of $H_\nu^{(1)}(\cdot)$ with respect to its argument,
 (5.8)

$$D_0(\nu) = \int_a^\infty H_\nu^{(1)}(k\rho) u(\rho, 0) \frac{d\rho}{\rho}, \quad N_0(\nu) = \int_a^\infty H_\nu^{(1)}(k\rho) u_\theta(\rho, 0) \frac{d\rho}{\rho}, \quad \nu \in \mathbb{C},$$

and

$$(5.9) \quad D(i\nu) = \int_0^{2\pi} e^{i\nu\phi} u(a, \phi) d\phi, \quad N(i\nu) = \int_0^{2\pi} e^{i\nu\phi} u_r(a, \phi) d\phi, \quad \nu \in \mathbb{C}.$$

The notation indicates that D and D_0 are transforms of the Dirichlet boundary values, and N and N_0 are transforms of the Neumann boundary values, with D and N the boundary values on the surface of the circle and D_0 and N_0 the boundary values on the cut.

5.2. Step 2. The integral representation of Example 1 was obtained by inverting one of the integrals in the global relation. Although one *can* still obtain the integral representation for elliptic PDEs from the global relation, it is simpler to use Green's integral representation. Furthermore, this latter method of obtaining the integral representation has the advantage that it shows that the integral representation of the new method is the analogue of Green's integral representation in the transform (or spectral) space. Green's integral representation for the solution of (1.10) in Ω is

$$(5.10) \quad u(\mathbf{x}) = \int_{\partial\Omega} \left(E(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial E}{\partial n}(\boldsymbol{\xi}, \mathbf{x}) \right) dS(\boldsymbol{\xi}), \quad \mathbf{x} \in \Omega,$$

where $E(\boldsymbol{\xi}, \mathbf{x})$ is the fundamental solution (or free-space Green's function) satisfying

$$(5.11) \quad (\Delta_{\boldsymbol{\xi}} + k^2) E(\boldsymbol{\xi}, \mathbf{x}) = -\delta(\boldsymbol{\xi} - \mathbf{x}), \quad \boldsymbol{\xi} \in \mathbb{R}^2.$$

For the Helmholtz equation in two dimensions with the radiation condition (1.13) the fundamental solution is given by

$$E(\boldsymbol{\xi}, \mathbf{x}) = \frac{i}{4} H_0^{(1)}(k|\boldsymbol{\xi} - \mathbf{x}|);$$

however, we will not use this fact directly, and instead start from the PDE defining E .

The integral representation of the new method can be obtained by substituting two different expressions for the fundamental solution into Green's integral representation (5.10). These two expressions are obtained by solving (5.11) by the two appropriate classical transforms (associated with each separated ODE) in the chosen coordinate system. Since the separated ODEs are considered in the whole space (as the equation for the fundamental solution is posed in \mathbb{R}^2), they usually have a continuous spectrum; thus the expressions for E are usually in the form of integrals. For the Helmholtz equation in polar coordinates these formulae are given in Proposition 5.2 below. In this respect we note that an expansion in radial eigenfunctions of the fundamental solution is impossible if the θ -variable is periodic (the details can be found in [102, Remark 3.4]); thus we must consider nonperiodic θ .

In order to emphasize that our fundamental solution is *not* the usual one, we denote it by E_s (following the notation in [105, p. 270], where Sommerfeld's use of the nonperiodic fundamental solution in a different context is discussed). The nonperiodic fundamental solution E_s satisfies (5.11) and the radiation condition (1.13) in the

domain defined in polar coordinates (ρ, ϕ) by $-\infty < \phi < \infty$, $0 < \rho < \infty$. The two expressions for E_s , obtained by the appropriate transforms, are given in the following proposition.

PROPOSITION 5.2 (expansions of E_s in radial and angular eigenfunctions). *The nonperiodic fundamental solution E_s for the Helmholtz equation can be expressed as expansions in terms of either the radial or the angular eigenfunctions. The radial expansion is*

$$(5.12) \quad E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} \left(\int_0^{i\infty} e^{\varepsilon\nu^2} H_\nu^{(1)}(kr_1) J_\nu(kr_2) e^{i\nu|\theta-\phi|} d\nu + \int_0^{-i\infty} e^{\varepsilon\nu^2} H_\nu^{(1)}(kr_1) J_\nu(kr_2) e^{-i\nu|\theta-\phi|} d\nu \right),$$

where either $r_1 = r$ and $r_2 = \rho$, or vice versa. The angular expansion is

$$(5.13) \quad E_s(\rho, \phi; r, \theta) = \frac{i}{4} \left(\int_0^\infty H_\nu^{(1)}(kr_>) J_\nu(kr_<) e^{i\nu(\theta-\phi)} d\nu + \int_0^\infty H_\nu^{(1)}(kr_>) J_\nu(kr_<) e^{-i\nu(\theta-\phi)} d\nu \right),$$

where $r_> = \max(r, \rho)$, $r_< = \min(r, \rho)$, and $0 < \theta, \phi < 2\pi$.

Equations (5.12) and (5.13) can be obtained using the two completeness relations

$$(5.14) \quad \delta(\theta - \phi) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{i\nu(\theta-\phi)} d\nu$$

and

$$(5.15) \quad \rho\delta(r - \rho) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{-i\infty}^{i\infty} e^{\varepsilon\nu^2} H_\nu^{(1)}(kr_1) J_\nu(kr_2) \nu d\nu,$$

where either $r_1 = r$ and $r_2 = \rho$, or vice versa: the details are given in [102, section 3]. The first completeness relation (5.14) is associated with the angular ODE under nonperiodicity and corresponds to the Fourier transform (as discussed in section 2.1), whereas (5.15) is associated with the radial ODE under the radiation condition (1.13) and corresponds to the Kontorovich–Lebedev transform.

Deforming the contours of integration in (5.12) from $(0, i\infty)$ to $(0, \infty)$ shows that the two expressions for E_s are equivalent. Thus, both expressions (5.12) and (5.13) can actually be obtained from either (5.14) or (5.15). This is analogous to the fact that in order to obtain the integral representation (4.10) in Example 1 we needed only the Fourier transform, which is the transform associated with the separated ODE in the x -variable considered in the whole space.

Both (5.15) and (5.12) contain the regularizing factor $\exp \varepsilon\nu^2$, which illustrates the technical complications that arise due to the non-self-adjointness of the Helmholtz equation in unbounded domains. Performing step 2 of the classical transform method algorithm in section 2.1 on the radial ODE with the radiation condition, one arrives at (5.15) *without* the regularizing factor $\exp \varepsilon\nu^2$. However, D. S. Jones showed in [61] that without this factor the transform is *not* complete (the transform even fails for the simple function $\exp(-a\rho)$, $a > 0$), and he established via a proof of completeness the version (5.15). Nevertheless, many of the classical solutions of BVPs obtained

via (5.15) without the term $\exp \varepsilon \nu^2$ are still correct because the contour is deformed (albeit illegally) and the resulting expression converges (the term $\exp \varepsilon \nu^2$ rigorously justifies the contour deformation, after which ε can be set to zero).

This difficulty with transforms associated with the Helmholtz equation illustrates another advantage of the new method in comparison to the classical transform method. Although we have chosen to work in polar coordinates, the two expansions of E_s (5.12) and (5.13) are *independent* of the particular domain and boundary conditions of the BVP under consideration (recall that the choice of the adjoint solutions for the global relation in Step 1 is also independent of the particular domain and boundary conditions). Thus, for *any* separable Helmholtz problem in polar coordinates we only need to use the completeness relations (5.14) and (5.15) (or actually only one of them, as noted earlier). However, in the classical transform method every time the domain or the boundary conditions are changed one needs to derive a new transform, which may pose new difficulties (such as the nonconvergence of the radial series solution for Example 2 discussed in section 2.3).

The integral representation of the new method is

$$\begin{aligned}
 u(r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} & \left[\int_0^{i\infty} e^{\varepsilon \nu^2} J_\nu(kr) \left[- \left(e^{i\nu\theta} + e^{i\nu(2\pi-\theta)} \right) i\nu D_0(\nu) \right. \right. \\
 & \left. \left. - \left(e^{i\nu\theta} - e^{i\nu(2\pi-\theta)} \right) N_0(\nu) \right] d\nu \right. \\
 & \left. + \int_0^{-i\infty} e^{\varepsilon \nu^2} J_\nu(kr) \left[\left(e^{-i\nu\theta} + e^{-i\nu(2\pi-\theta)} \right) i\nu D_0(\nu) \right. \right. \\
 & \left. \left. - \left(e^{-i\nu\theta} - e^{-i\nu(2\pi-\theta)} \right) N_0(\nu) \right] d\nu \right] \\
 & - \frac{ia}{4} \left[\int_0^\infty e^{i\nu\theta} H_\nu^{(1)}(kr) \left(J_\nu(ka)N(-i\nu) - kJ'_\nu(ka)D(-i\nu) \right) d\nu \right. \\
 (5.16) \quad & \left. + \int_0^\infty e^{-i\nu\theta} H_\nu^{(1)}(kr) \left(J_\nu(ka)N(i\nu) - kJ'_\nu(ka)D(i\nu) \right) d\nu \right],
 \end{aligned}$$

where \prime denotes differentiation of $J_\nu(\cdot)$ with respect to its argument. In (5.16) the solution u is given in terms of integrals of the *transforms* D , N , D_0 , N_0 of the boundary values. To obtain (5.16) we write Green's integral representation (5.10) in polar coordinates, and then substitute into the resulting expression the expansions (5.12) and (5.13); on the boundaries where ϕ is fixed and ρ varies we use the radial expansion (5.12), whereas on the boundaries where ρ is fixed and ϕ varies we use the angular expansion (5.13). Changing the order of the physical integrals (in ρ or ϕ) and the spectral integrals (in ν) we find (5.16), provided we choose $r_1 = \rho$ and $r_2 = r$ in the radial representation (5.12). The alternative choice $r_1 = r$, $r_2 = \rho$ still leads to an integral representation, but the transforms of the boundary values are not the same transforms that appear in the global relation (5.7). Note that we continue *not* to favor either the r - or the θ -variable, with the integral representation (5.16) using *both* the radial expansion of E_s (5.12) and the angular expansion (5.13).

Equation (5.16) provides the analogue of the integral representation (4.10) for Example 1. (In [49, section 3] and [14, section 4] it is shown how to obtain (4.10) from Green's integral representation instead of employing the inverse Fourier transform used in section 4.1; however, the latter derivation is simpler.)

5.3. Step 3. Our task is to eliminate the transforms of the unknown boundary values $D_0(\nu)$, $N_0(\nu)$, and $N(\pm i\nu)$ from the integral representation (5.16), using the global relation (5.7), and obtain (5.2). Recall that in Example 1 we applied certain transformations in ν to the global relation that leave some of the transforms of the boundary values invariant. Now the appropriate transformation is $\nu \mapsto -\nu$; this is a consequence of the symmetry

$$(5.17) \quad H_{-\nu}^{(1)}(x) = e^{i\pi\nu} H_{\nu}^{(1)}(x)$$

and the fact that the integral representation (5.16) involves $D(-i\nu)$ and $N(-i\nu)$, whereas the global relation (5.7) involves only $D(i\nu)$, $N(i\nu)$. Letting $\nu \mapsto -\nu$ in (5.7) and using (5.17) yields

$$(5.18) \quad -aH_{\nu}^{(1)}(ka)N(\pm i\nu) + akH_{\nu}^{(1)'}(ka)D(\pm i\nu) + (1 - e^{\pm 2\pi i\nu}) \left(\pm i\nu D_0(\nu) - N_0(\nu) \right) = 0.$$

Using this equation and performing appropriate contour deformations in (5.16) it is possible to obtain (5.2). For brevity of presentation we do not give the details; these can be found in [101]. We only note that the relevant approach is conceptually similar to Step 3 for Example 1 in section 4.3, namely, it is possible to eliminate the transforms of the unknown boundary values from the integral representation using both the algebraic manipulation of (5.18) and analyticity.

For completeness we recall the classical radial series expansion

$$(5.19) \quad u(r, \theta) = i \sum_{n=1}^{\infty} \frac{H_{\nu_n}^{(1)}(kr)}{\dot{H}_{\nu_n}^{(1)}(ka)} \left(e^{i\nu_n\theta} D_L(-i\nu_n) + e^{-i\nu_n\theta} D_R(i\nu_n) - \frac{e^{i\nu_n\theta} D(-i\nu_n) + e^{-i\nu_n\theta} D(i\nu_n)}{1 - e^{-2\pi i\nu_n}} \right),$$

where

$$\dot{H}_{\nu_n}^{(1)}(ka) = \left. \frac{d}{d\nu} H_{\nu}^{(1)}(ka) \right|_{\nu=\nu_n}$$

and

$$D_L(\pm i\nu) = \int_0^{\theta} e^{\pm i\nu\phi} d(\phi) d\phi, \quad D_R(\pm i\nu) = \int_{\theta}^{2\pi} e^{\pm i\nu\phi} d(\phi) d\phi,$$

with L and R denoting “left” and “right,” respectively [18]. Note that, as expected, the series (5.19) is *not* uniformly convergent in the parameter r , since each term is zero when $r = a$, but $u(a, \theta) \neq 0$. This series is obtained classically through the algorithm of section 2.1, where the spectral analysis of the radial ODE with the boundary condition $u(a, \theta) = 0$ and the radiation condition (1.13) yields the candidate completeness relation

$$(5.20) \quad \rho \delta(r - \rho) = -\pi i \sum_{n=1}^{\infty} \frac{\nu_n H_{\nu_n}^{(1)}(k\rho) H_{\nu_n}^{(1)}(kr) J_{\nu_n}(ka)}{\dot{H}_{\nu_n}^{(1)}(ka)},$$

see [76, p. 299], [33, p. 116] (actually these papers present the analogous completeness relation in the three-dimensional case with Neumann boundary conditions, but the

derivation of (5.20) is very similar). However, Cohen [18] proved that the transform pair obtained from (5.20) is *not* complete, and also that the series (5.19) *diverges* if $d(\theta) \neq 0$.

In order to show the relation between the *divergent* radial series expansion (5.19) and the *convergent* integral expression (5.2), we split $D(-i\nu)$ into $D_L(-i\nu) + D_R(-i\nu)$ in the first integral of (5.2), we let $\nu \mapsto -\nu$ in the term involving D_R (using the symmetry (5.17)), and then we deform the contour from $(-\infty, \infty)$ to $(-\infty + ic, \infty + ic)$, where $\Im c < \Im \nu_1$. This yields

$$(5.21) \quad u(r, \theta) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \frac{H_\nu^{(1)}(kr)}{H_\nu^{(1)}(ka)} \left(e^{i\nu\theta} D_L(-i\nu) + e^{-i\nu\theta} D_R(i\nu) - \frac{e^{i\nu\theta} D(-i\nu) + e^{-i\nu\theta} D(i\nu)}{1 - e^{-2\pi i\nu}} \right) d\nu.$$

The radial series follows by evaluating the integral in (5.21) as residues in the upper half-plane. However, if $d(\theta) \neq 0$, the contribution from the contour at infinity is unbounded [101] and hence Cauchy's theorem *cannot* be applied.

6. Discussion of the New Method.

6.1. Comparison with the Classical Transform Method and the Watson Transformation. For BVPs that *can* be solved using the classical transform method the new method has the following advantages:

1. It yields the solution as an integral in the complex plane which can be deformed to either of the two classical expressions for the solution. This avoids the need to use ad hoc approaches, such as the Watson transformation, in order to convert the classical solution expressions into forms that are computationally more efficient (see, e.g., [28, Chapter 5] for an overview of such approaches).
2. The solution given by the new method is uniformly convergent on the whole of the boundary of the domain. This is not the case for the classical expressions for the solution when *nonzero* boundary conditions are prescribed, since by construction the classical expressions are expansions in eigenfunctions of one of the homogeneous ODEs, i.e., ODEs with *zero* boundary conditions.
3. For constructing the expression of the solution via the new method, one only requires the completeness relation for one separated ODE in the *whole space*, i.e., *without* any boundary conditions. In contrast, the classical transform method relies on the completeness relation for one of the separated ODEs *with* boundary conditions; such completeness relations are more difficult to construct and also may not even exist. Furthermore, if the domain or boundary conditions are changed, a new completeness relation must be derived.

In addition, in some applications (including finding artificial boundary conditions for a numerical scheme) one is only interested in obtaining the unknown boundary values, i.e., the *Dirichlet to Neumann map*. In the classical transform method this requires first finding the solution and then evaluating it or its derivatives on the boundary. In the new method, the global relation can be used to obtain the Dirichlet to Neumann map directly, *without* going through the solution (see, e.g., [41, section I.1.1.4]).

Of course, for very simple BVPs it is immediately clear which are the appropriate transforms, as well as which of them provides the best expression for the solution; for such simple BVPs it may be easier to apply the standard procedure instead of the new method.

Elaborating further on points 2 and 3, we note that the classical transform method represents the solution as a superposition of separable solutions of the PDE that satisfy certain boundary conditions. For example, for the heat equation on the half-line, the classical sine transform solution (1.5) is a superposition of separable solutions $\exp(-\nu^2 t) \sin \nu x$ for $\nu \in \mathbb{R}^+$, where $\sin \nu x$ is chosen because it vanishes when $x = 0$. On the other hand, the new method represents the solution as a superposition of separable solutions of the PDE *without* requiring these solutions to satisfy any boundary conditions. In the heat equation example these are $\exp(-\nu^2 t + i\nu x)$ for ν both on \mathbb{R} and also on ∂D^+ in \mathbb{C} . Thus, the new method constructs the solution as a superposition of a *wider class* of separable solutions than the classical transform method.

Elaborating further on point 1, we note that the Watson transformation demonstrated, in the context of PDEs describing wave propagation, that the best expression for the solution of a separable BVP is as an *integral in the complex plane*. However, surprisingly, this understanding has *not* been systematically applied to other PDEs. For example, consider the heat equation (1.2) posed on a finite interval $0 < x < L$. If Dirichlet boundary conditions are imposed on both ends, then the classical transform method yields the familiar sine series solution, with the sum taken over the eigenvalues of the x -ODE, $n\pi/L$, $n \in \mathbb{Z}$. However, if Robin boundary conditions are applied at both ends, namely,

$$u_x(0, t) - \gamma_1 u(0, t) = g_R(t), \quad u_x(L, t) + \gamma_2 u(L, t) = h_R(t), \quad 0 < t < T, \quad \gamma_1 > 0, \quad \gamma_2 > 0,$$

the classical solution involves a series over $\{\nu_n\}_{n=1}^\infty$, where ν_n are the (real) zeros of

$$(6.1) \quad \Delta(\nu) := (i\nu - \gamma_1)(i\nu - \gamma_2)e^{-i\nu L} - (i\nu + \gamma_1)(i\nu + \gamma_2)e^{i\nu L}.$$

The equation $\Delta(\nu) = 0$ is a transcendental equation, thus the classical expression for the solution to the Robin problem is significantly less useful than the classical expression for the solution to the Dirichlet problem. On the other hand, the new method expresses the solution of the heat equation (and indeed of a general evolution PDE) as a collection of three *integrals*, which not only overcome the problem of nonuniform convergence at the endpoints $x = 0$ and $x = L$, but also retain their form when the boundary conditions become more complicated. Indeed, for the above Robin problem,

$$(6.2) \quad u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\nu x - \nu^2 t} \hat{u}_0(\nu) d\nu - \frac{1}{2\pi} \int_{\partial D^+} e^{i\nu x - \nu^2 t} \frac{\tilde{g}(\nu)}{\Delta(\nu)} d\nu \\ - \frac{1}{2\pi} \int_{\partial D^-} e^{i\nu x - \nu^2 t} \frac{\tilde{h}(\nu)}{\Delta(\nu)} d\nu,$$

where $\hat{u}_0(\nu)$ is the Fourier transform of $u_0(x)$, $\Delta(\nu)$ is defined by (6.1), ∂D^+ is shown in Figure 1.1, and ∂D^- is the reflection of ∂D^+ across the real axis, i.e., the union of the rays $\arg \nu = -\pi/4$ and $\arg \nu = -3\pi/4$. The transforms \tilde{g}, \tilde{h} are *explicitly* given in terms of $\hat{u}_0(\pm\nu)$, \tilde{g}_R , and \tilde{h}_R , where the last two are the t -transforms of g_R and h_R (analogous to the second equation in (1.8)). For example, if $\gamma_2 = h_R = 0$, then

$$\tilde{g}(\nu) = i\nu [2i\nu e^{-i\nu L} \tilde{g}_R(\nu^2) - (i\nu + \gamma_1)(e^{i\nu L} \hat{u}_0(\nu) + e^{-i\nu L} \hat{u}_0(-\nu))], \\ \tilde{h}(\nu) = i\nu [2i\nu \tilde{g}_R(\nu^2) - (i\nu - \gamma_1) \hat{u}_0(\nu) - (i\nu + \gamma_1) \hat{u}_0(-\nu)].$$

For the Dirichlet problem, the solution is similar to (6.2), with \tilde{g} and \tilde{h} given in terms of the boundary conditions, and $\Delta(\nu) = \exp(i\nu L) - \exp(-i\nu L)$, the zeros of

which are the familiar eigenvalues $\nu = n\pi/L$, $n \in \mathbb{Z}$. Although in principle (6.2) can be obtained from the classical series solution by using a Watson-type transformation similar to (2.4), apparently this was not done, and hence the cumbersome classical series solution over eigenvalues satisfying a transcendental equation appears in many reference texts, e.g., [89, section 1.1.1-11], [28, Example 4.2.3], [58, section 5.8].

Finally, we emphasize that, just like the classical transform method, the new method is also applicable in three dimensions. For example, it has been applied to evolution PDEs in two space dimensions in [40] and [64]. Although the global relation now contains two complex variables, these variables are not coupled, and this avoids the subtleties of the theory of several complex variables.

6.2. Breaking the Separable/Self-Adjoint Framework. As discussed in section 2, the classical transform method requires that the PDE, domain, and boundary conditions are separable. Furthermore it may fail when the separated ODEs are non-self-adjoint. In contrast, the new method can be applied to certain nonseparable and non-self-adjoint problems.

6.2.1. Nonseparable Boundary Conditions. The new method can be applied to large classes of nonseparable boundary conditions for separable PDEs in separable domains. For example, problems with oblique Robin boundary conditions (which involve derivatives at angles to the boundary) are solved in [100, section 4.2] for the Helmholtz equation in a wedge domain and in [41, Chapter 12], [45], [4] for other elliptic equations in various separable domains. This demonstrates the advantage of representing the solution of a BVP as a superposition of separable solutions, *without* imposing any boundary conditions on those separable solutions (recall the discussion in section 6.1).

6.2.2. Nonseparable PDEs. Although separable solutions can be constructed for the biharmonic equation $\Delta^2 u = 0$, this PDE is not separable in Cartesian and polar coordinates in the standard sense. Thus, although certain transforms can be applied to some simple problems, questions regarding completeness immediately arise. The most famous such completeness question involved the *Papkovich–Fadle* eigenfunctions, which can be used to express the solution of various BVPs for the biharmonic equation in a semi-infinite strip. The completeness of these eigenfunctions was established simultaneously by Gregory [56] and Spence [99]. Interestingly, whereas the results in [56] relied on a direct and involved calculation with the relevant series, the key step in the stronger results in [99] was to treat the series as the residue sum of an *integral in the complex plane* (a completeness proof using explicitly the method (i) of section 2.3, integration in the complex plane, was then presented in [62, section 4]).

The biharmonic equation is self-adjoint and, despite the lack of standard separability, several one-parameter families of adjoint solutions can be constructed from combining separable solutions of the Laplace equation. These one-parameter families lead to several global relations which can then be used to eliminate the transforms of the unknown boundary values. Several BVPs for the biharmonic equation in a semi-infinite strip are solved in [22] and [26] and these solutions are expressed as *integrals* in the complex plane. Thus, one avoids the Papkovich–Fadle eigenfunctions and the issue of their completeness. Furthermore, a general method is described in [41, Chapter 4] of how to use the new method to derive completeness relations which can be used to establish the completeness of the Papkovich–Fadle eigenfunctions in a straightforward manner.

6.2.3. Nonseparable Domains. There exist certain domains that are *not* separable but in which BVPs can nevertheless be solved by the method of images. Essentially this method uses reflections to transform a BVP in the nonseparable domain into a BVP in a separable domain. In two dimensions the special nonseparable domains for which this procedure is possible are the equilateral triangle, the right isosceles triangle, and half an equilateral triangle. The characterization of these domains can be understood either in terms of reflection groups [66] or in terms of root systems of Lie algebras (see, e.g., [3]). In the new method there is no conceptual difference between solving BVPs in these “separable-after-reflection” domains and separable domains, either in two dimensions [100, section 5], [63, section 2], [24], [44], [9] or in three dimensions [64]. Although the new method has led to the solution of certain BVPs for these particular domains for which the method of images fails, the question of whether the new method can be applied to additional nonseparable domains remains open.

6.2.4. Non-Self-Adjoint Problems. The problems solved in sections 4 and 5 are examples of non-self-adjoint BVPs. In what follows we briefly discuss several other non-self-adjoint problems that can be solved by the new method.

BVPs for the evolution PDE (4.13), which involves an arbitrary number of x -derivatives, posed both on the half-line and the finite interval, have been extensively investigated with the new method [41, Chapters 2 and 3], [29] (see Remark 4.4 which discusses the half-line case). An interesting feature of the finite interval case is that effective expressions for the solution involving integrals, similar to (6.2), can always be obtained via the new method, whereas in many cases, including the linearized KdV equation (1.1) with generic boundary conditions, the solution *cannot* be expressed as an infinite series. In this case the eigenfunctions of the associated x -differential operator do *not* form a basis [87], [95], [96]. For the linearized Benjamin–Bona–Mahony equation (4.15) on the finite interval (studied in [111]), the associated eigenvalue problem for the x -differential operator is $-d^2u/dx^2 + u = \lambda du/dx$, with a Robin boundary condition at both $x = 0$ and $x = L$, which is highly nonstandard.

The reason that the new method can tackle these problems involving evolution PDEs is a consequence of the fact that, whereas the spectral theory of the x -differential operators on the half-line and finite interval is difficult (with the eigenfunctions often *not* forming a basis), the new method only requires the existence of the Fourier transform, which is the appropriate transform for the x -differential operators on the line (recall point 3 of section 6.1).

6.3. Extensions of the Method, and Connections with Other Techniques.

New Numerical Schemes. Just as both the classical transform method and Green’s integral representation give rise to numerical methods, the new method has also given rise to novel techniques. Such techniques for linear evolution PDEs, linear elliptic PDEs, and nonlinear evolution PDEs are presented in [35], in [97] and [50] (following on from [52] and [94]), and in [114], respectively.

The Method of Images. Another classical method for finding explicit solutions to PDEs, other than the classical transform method, is the method of images. This method is applicable when both the PDE and the domain have certain symmetries; a characterization of such PDEs, domains, and boundary conditions is given in [66]. A connection between the new method and the method of images was already noted in section 6.2.3. In fact, the new method can be viewed as the analogue of the method of images in the transform (or spectral) space instead of the physical space.

The transformations in ν used in Step 3 of the new method are the analogue of the reflections used in the method of images. It appears that the advantage of the new method over the method of images is due to the fact that there exist more symmetries in the complex ν -plane than in the physical plane; see [41, section I.4.2], [100, section 1.4.5].

The Wiener–Hopf Technique and Related Approaches. Mixed BVPs (where the boundary conditions change on part of the boundary) cannot be solved in general by the classic transform method. Nevertheless, for some of these problems it is possible, after using a suitable transform, to express their solution in terms of a Wiener–Hopf (or Riemann–Hilbert) problem [83], [70]. The application of the new method to these types of problems is work in progress; preliminary investigations have shown that the global relation yields immediately the relevant Wiener–Hopf problem, thus eliminating the need to first choose a suitable transform. It also appears that the new method can be linked to other transform-based approaches, including the Sommerfeld–Malyuzhinets technique [8] and approaches involving combining transform methods with representations of the solution via Green’s functions [54], [21].

Coupled Linear Systems. Many physical applications give rise to coupled linear systems of PDEs. In some situations these can be reduced to separable equations coupled only by their boundary conditions (for example, by the Helmholtz decomposition in elastodynamics; see, e.g., [15, Chapter 4]), but even in these cases the application of transform methods to solve these problems is notoriously difficult, and many important problems remain essentially open. Recently, the new method has been applied to some of these problems in both fluid dynamics [6] and elastodynamics [60], [48].

The Ehrenpreis Fundamental Principle. The fact that the solution of the ODE

$$P\left(\frac{d}{dx}\right)u(x) = 0,$$

where P is a polynomial, can be written as a linear combination of exponentials is known as the *Euler principle*. The extension of this result to systems of differential equations in more than one dimension was established by Ehrenpreis and Palamodov, and this extension was called by Ehrenpreis the *fundamental principle*; see, e.g., [59, section 15.3 and notes to Chapter 15], [32, Chapter 4, section 4], [86, Chapter 4, section 5]. This deep result implies that for the evolution PDE (4.13) there exists a measure $\mu(\nu)$ with support L such that

$$u(x, t) = \int_L e^{i\nu x - \omega(\nu)t} d\mu(\nu);$$

however, the measure μ is *not* constructed explicitly. The expressions for the solutions of BVPs obtained by the new method are consistent with this result; furthermore, the new method expresses $d\mu(\nu)$ *explicitly* in terms of the given initial and boundary data. It is interesting to note that recent results of Ashton [5] suggest that the new method provides an alternative, apparently simpler approach for obtaining rigorous estimates concerning elliptic PDEs.

7. From Green to Lax. As noted in the introduction, the new method arose out of the theory of certain nonlinear PDEs called *integrable*. These PDEs have the defining property of possessing a *Lax pair*. In this section we briefly show that Lax

pairs for separable linear PDEs arise naturally from Green's divergence form; thus Lax pairs provide the analogue of Green's divergence form for nonlinear integrable PDEs. We focus only on the two-dimensional case, since the three-dimensional case is more complicated (some of this material is approached from the nonlinear point of view in [42]).

In 1967 a method called the inverse scattering transform (IST) method was introduced to solve the IVP of the KdV equation (1.3) [53]. Shortly afterwards, Lax realized that this method relied on the fact that the KdV equation can be written as the compatibility condition of a pair of equations that were subsequently called a "Lax pair" [71].

A PDE for the function $u(x, t)$ has a Lax pair formulation if the PDE can be written in the form

$$(7.1) \quad A_t - B_x + [A, B] = 0,$$

where $[A, B] := AB - BA$ and both $A(x, t, u, \nu)$ and $B(x, t, u, \nu)$ are matrix functions of $x, t, u(x, t)$, and $\nu \in \mathbb{C}$. If (7.1) holds, then the PDE is the *compatibility condition* of the following (Lax) pair of equations:

$$(7.2a) \quad \Phi_x = A\Phi,$$

$$(7.2b) \quad \Phi_t = B\Phi,$$

which involve a vector function $\Phi(x, t, u, \nu)$. The condition $\Phi_{xt} = \Phi_{tx}$, which is equivalent to (7.1), ensures that (7.2) describe the same Φ . The matrices A and B in (7.2) depend on a complex parameter ν , and this plays a crucial role in using the Lax pair to obtain useful information about the solution u of the PDE (7.1). Possessing a Lax pair is a very strong property; nevertheless, integrable PDEs arise in many applications; see, e.g., [1, section 1.1], [27, section 8.2] (see [16] for a discussion of why certain PDEs are both integrable and ubiquitous in applications).

The IST method solves the IVP of integrable nonlinear evolution PDEs by first analyzing (7.2a) in order to obtain a nonlinear transform, and then by using (7.2b) to obtain the time evolution of this transform. Each of these two steps involves the solution of a linear problem, thus the IVP of a *nonlinear* integrable PDE can be solved through a sequence of *linear* steps. Although the IST method has conceptual similarities with the classical Fourier transform method for solving the IVP for linear evolution PDEs [2], the IST method originally had no linear analogue. The proper linear analogue of the IST was obtained by the late Israel Gel'fand and one of the authors who constructed Lax pairs for linear evolution PDEs in [43] and demonstrated that the analysis of the x -parts of all these Lax pairs yields the usual Fourier transform. Thus the IST method applied to linear evolution PDEs produces the familiar expression for the solution obtained by the Fourier transform, i.e., (1.4) in the case of the linearized KdV equation. In this way, the IST method can be rigorously understood as a nonlinear Fourier transform: the fundamental step in the IST is the spectral analysis of the linear eigenvalue equation (7.2a), which yields a nonlinear analogue of the Fourier transform "custom made" for the nonlinear PDE associated with (7.2a); after this pair is determined, it is straightforward using (7.2b) (or the PDE itself) to determine the time evolution of this nonlinear transform.

It was shown in [38] that in order to analyze IBVPs (as opposed to IVPs) for integrable nonlinear PDEs, it is now necessary to implement the *simultaneous* spectral analysis of both equations defining the Lax pair as opposed to the spectral analysis

of only the x -part (7.2a) used in the IST method. The easiest way to carry out this simultaneous spectral analysis is to consider the differential form

$$(7.3) \quad d\Phi = A\Phi dx + B\Phi dt,$$

which according to (7.1) is a *closed* form (thus its integral is path-independent). A key role is played by the *global relation*, which arises from the integration of (7.3) around the boundary of the domain in which the PDE is posed (in the linear limit this yields the global relation of the associated linear PDE).

Lax pairs for separable linear PDEs arise naturally from Green's divergence form. In particular, recall that the divergence form of the linearized KdV equation (1.1) is

$$(7.4) \quad (vu)_t + (vu + v_{xx}u - v_xu_x + vu_{xx})_x = 0,$$

where v is a solution of the adjoint equation. Introducing a potential ψ , we immediately obtain the following Lax pair for the linearized KdV equation:

$$(7.5a) \quad \psi_x = -vu,$$

$$(7.5b) \quad \psi_t = vu + v_{xx}u - v_xu_x + vu_{xx},$$

where v is any suitable one-parameter family of solutions to the adjoint equation, such as $v = \exp(-i\nu x + i(\nu^3 - \nu)t)$. The pair (7.5) can be written in the vector form (7.2) by letting

$$\Phi = \begin{pmatrix} \psi \\ 1 \end{pmatrix}, A = \begin{pmatrix} 0 & -vu \\ 0 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & v_{xx}u - v_xu_x + vu_{xx} \\ 0 & 0 \end{pmatrix}.$$

Similarly, recall that the divergence form of the Helmholtz equation is (5.4). Using Cartesian instead of polar coordinates, we find

$$(vu_x - uv_x)_x + (vu_y - uv_y)_y = 0,$$

where v is also a solution of the Helmholtz equation. Hence, we find the Lax pair

$$(7.6a) \quad \psi_x = -(vu_y - uv_y),$$

$$(7.6b) \quad \psi_y = vu_x - uv_x,$$

where v is any suitable one-parameter family of solutions to the Helmholtz equation, such as $v = \exp(i\nu_1 x + i\nu_2 y)$ with $\nu_1^2 + \nu_2^2 = k^2$.

This relationship among Green's divergence form, Lax pairs, the new method, and the method of images is shown schematically in Figure 7.1.

In conclusion, Lax pairs provide the analogue of the classical concept of separation of variables for certain nonlinear PDEs. Furthermore "synthesis" as opposed to "separation" confers an advantage. Indeed, the global relation, which is obtained by combining the classical separable framework with Green's divergence form, allows one to analyze linear separable PDEs *without* choosing between the variables (the first manifestation of synthesis as opposed to separation). Furthermore, for linear PDEs the global relation immediately gives rise to a Lax pair, which provides an alternative, deeper type of separability, and which is the only separability applicable to certain nonlinear PDEs. For the solution of BVPs, the Lax pair of a PDE should be analyzed *simultaneously* via the associated differential form (the second manifestation of synthesis as opposed to separation of variables).

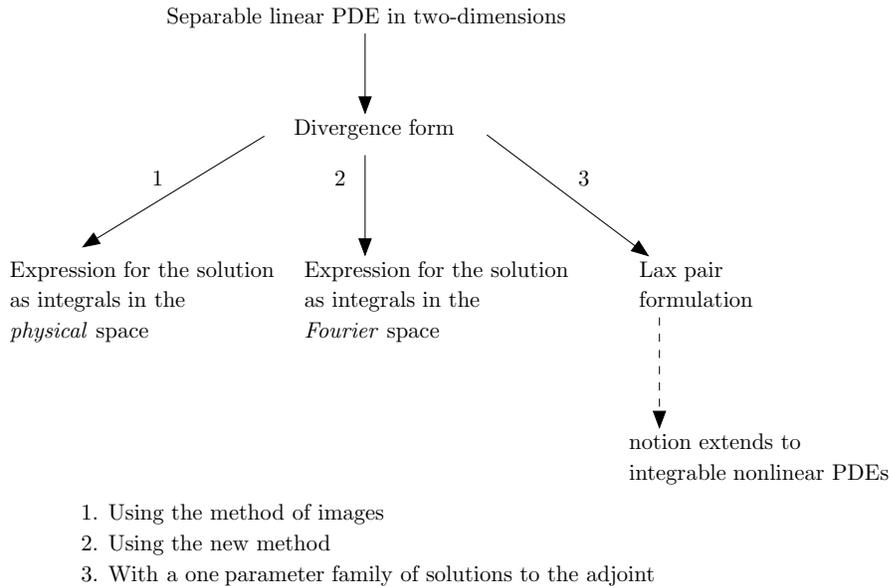


Fig. 7.1 *The relationship among Green's divergence form, Lax pairs, the new method, and the method of images.*

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