Title: Transform Methods for Linear PDEs

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Transform Methods for Linear PDEs

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Transform methods, separation of variables, eigenfunction expansions, spectral representations.

Mathematics Subject Classification

35A22, 35C05, 35C15, 35P10.

Short Definition

Transform methods replace differentiation in one variable with multiplication by a "transform" variable.

Description

The utility of transform methods essentially stems from the fact that they replace differentiation in one variable with multiplication by a transform variable. Hence, a PDE in m variables can be converted into a PDE in $m-1$ variables, and thus ultimately to an ODE, or algebraic equation.

This is best illustrated by example. Possibly the most well-known transform is the Fourier transform: given a smooth function f on $\mathbb{R}$ with sufficient decay at infinity, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\nu) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\nu x} f(x)\,\mathrm{d}x, \quad \nu \in \mathbb{R}.$$

If we are given the Fourier transform of f, then the function itself can be recovered through the inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu x} \widehat{f}(\nu)\,\mathrm{d}\nu, \quad x \in \mathbb{R}. \tag{1}$$

Using integration by parts and the fact that f vanishes at $\pm\infty$, we have that

$$\widehat{\frac{\mathrm{d}f}{\mathrm{d}x}}(\nu) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\nu x} \frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x = \mathrm{i}\nu \widehat{f}(\nu), \tag{2}$$

i.e. when taking the Fourier transform, differentiation is replaced by multiplication by $\mathrm{i}\nu$, where ν is the transform variable.

Example 1: the heat/diffusion equation on the infinite line

To illustrate one of the uses of the Fourier transform (and transform methods in general), consider the following boundary value problem (BVP) for the heat (or diffusion) equation in one space and one time dimension:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in (-\infty, \infty), \quad t \in (0, \infty), \tag{3}$$

with the initial condition $u(x,0) = u_0(x)$ (for a given function $u_0(x)$ that decays as $|x| \to \infty$), and the condition that $u(x,t)$ and all its derivatives tend to zero as $|x| \to \infty$ for all $t > 0$.

Taking the Fourier transform of (3) in the variable x (i.e. multiplying (3) by $\mathrm{e}^{-\mathrm{i}\nu x}$ and integrating over $\mathbb{R}$), and using the rule (2) twice, we obtain

$$\frac{\mathrm{d}\widehat{u}}{\mathrm{d}t}(\nu,t) = -\nu^2 \widehat{u}(\nu,t), \tag{4}$$

where

$$\widehat{u}(\nu,t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\nu x} u(x,t)\,\mathrm{d}x.$$

Thus, by taking the Fourier transform, we have reduced the PDE (3) to the ODE (4). Solving (4) and using the inversion formula (1), we obtain the following expression for the solution of the BVP :

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu x - \nu^2 t} \widehat{u}_0(\nu)\,\mathrm{d}\nu. \tag{5}$$

From this expression we can then extract information about the solution $u(x,t)$ (we will return to this below).

Why was the Fourier transform an appropriate transform to use? The answer is that the functions $\mathrm{e}^{\mathrm{i}\nu x}$ are eigenfunctions of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ with eigenvalue $-\nu^2$, and the expression (5) is then the expansion of the solution $u(x,t)$ in terms of these eigenfunctions (i.e. a linear superposition of them, in this case an integral).

Instead of expanding the solution in terms of the eigenfunctions of $\mathrm{d}^2/\mathrm{d}x^2$, we could have chosen to expand the solution is terms of the eigenfunctions of $\mathrm{d}/\mathrm{d}t$. These eigenfunctions are again exponentials, and it turns out that the relevant transform is the *Laplace transform*. Given a smooth function g on $(0,\infty)$ with sufficient decay at infinity, its Laplace transform, $\widetilde{g}$, and inverse are given by

$$\widetilde{g}(s) := \int_0^{\infty} \mathrm{e}^{-st} g(t)\,\mathrm{d}t, \quad \Re(s) \geq 0, \qquad g(t) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{st} \widetilde{g}(s)\,\mathrm{d}s, \quad t \in (0,\infty). \tag{6}$$

Integration by parts shows that

$$\widetilde{\frac{\mathrm{d}g}{\mathrm{d}t}}(s) = s\widetilde{g}(s) - g(0),$$

and thus, similar to the Fourier transform, the Laplace transform replaces differentiation with multiplication.

Applying the Laplace transform to (3) yields an inhomogeneous ODE in x. Solving this ODE using standard, but slightly involved, calculation, and then using the inversion formula in (6), we eventually obtain the expression for the solution

$$u(x,t) = \frac{1}{4\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{e}^{st}}{\sqrt{s}} \left(\int_{-\infty}^{x} \mathrm{e}^{-\sqrt{s}(x-\xi)} u_0(\xi)\,\mathrm{d}\xi + \int_{x}^{\infty} \mathrm{e}^{\sqrt{s}(x-\xi)} u_0(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s, \tag{7}$$

(see, e.g. Ockendon et al (2003, Example 6.4)). Since s is now a complex variable we need to specify what branch of $\sqrt{s}$ we have chosen; in the expression (7) the branch cut for $\sqrt{s}$ is on the negative imaginary axis and the real part of $\sqrt{s}$ is positive. Deforming the contour to enclose the branch cut (using Cauchy's theorem) and making the change of variables $s = -\nu^2$ we obtain the expression (5).

We have gone through this particular example involving the heat equation in some detail because it illustrates the following general features of the classical transform method:

1. The solution is expressed as an expansion in eigenfunctions of one of the ODEs.
2. If the PDE has m variables, then there are m different transforms one can apply.
3. Understanding how the different expressions for the solution (obtained via different transforms) are related to one another requires considering the transform variables as complex variables, and deforming contours in the complex plane.

Example 2: the heat/diffusion equation on the finite interval

We now give another example, which emphasises the fact that the appropriate transform to use is an expansion in eigenfunctions. Consider the heat equation (3), but posed on a finite interval, $0 < x < L$, with boundary conditions $u(0,t) = u(L,t) = 0$.

Now the appropriate transform in x is the discrete sine transform, i.e.

$$\widehat{f}(n) := \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,\mathrm{d}x, \quad n \in \mathbb{Z}^+, \qquad f(x) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \widehat{f}(n), \quad 0 < x < L. \tag{8}$$

The functions $\sin(n\pi x/L)$, $n \in \mathbb{Z}^+$, are eigenfunctions of the differential operator $\mathrm{d}^2/\mathrm{d}x^2$ on $0 < x < L$, with zero boundary conditions at the endpoints.

Applying the transform (8) to the PDE, we obtain an ODE similar to (4). Solving this ODE and then using the inversion formula yields the expression for the solution

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right) \widehat{u}_0(n), \tag{9}$$

where $\widehat{u}_0(n)$ is the discrete sine transform of the initial condition $u_0(x)$.

Similar to the case of the infinite line, the appropriate transform in t is the Laplace transform, and this yields an expression for the solution as an integral over the imaginary axis, similar to (7).

Having illustrated the classical transform method for solving separable PDEs in these two examples, we now discuss it more generally.

The classical transform method

The algorithm

For simplicity, consider BVPs in two dimensions (by the very nature of *separation* of variables, the three dimensional case is similar!). The method requires that the domain, PDE, and boundary conditions are all separable; see Moon and Spencer (1971) or Morse and Feshbach (1953, §5.1) for accounts of the various coordinate systems in which the Laplacian (the higher dimensional analogue of $\mathrm{d}^2/\mathrm{d}x^2$) is separable (these include, e.g., cartesian coordinates, polar coordinates, and elliptic coordinates). The classical transform method then consists of the following four steps (see, e.g., Keener (1995, §8.1.3), Friedman (1956, p.259), Ockendon et al (2003, §4.4,§5.7,§5.8)).

1. Separate the PDE into 2 ODEs.
2. Choose one of the ODEs and derive the associated transform pair (which depends on the ODE, the domain, and the boundary conditions) by *spectral analysis* of

the ODE, see e.g. Keener (1995, Chapter 7), Stakgold (1967, Chapter 4), Stakgold (1979, Chapter 7).

3. Apply the transform to the PDE and use integration by parts to derive the ODE associated with this transform (thus one differential operator in the PDE is replaced by multiplication by a transform variable).
4. Solve the ODE of Step 3 and then apply the appropriate inverse transform.

In many cases it is possible to guess the appropriate transform pair in Step 2, and thus the spectral analysis can be avoided. We emphasise, however, that one always has the option of deriving the appropriate transform pair algorithmically via spectral analysis, since many texts on transform methods just list different transform pairs without explaining that each one is tailor-made for a particular BVP and can be found without any guesswork.

As emphasised in the examples, the solution to the given BVP is expressed as a superposition of eigenfunctions of the ODE chosen in Step 2, involving either an integral or a series depending on whether this ODE has a continuous or discrete spectrum.

Into the complex plane

As noted in Example 1, the different expressions for the solution obtained by different transforms can be shown to be equivalent by going into the complex plane (i.e. considering the transform variables as complex variables). If the two expressions are both integrals (like (5) and (7)) this procedure only requires deforming the contours of integration and possibly making a change of variables (as in Example 1). If one of the expressions is a sum, and the other an integral (like (9) and the analogue of (7) for this case), then deforming the contour of integration and evaluating the integral as residues gives the sum (see e.g. Stakgold (1968, p.161, p 219)). If both the expressions are sums, then they can be converted into integrals via a “reverse” residue calculation

(i.e. finding an integral in the complex plane that can be evaulated as residues to give the sum), and then, in principle, their contours deformed to show that they are equal (see, e.g., Friedman (1956, p. 274)). Given a sum, however, there are many different integrals that evaluate as residues to the sum, and thus choosing one whose integrand has the right analyticity properties in the complex plane is often difficult.

Recently an extension of the classical transform method has been developed that can obtain explicit expressions for the solution of certain non-separable problems (Fokas, 2008). In addition, for a separable BVP, this method provides an algorithmic way to obtain directly the expression for the solution as an integral in the complex plane, which can then be deformed (and evaluated as residues if necessary) to give the two expressions for the solution obtained by the classical transforms; see Fokas and Spence (2012) for an introduction to this method.

Using the expressions for the solution

Having obtained an explicit expression for the solution of a PDE via the classical transform method, one often wants to either (i) compute the solution via evaluating the integral or sum numerically, or (ii) obtain the asymptotic behaviour of the solution as some parameter becomes either large or small.

It is difficult to make any remarks on how to do either of these tasks in general (since the expressions for the solutions vary widely); however we note that, both for numerics and asymptotics, the fact that we can always express the solution as an integral in the complex plane (with the possibility of deforming the contour so that the integrand decays exponentially) is usually advantageous. For example, such deformations are the basis of the method of steepest descent for obtaining asymptotics of integrals (see, e.g., Bender and Orszag (1978, §6.6)) and Talbot's method for inverting Laplace transforms (see, e.g., Cohen (2007, Chapter 6)).

Generalisations and extensions

So far we have only discussed the case when, after taking an appropriate transform, the BVP reduces to an ODE that can be solved explicitly. In some situations, for example when certain mixed boundary conditions are prescribed, the resulting ODE *cannot* be solved explicitly, but instead the transform of the solution can be expressed in terms of a *Wiener–Hopf problem* (see, e.g., Noble (1988)), or, more generally, a *Riemann–Hilbert problem* [link to "Riemann–Hilbert problems" article] (note that these problems can only be formulated if the transform variable is thought of as a complex variable). In a similar vein, BVPs for the Helmholtz equation in wedge and cone geometries can be expressed in terms of *functional–difference* equations by the *Sommerfeld–Malyuzhinets technique* (see, e.g., Babich et al (2008)). In both these cases we obtain an expression for the solution of the BVP not in terms of an integral or a sum, but instead in terms of a more complicated mathematical object. It is often still possible to obtain useful asymptotic or numerical information about the solution of the BVP, but this is considerably harder than in the case of an integral or sum.

In another direction, we can abandon trying to find an explicit (or semi–explicit) expression for the solution, and instead concentrate on designing efficient ways to compute the solution (which one would hope would then be applicable to a wider range of BVPs). The ideas behind transform methods give rise to *spectral methods* [link to "Spectral methods" article].

Finally we note that transform methods are used more widely in the analysis of PDEs (i.e. not just for obtaining explicit expressions for the solution) [link to "Distributions and the Fourier Transform"].

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