Boundary Value Problems for Linear Elliptic PDEs

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Declaration

This dissertation is based on research done at the Department of Applied Mathematics and Theoretical Physics from October 2005 to March 2009.

This dissertation is the result of my own work. and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Chapter 6 of this dissertation contains the paper

S. A. Smitheman, E. A. Spence, A. S. Fokas, "A spectral collocation method for the Laplace and modified Helmholtz equations in a convex polygon" IMA J. Num. Anal. to appear

This paper contains a numerical method which was designed by Spence and Fokas, and programmed by Smitheman.

Euan Spence

Cambridge,

23 March 2009

Abstract

This thesis is concerned with new analytical and numerical methods for solving boundary value problems for the 2nd order linear elliptic PDEs of Poisson, Helmholtz, and modified Helmholtz in two dimensions.

In 1967 a new method called the Inverse Scattering Transform (IST) method was introduced to solve the initial value problem of certain non-linear PDEs (so-called "integrable" PDEs) including the celebrated Korteweg-de Vries (KdV) and nonlinear Schrödinger (NLS) equation. The extension of the IST method from initial value problems to boundary value problems (BVPs) was achieved by Fokas in 1997 when a unified method for solving BVPs for integrable nonlinear *and* linear PDEs was introduced. This thesis applies "the Fokas method" to the basic elliptic PDEs in two dimensions.

It is perhaps suprising that ideas from the theory of integrable nonlinear PDEs can be used to obtain new results in the classical theory of linear PDEs. In fact, the new method has a beautiful connection with the classical integral representations of the solutions of these PDEs due to Green. Indeed, this thesis shows that the Fokas method provides the analogue of Green's integral representation (IR) in the transform, or spectral, space. Both IRs contain boundary values which are not given as boundary conditions, and the main difficulty with BVPs is determining these unknown boundary values. In addition to the novel IR, the Fokas method provides a relation coupling the transforms of both the known and unknown boundary values known as "the global relation", which is then used to determine the contribution of the unknown boundary values to the solution.

One of the conclusions of this thesis is that the new method (applied to these 2nd order linear elliptic PDEs) does three things: (a) solves certain BVPs which cannot be solved by classical techniques, (b) yields novel expressions for the solutions of BVPs which have both analytical and computational advantages over the classical ones, and (c) provides

an alternative, simpler, method for obtaining the classical solutions.

Chapter 2 is about the novel integral representations. Chapter 3 is about the global relation. In Chapter 4, a variety of boundary value problems in the separable domains of the half plane, quarter plane and the exterior of the circle are solved. In Chapter 5, boundary value problems are solved in a non-separable domain, the interior of a right isosceles triangle. Just as Green's integral representation gives rise to a numerical method for solving these PDEs (the boundary integral method), the Fokas method can also be used to design new numerical schemes; Chapter 6 presents these for the Laplace and modified Helmholtz equation in the interior of a convex polygon.

Contents

Chapter 1

Introduction

1.1 The problem

The most famous second order linear elliptic PDE is

$$\Delta u(\boldsymbol{x}) + \lambda u(\boldsymbol{x}) = -f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1.1}$$

where λ is a real constant, $f(\boldsymbol{x})$ a given function, and Ω is some 2 dimensional domain with a piecewise smooth boundary. For $\lambda = 0$ this is Poisson's equation, $\lambda > 0$ the Helmholtz equation, and $\lambda < 0$ the Modified Helmholtz equation.

These PDEs appear in a myriad of applications. The Helmholtz equation with $\lambda = \omega^2/c^2$ arises from the wave equation

$$\frac{\partial^2}{\partial t^2} U(\boldsymbol{x}, t) - c^2 \Delta U(\boldsymbol{x}, t) = 0$$

under the assumption that the solution is harmonic in time with frequency ω: $U(\boldsymbol{x}, t) = u(\boldsymbol{x}) e^{-i\omega t}$. In unbounded domains the behaviour of u needs to be prescribed as $r = |\boldsymbol{x}| \to \infty$: $u \sim e^{-i\sqrt{\lambda} r}$ (multiplied by some function that decays algebraically in r) corresponds to incoming waves and $u \sim e^{i\sqrt{\lambda} r}$ corresponds to outgoing waves.

For a boundary value problem (BVP) to be well-posed, certain *boundary conditions* must be prescribed; the ones of most physical importance are:

- Dirichlet: $u(\boldsymbol{x}) = \text{known}$, $\boldsymbol{x} \in \partial\Omega$

- Neumann: $\frac{\partial u}{\partial n}(\boldsymbol{x}) = \text{known}$, $\boldsymbol{x} \in \partial\Omega$

- Robin: $\frac{\partial u}{\partial n}(\boldsymbol{x}) + \alpha u(\boldsymbol{x}) = \text{known}$, $\alpha = \text{constant}$, $\quad \boldsymbol{x} \in \partial\Omega$,

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit outward-pointing normal to Ω. More complicated boundary conditions can involve derivatives at angles to the boundary. One can also prescribe mixed boundary conditions, such as Dirichlet on part of the domain, and Neumann on another part.

In this thesis we will mainly be concerned with finding "explicit" expressions (i.e. given in terms of integrals or sums) for the solutions of BVPs involving (1.1.1). Obviously the class of BVPs for which this is possible is very restrictive, nevertheless it still contains many problems of physical interest. The penultimate chapter of the thesis (Chapter 6) is concerned with designing a numerical method for certain BVPs involving (1.1.1) (i.e. a method that computes an approximation to the solution).

Given that the majority of the thesis is concerned with "analytical" (as opposed to numerical) methods, the introduction will reflect this. A brief discussion of numerical methods for solving the PDE (1.1.1) is presented in §6.4.1. We note that the distinction between "analytical" and "numerical" methods is often blurred; for example, even if one has an expression for the solution of a BVP as an integral, in general this integral must be evaluated numerically to obtain the value of the solution at a particular point in the domain Ω.

Outline of this Chapter In Section §1.2 we review the classical "analytical" methods for finding expressions for the solution of (1.1.1). In Section §1.3 we explain the Fokas method applied to (1.1.1). In Section §1.4 we compare the Fokas method to classical techniques. Section §1.5 is concerned with "the bigger picture" of the Fokas method. Section §1.6 states how the results of this thesis are related to previous results obtained with the Fokas method. Section §1.7 summarises the thesis, with several bullet points per Chapter; these are then repeated at the beginning of the relevant chapters.

1.2 The classical theory and techniques

In this Section we review the classical techniques for finding explicit expressions for the solution of (1.1.1). We discuss only the techniques that can be applied to the boundary value problems considered in the thesis (other techniques, such as the Wiener-Hopf technique, see e.g. [Nob88], and the Sommerfeld–Malyuzhinets technique, see e.g. [BLG08], are discussed in Chapter 7). [Kel79] provides an excellent survey of these techniques, as well as many other exact and approximate methods for solving boundary value problems for linear PDEs.

The subsections on separation of variables/transform methods §1.2.2 and the method of images §1.2.3 contain some historical remarks. In §1.2.2 our aim is *not* to provide a comprehensive account of the history, but to highlight the developments since 1918 which centre on the boundary value problem of the Helmholtz equation in the exterior of the circle (which is solved using the new method in Chapter 4 §4.3). In §1.2.3 our aim *is* to provide an accurate account of the history of this method since results using this method continue to be rediscovered with the authors seemingly oblivious to what has been done before!

The Figure 1.1 gives an overview of the classical techniques, and shows how they fit together.

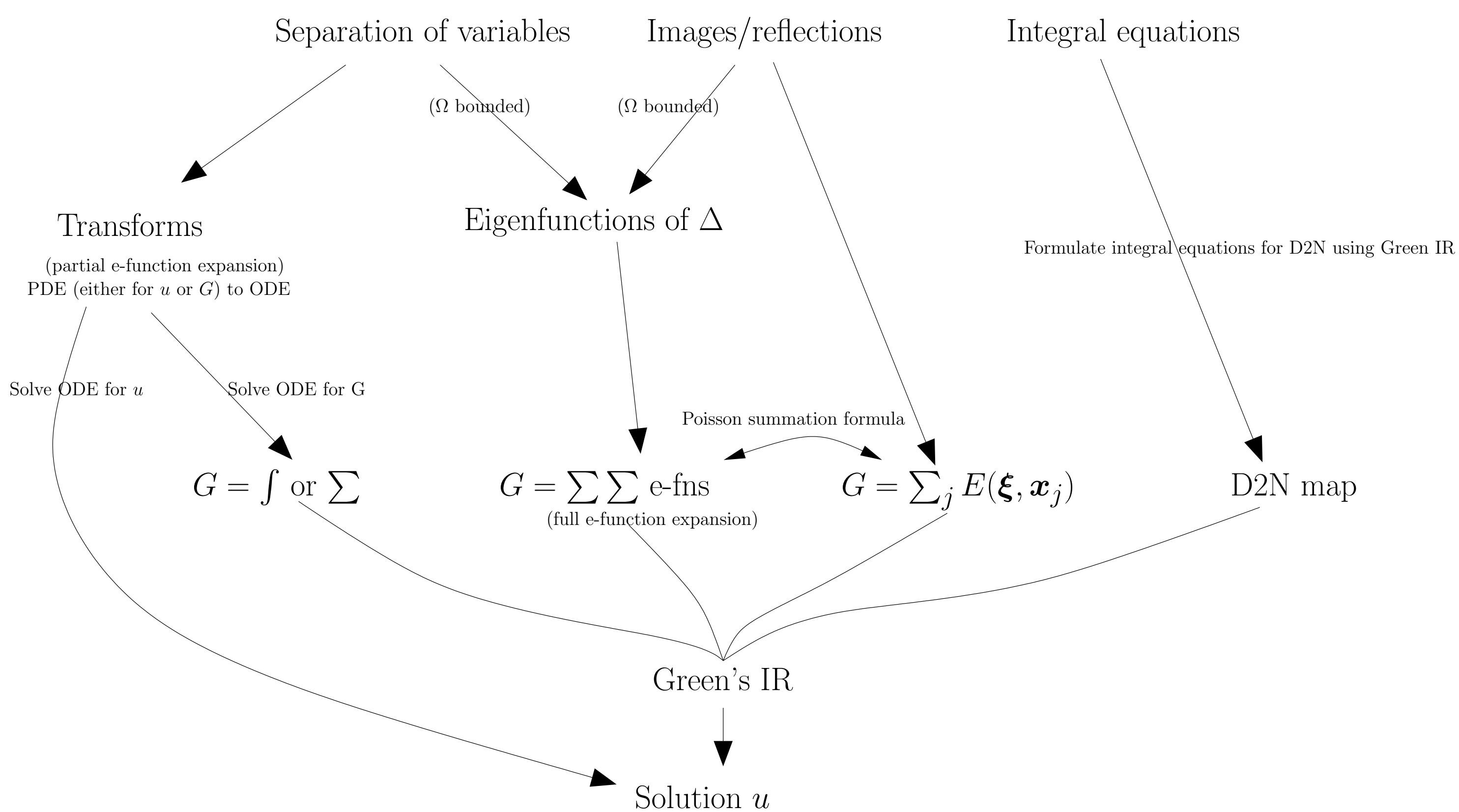


Figure 1.1: The classical techniques and how they are related.

1.2.1 Green's integral representation

Green's theorem gives the following integral representation of the solution of (1.1.1)

$$u(\boldsymbol{x}) = \int_{\partial\Omega} \left(E(\boldsymbol{\xi}, \boldsymbol{x}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial E}{\partial n}(\boldsymbol{\xi}, \boldsymbol{x}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) E(\boldsymbol{\xi}, \boldsymbol{x}) dV(\boldsymbol{\xi}), \quad \boldsymbol{x} \in \Omega \tag{1.2.1}$$

where E is the fundamental solution (sometimes known as the free space Green's function) satisfying

$$\left(\Delta_{\boldsymbol{\xi}} + \lambda \right) E(\boldsymbol{\xi}, \boldsymbol{x}) = -\delta(\boldsymbol{\xi} - \boldsymbol{x}), \quad \boldsymbol{\xi} \in \Omega. \tag{1.2.2}$$

For the different values of λ, E is given by

- Laplace/Poisson ($\lambda = 0$): $E = -\frac{1}{2\pi} \log |\boldsymbol{\xi} - \boldsymbol{x}|$,
- Helmholtz ($\lambda > 0$): $E = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda}|\boldsymbol{\xi} - \boldsymbol{x}|)$ (for outgoing waves),
- Modified Helmholtz ($\lambda < 0$): $E = \frac{1}{2\pi} K_0(\sqrt{-\lambda}|\boldsymbol{\xi} - \boldsymbol{x}|)$,

where $H_0^{(1)}$ is a Hankel function and K_0 a modified Bessel function.

The integral representation is obtained by forming the *divergence form* of (1.1.1) and (1.2.2)

$$\nabla \cdot (E \nabla u - u \nabla E) = -fE + u\delta, \tag{1.2.3}$$

integrating over Ω and using the divergence theorem (in 2-d Green's theorem in the plane) to replace the area integral by an integral over the boundary.

The integral representation (1.2.1) involves both u and its normal derivative on the boundary, that is the *Dirichlet* and *Neumann* boundary values respectively. However for a well posed problem only one of these boundary values (or a linear combination of the two) is given as boundary conditions. In some applications it is precisely the unknown boundary values that are required. If Dirichlet boundary conditions are given, the determination of the unknown Neumann boundary values is achieved by finding the **Dirichlet to Neumann (D2N) map**. Actually, the term "Dirichlet to Neumann" map is often used to mean the map from the known to the unknown boundary values in general.

Consider the Dirichlet problem: (1.1.1) with boundary conditions

$$u(\boldsymbol{x}) = D(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega, \tag{1.2.4}$$

where $D(\boldsymbol{x})$ is known. **If** we can find a function $G(\boldsymbol{\xi}, \boldsymbol{x})$ such that

$$\left(\Delta_{\boldsymbol{\xi}} + \lambda\right) G(\boldsymbol{\xi}, \boldsymbol{x}) = \delta(\boldsymbol{\xi} - \boldsymbol{x}), \quad \boldsymbol{\xi} \in \Omega, \tag{1.2.5a}$$

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = 0, \quad \boldsymbol{\xi} \in \partial\Omega, \tag{1.2.5b}$$

then

$$u(\boldsymbol{x}) = -\int_{\partial\Omega} D(\boldsymbol{\xi}) \frac{\partial G}{\partial n}(\boldsymbol{\xi}, \boldsymbol{x}) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) G(\boldsymbol{\xi}, \boldsymbol{x}) dV(\boldsymbol{\xi}), \quad \boldsymbol{x} \in \Omega, \tag{1.2.6}$$

and the problem of finding u has been reduced to that of finding $G(\boldsymbol{\xi}, \boldsymbol{x})$ - **the Green's function**. If the eigenvalues and eigenfunctions of the Laplacian are known in Ω then the problem is solved since the Green's function can be constructed as the sum

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = \sum_n \frac{u_n(\boldsymbol{\xi}) u_n(\boldsymbol{x})}{\lambda_n - \lambda},$$

where u_n and λ_n denote the normalised eigenfunctions and eigenvalues of the Dirichlet Laplacian in Ω respectively, that is

$$-\Delta u_n(\boldsymbol{x}) = \lambda_n u_n(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$
$$u_n(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega,$$
$$\int_{\Omega} |u(\boldsymbol{x})|^2 d\boldsymbol{x} = 1.$$

1.2.2 Separation of variables, a.k.a. transform methods

1.2.2.1 Outline

Starting with a given boundary value problem in a separable domain, i.e. a domain of the form $\Omega = \{a_1 \leq \xi_1 \leq b_1\} \times \{a_2 \leq \xi_2 \leq b_2\}$ where ξ_j are the co-ordinates under which the differential operator is separable, the method of **separation of variables** consists of the following steps:

1. Separate the PDE into two ODEs.

2. Concentrate on one of these ODEs and derive the associated completeness relation (i.e. transform pair) depending on the boundary conditions. This is achieved by finding the 1-dimensional Green's function, $g(x, \xi; \nu)$, with eigenvalue ν and integrating g over a large circle in the complex ν plane:

$$\delta(x - \xi) = -\lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|\nu|=R} g(x, \xi; \nu)\, d\nu. \tag{1.2.7}$$

3. Apply this transform to the PDE and use integration by parts to derive the ODE associated with this transform.

4. Solve this ODE using an appropriate 1-dimensional Green's function, or variation of parameters.

The solution of the boundary value problem is given as a superposition of eigenfunctions of the ODE considered in step 2 (either an integral or a sum depending on whether the ODE has a continuous or discrete spectrum).

For each boundary value problem there exist two different representations of the solution depending on which ODE was considered in step 2. To show that these two representations are equivalent requires two steps:

1. Go into the complex plane, either by deforming contours (if the solution is given as an integral), or by converting the series solution into an integral using the identity

$$\sum_{n=-\infty}^{\infty} f(n) = \int_C \frac{f(k)}{1 - e^{-2\pi i k}} dk, \tag{1.2.8}$$

   where C is a contour which encloses the real k axis (in the positive sense) but no singularities of $f(k)$, see Figure 1.2. This latter procedure is known as **the Watson transformation** since it was first introduced in [Wat18].

2. Deform contours to enclose the singularities of $f(k)$ and evaluate the integral as residues/branch cut integrals.

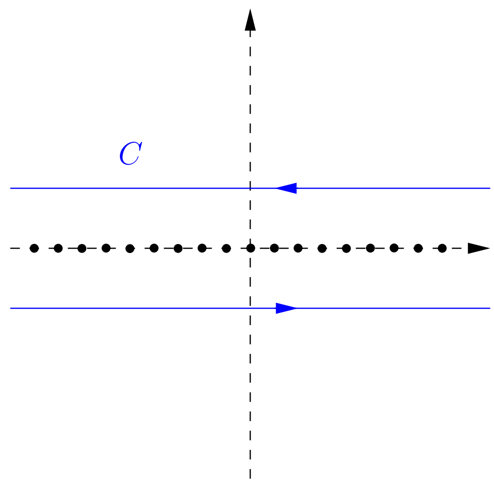


Figure 1.2: The contour C involved in the Watson transformation.

Roughly speaking, there are two different approaches for proving rigorously that a transform derived from the spectral analysis of an ODE (step 2 above) is complete:

(i) Provide a direct proof of the validity of the formula (1.2.7) for a given Green's function (for example using integration in the complex plane [Tit62]). There are some general theorems that guarantee the validity of (1.2.7) for a wide class of both self-adjoint and non-self-adjoint problems under certain subtle constraints on the boundary conditions, see for example [Nai67, §5].

(ii) If the operator is self-adjoint, then results about self-adjoint operators on Hilbert spaces are directly applicable, see for example [Nai68], [Sta67, §3.3], [RS72, §VII].

The main limitations of this method for solving boundary value problems are the following:

- It fails for BVPs with *non-separable* boundary conditions (for example, those that include a derivative at an angle to the boundary).
- The appropriate transform depends on the boundary conditions and so the process must be repeated for different boundary conditions.

- The solution is not necessarily uniformly convergent on the whole boundary of the domain (since it is given as a superposition of eigenfunctions of one of the ODEs).

- A priori it is not clear which of the two representations is better for practical purposes. In many situations, if an integral representation is possible this is preferable to an infinite series from the point of view of computing the solution*. In the cases where only an infinite series is possible, many techniques have been used to try and improve the convergence of this series, an introduction to some of these methods is given in [Duf01, §5.8 page 344.].

- BVPs involving the Helmholtz equation in unbounded domains, in order to be well posed, require a radiation condition at infinity. Therefore, at least one of the separated ODEs must include a radiation condition, which means that the BVP involving this ODE is non-self-adjoint, and thus the second approach mentioned above for proving completeness of the associated transform cannot be applied. In addition, the general results using the first approach, described in [Nai67, §5], are also not applicable since they do not apply to problems with a radiation condition. Thus, in this case the only known approach to investigate completeness is to analyse directly the given transform, this was performed for a large class of BVPs for the Helmholtz equation by Cohen [Coh64b, Coh64a, Coh65].

In the author's opinion the best references on separation of variables/transform methods are: [Sta67, Chapter 4] (spectral analysis of differential operators), [MF53] S5.1 (separable co-ordinates), [Fri56] Chapter 4 (spectral analysis), Chapter 5 (transforms and switching between the alternative representations), [Kee95] Chapter 7 (spectral analysis) Chapter 8 §8.1.3 (transform methods) [OHLM03] §4.4, 5.7, 5.8 (transform methods)

*Of course there are many exceptions to this, for example the case when the series can be computed using the Fast Fourier Transform.

1.2.2.2 Historical remarks

The method of separation of variables was introduced in the 1750's by d'Alembert and Euler in their attempts to solve the wave equation. It was further developed by Fourier, Daniel Bernoulli, Lamé (who introduced curvilinear co-ordinates), and Sturm and Liouville (who investigated general eigenfunction expansions). (This historical material is recounted in [GG03, Part 3.15 §5 page 459] and [Jah03, §7.1.2 page 199]). The systematic investigation of the co-ordinate systems in which the classic PDEs of mathematical physics are separable began around 1890 (Chapter 5 of [MF53] gives a detailed account). The investigation of expanding a function in terms of eigenvalues of a differential operator began with Sturm and Liouville in the 19th Century and continued with work by, among others, Birkhoff [Bir08b, Bir08a] (with some authors calling these expansions "Birkhoff expansions"), Weyl [Wey10], Titchmarsh [Tit62], and Naimark [Nai67, Nai68].

One boundary value problem that is of particular significance in the development of transform methods is the Helmholtz equation in the exterior of the circle (in 2-d) or sphere (in 3-d). Around 1900 there was interest in this problem because it models the propagation of radio waves around the Earth. This BVP is separable in polar co-ordinates, and an expression for the solution was obtained using the appropriate transform in the *angular* variable – a Fourier series. However this series converged extremely slowly for large frequency (Love estimated that 8000 terms were required for the desired accuracy [Kel79]).

In 1918 Watson overcame this difficulty by using (1.2.8) to transform the angular series into an integral, and then evaluate the integral as a second series of which one or two terms gave the desired accuracy [Wat18]. Around 1950 Sommerfeld showed that this second series could be obtained *directly* by considering the *radial* ODE [Som64b, Appendix 2 of Chapter 5 p.214, Appendix to Chapter 6 p.279]. This led to the realisation that "the technique of separation of variables has not yet been fully exploited" [Coh64b]. A systematic derivation of the 2 different representations for boundary value problems in 2-dimensions (3 in 3-d) was undertaken [Mar51], [Fel57], [FM96, Chapters 3 and 6], [Coh64b], [Coh64a].

The situation became even more interesting when Cohen proved that the radial transform for solving the Helmholtz equation in the exterior of the circle is *not* complete [Coh64a]. Thus, although the best representation of the solution *can* be obtained by the radial transform, this procedure is not justified, and this representation must be obtained by first using the angular transform, replacing the series by an integral using the Watson transformation, deforming contours, and evaluating the integral on the "radial" poles. ([BSU87] §I.2.13.6 provides a more detailed account of the history of the Watson transformation.)

1.2.3 The method of images/reflections

1.2.3.1 Outline

This technique can be used to find *either* the Green's function *or* the eigenfunctions and eigenvalues. We first discuss the former.

Let Ω be the upper half plane

$$\Omega = \{-\infty < x < \infty,\, 0 < y < \infty\}.$$

Since $E(\boldsymbol{\xi}, \boldsymbol{x})$ is a function of $|\boldsymbol{\xi} - \boldsymbol{x}|$, subtracting the fundamental solution at the point $\boldsymbol{x}'$ from the fundamental solution at $\boldsymbol{x}$, where $\boldsymbol{x}' = (x, -y)$ is the *reflection* of the point $\boldsymbol{x} = (x, y)$ in the boundary $y = 0$ (see fig 1.2.3.1), means that this combination satisfies (1.2.5). Indeed, the differential equation (1.2.5a) is still satisfied as the reflected point is outside Ω, and for $\boldsymbol{\xi}$ on the boundary $E(\boldsymbol{\xi}, \boldsymbol{x})$ and $E(\boldsymbol{\xi}, \boldsymbol{x}')$ are equal. Hence

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = E(\boldsymbol{\xi}, \boldsymbol{x}) - E(\boldsymbol{\xi}, \boldsymbol{x}').$$

For the Neumann problem, adding the fundamental solution at the point $\boldsymbol{x}'$ satisfies the Neumann boundary condition, hence

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = E(\boldsymbol{\xi}, \boldsymbol{x}) + E(\boldsymbol{\xi}, \boldsymbol{x}').$$

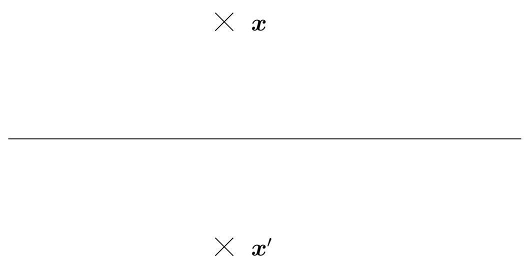


Figure 1.3: The upper half plane with source and image points.

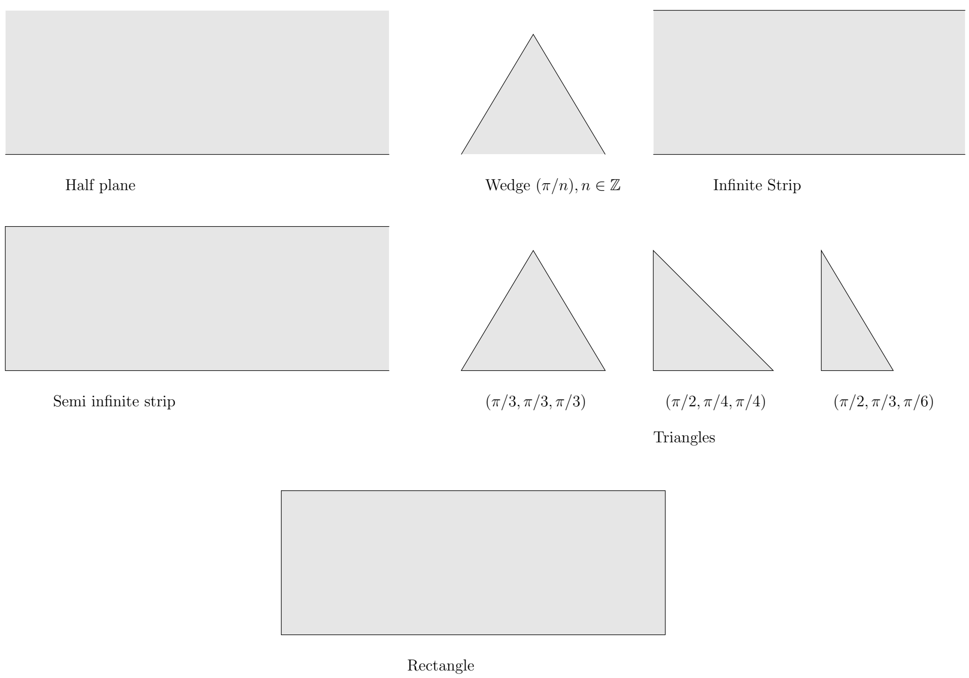


Figure 1.4: The admissible domains for the method of images in two dimensions.

The domains for which the **Green's function** can be found using this method are those which are fundamental domains of discrete groups of reflections, [Cox73], with the additional requirement that no image lies inside the domain except the source point $\boldsymbol{x}$ itself (as this would violate (1.2.5a)),[Kel53]. Thus the admissible domains for the method of images in two dimensions are given by Figure 1.4. (Alternatively the characterisation of these domains can be understood in terms of root systems of Lie algebras, see e.g. [AH10].)

This applies to Dirichlet and Neumann boundary conditions, as well as some mixed boundary conditions where Dirichlet conditions are posed on part of the boundary and

Neumann conditions on the rest (the mixed boundary conditions which are allowed for each domain are detailed in [Kel53]). The Green's function is then given as a sum

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = E(\boldsymbol{\xi}, \boldsymbol{x}) + \sum_j a_j E(\boldsymbol{\xi}, \boldsymbol{x}_j) \tag{1.2.9}$$

where $\boldsymbol{x}_j$ are the image points obtained by reflecting the source point $\boldsymbol{x}$ in the boundaries of the domain, and a_j are chosen such that $G(\boldsymbol{\xi}, \boldsymbol{x})$ satisfies the required boundary condition. For all the domains except for the half plane and wedge, an infinite number of images is required, and so the Green's function is given as an *infinite sum*. In some cases this sum does not converge, for example for the Helmholtz equation in all 4 bounded domains.† In the cases where it does, many techniques have been used to try and improve the rate of convergence, a good introduction is given in [Duf01, §5.8 page 344].

The extension of the method to Robin and oblique Robin boundary conditions in the upper half plane is given in [GT01] and [Kel81]. The Green's function is given as the source, plus one image, plus an semi-infinite line of images. Robin and oblique Robin boundary conditions in a wedge of angle π/n, $n \in \mathbb{Z}^+$ are considered in [Gau88]. For the Robin problem the Green's function is given as a source point, plus infinite lines of images, plus infinite regions of images. The oblique Robin problem can only be solved if n is odd and under some restrictions on the angle of derivative in the boundary conditions (this is to ensure no images lie inside the domain).

For the 4 bounded domains in Figure 1.4, the method of images can be used to find their **eigenfunctions and eigenvalues** under Dirichlet, or Neumann, or some mixed Dirichlet-Neumann boundary conditions (the same ones for which the Green's function can be found) by reflecting to one of

- the whole space [TS80a], [TS80b], [MW70]
- a parallelogram [Pin80],

†A neat way to see this is that the Helmholtz equation has 2 linearly independent fundamental solutions (incoming and outgoing), yet the solution is unique in a bounded domain (for λ not an eigenvalue), so if the sum converged there would be a contradiction.

- a rectangle [Prá98],

where one can use separation of variables in cartesian co-ordinates, then reflecting back. This reflection technique does *not* work for Robin boundary conditions. However for the equilateral triangle these have been found by using an ansatz based on the form of the Dirichlet and Neumann eigenfunctions, and proving completeness using the fact that the eigenfunctions are analytic functions of the Robin parameter α and using results about perturbations of spectra in Hilbert spaces to obtain completeness of the Robin eigenfunctions from completeness of the Neumann eigenfunctions [McC04].

The Poisson equation in polar co-ordinates is unique in admitting another type of image (other than reflection) known as Kelvin inversion, see e.g. [DK89]. Thus, for this equation alone, the method of images can be used to find the Green's function for certain BVPs in polar co-ordinates.

1.2.3.2 Historical remarks

In 1833 Lamé reflected the equilateral triangle to cover the whole plane, and hence determined its eigenfunctions and eigenvalues [Lam33]. The method of images was first introduced for the upper half plane problem by Sir William Thompson (Lord Kelvin) in 1847 [Tho47]. In 1953 J.B. Keller wrote the definitive work on the method of images, characterising exactly which domains and boundary conditions (Dirichlet, Neumann and mixed) the method is applicable to in both two and three dimensions [Kel53] (incredibly, despite its importance, this paper has only been cited 11 times!). In 1970 obtaining the eigenfunctions and eigenvalues of the equilateral triangle by reflecting to the whole plane appeared in the textbook [MW70]. In 1979, Terras and Swanson applied the method of images to find both the Green's function (reproducing Keller's result) and eigenfunctions and eigenvalues [TS80a], [TS80b]. They appear to give the first rigorous proof of completeness of the eigenfunctions by reflecting to the whole plane, although they state that completeness for the equilateral triangle was proved by C.G. Nooney "On the vibrations of triangular membranes", dissertation, Stanford University, 1953.

In 1980 Pinsky reflected the equilateral triangle to a parallelogram to obtain the eigenfunctions and proved several number-theoretic results on the eigenvalues [Pin80]. In 1985 he proved completeness via the parallelogram [Pin85]. In 1998 Pragér solved boundary value problems in the equilateral triangle by first splitting them into odd and even parts about the midline, then reflecting the corresponding boundary value problems in the half-equilateral triangle, see Figure 1.4, into boundary value problems in the rectangle [Prá98]. He also found the eigenfunctions and eigenvalues of the equilateral triangle and proved completeness in this way.

In 2003 McCartin obtained the eigenfunctions and eigenvalues by introducing a "triangular co-ordinate system" in which separable solutions to the eigenproblem can be obtained [McC03]. To prove completeness he had to use the reflection method, citing [Prá98]. He then repeated this for the Neumann problem [McC02]. For the Robin problem McCartin introduced the technique discussed above for proving completeness [McC04]. He investigated the absorbing and impedance boundary conditions in [McC07] [McC08a]. In [McC08b] he obtained the analogue of Keller's result about Green's functions for eigenfunctions (although apparently unaware of Keller's paper), that the only domains with a complete basis of eigenfunctions which are linear combinations of exponentials with complex arguments are the four bounded domains in Figure 1.4.

1.2.3.3 Sommerfeld's extension: non-periodic fundamental solutions

In 1896 Sommerfeld extended the image method to solve the problem of diffraction by a half-line (in 3-d, a half-plane), that is the Helmholtz equation in the domain

$$\Omega = \{\mathbb{R}^2 \backslash \{x > 0\}\} = \{0 < r < \infty,\, 0 < \theta < 2\pi\}.$$

To understand his idea, first consider the solution in the upper half-plane by the method of images. Viewed in polar co-ordinates, the source is at angle $\phi = \theta$, the image is at

$\phi = 2\pi - \theta$ (the reflection of the source in $\phi = \pi$), and the fundamental solution is 2π periodic. Sommerfeld's idea for the solution in Ω was to make the space 4π *periodic*, since the half line is to 4π periodic space what the half plane is to 2π periodic space: now the source is at angle $\phi = \theta$, and the image is at $\phi = 4\pi - \theta$ (the reflection of the source in $\phi = 2\pi$).

Similarly, considering space as non-periodic in θ, i.e. $-\infty < \phi, \theta < \infty$, converts a wedge of arbitrary angle (less than 2π) into an infinite strip, which can be solved using an infinite number of images. (Of course both the half-line and wedge problems can be solved by separation of variables in polar co-ordinates.)

Sommerfeld expressed his 4π periodic and non-periodic fundamental solution (called "Riemann surface" or "branched" solutions) as integrals of exponentials over the so-called Sommerfeld contours, [Som64a, p.249]. Stakgold [Sta68, page 270] expressed them in expansions using angular eigenfunctions: the 2π periodic fundamental solution is given by

$$E(\rho, \phi; r, \theta) = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(\beta r_>) J_n(\beta r_<) e^{in(\theta-\phi)}$$

where $r_> = \max(r, \rho)$, $r_< = \min(r, \rho)$, whereas the non-periodic fundamental solution, denoted E_s by Stakgold, is given as

$$E_s(\rho, \phi; r, \theta) = \frac{i}{4} \int_{-\infty}^{\infty} dk\, H_{|k|}^{(1)}(\beta r_>) J_{|k|}(\beta r_<) e^{ik(\theta-\phi)}. \tag{1.2.10}$$

We will return to (1.2.10) in Section 2.2.1, because (to give the punchline away) one of the arguments of this thesis is: *the best representations of solutions to boundary value problems in polar co-ordinates are obtained by considering the angular variable, θ, to be non-periodic.*

1.2.4 Integral equations

Taking the limit of Green's IR (1.2.1) as $\boldsymbol{x}$ tends to the boundary of the domain gives a linear integral equation for the unknown boundary values. Solving this integral equation is known as the **boundary integral method**. In separable domains this equation can be solved explicitly using the appropriate transform, but since the original problem can be solved in this way this is not useful. In general the integral equation must be solved numerically, see e.g. [CK83], and is a major technique in numerical analysis.

1.2.5 Conformal mapping

The Dirichlet and Neumann BVPs for the Poisson equation in two dimensions can be solved using conformal mapping. This is a consequence of the following two facts:

- the real and imaginary parts of an analytic (complex–differentiable) function satisify the Laplace equation, and
- the Laplace equation is “conformally invariant”, that is, if $\zeta = f(z)$, where $\zeta = \xi + i\eta$ and $z = x + iy$, then

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = |f'(z)|^2 \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right).$$

Thus, to find an expression for the solution of either the Dirichlet problem or the Neumann problem for the Poisson equation in a domain D, one only needs to find a conformal map (i.e. an analytic function $f(z)$ with $f'(z) \neq 0$ for $z \in D$) that maps D to the unit circle (or any other domain where the Green’s function is known explicitly). The existence of such a map is guaranteed for a large class of domains by the Riemann Mapping Theorem, which states that if D is simply connected then there exists an analytic function $f(z)$ such that $\zeta = f(z)$ maps D onto the disc $|\zeta| < 1$ and f has a single–valued inverse on the disc. However, this theorem is not constructive, and thus it does not tell us how to find the crucial map. For polygonal domains such a map is given as an integral by the Schwarz-Christoffel transformation, see e.g. [DT02]. (Using conformal mapping to solve the Poisson equation is covered in many books, in particular [AF03, Chapter 5], [Hen93, Chapter 15], and [OHLM03, §5.9.1].)

When the mapping function can be found explicitly, it is extremely hard to compete with conformal mapping as a method for solving the Poisson equation in two dimensions. Unfortunately, however, the method does not generalise to other second order elliptic PDEs, or to the Poisson equation in three dimensions.

1.3 The Fokas method

The Fokas method has two basic ingredients:

1. The integral representation.
2. The global relation.

The integral representation (IR). This is the analogue of Green's IR in the transform space. Indeed, the solution u is given as an integral involving *transforms* of the boundary values, whereas Green's IR expresses u as an integral involving directly the boundary values. The IR can be obtained by first constructing particular integral representations depending on the domain of the fundamental solution E ("domain-dependent fundamental solutions") and then substituting these representations into Green's integral formula of the solution and interchanging the orders of integration.

The global relation (GR). The global relation is Green's divergence form of the equation integrated over the domain, where one employs the solution of the adjoint equation instead of the fundamental solution. Indeed

$$0 = \int_{\partial\Omega} \left(v(\boldsymbol{\xi}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial v}{\partial n}(\boldsymbol{\xi}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) v(\boldsymbol{\xi}) dV(\boldsymbol{\xi}), \tag{1.3.1}$$

where v is any solution of the adjoint of (1.1.1):

$$\Delta v(\boldsymbol{x}) + \lambda v(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega; \tag{1.3.2}$$

(equation (1.3.2) is (1.2.2) without the delta function on the right hand side). Separation of variables gives a one-parameter family of solutions of the adjoint equation depending on the parameter $k \in \mathbb{C}$ (the separation constant). For example, for the the Poisson equation equation, separation of variables in Cartesian co-ordinates yields four solutions of the adjoint equation, $v = e^{\pm ikx \pm ky}$, $k \in \mathbb{C}$. All equations (1.3.1) obtained from these different choices of v shall be refered to as "the global relation". This relation was called the global relation by Fokas since it contains global, as opposed to local, information about the boundary values.

Just like Green's IR, the new IR contains contributions from both known and unknown boundary values. However, it turns out that the GR involves *precisely* the transforms of the boundary values appearing in the IR. The main idea of the Fokas method is that, for certain boundary value problems, one can **use the information given in the GR about the transforms of the unknown boundary values to eliminate these unknowns from the IR.** This can be achieved in three steps:

1. **Use the GR to express the transforms of the unknown boundary values appearing in the IR in terms of the smallest possible subset of the other functions appearing in the GR** (if there exist different possibilities, use the one which yields the smallest number of unknowns in each equation).

2. **Identify the domains in the complex k plane where the integrands are bounded and also identify the location of any singularities.** At this stage some unknowns can be eliminated directly using analyticity and employing Cauchy's theorem.

3. **Deform contours and use the GR again so that the contribution from the unknown boundary values vanish by analyticity** (employing Cauchy's theorem).

For certain domains it is possible to solve the given BVP by using only a subset of the above three steps. For example, the half plane only requires step 1 §4.1, the quarter plane steps 1 and 2 §4.2, and both the exterior of the circle §4.3 and right isosceles triangle §5 require all 3 steps.

This method yields the solution as an integral in the complex k plane involving transforms of the known boundary values. This novel solution formula has two significant features:

1. The integrals can be deformed to involve exponentially decaying integrands.
2. The expression is uniformly convergent at the boundary of the domain.

These features give rise to both analytical and numerical advantages in comparison with classical methods. In particular, for linear evolution PDEs, the effective numerical evaluation of the solution is given in [FF08].

Remark 1.3.1 (Philosophical Remark 1: "Green + Fourier") *As Figure 1.1 shows, the method of separation of variables/transform is completely independent from Green's integral representations. Thus, the Fokas method is the first time these two classical techniques have been genuinely used together.*

Remark 1.3.2 (Philosophical Remark 2: "Go into the complex plane.") *Representations of solutions to* ODEs *as integrals in the complex plane were pioneered by Laplace in 1782, following earlier investigation by Euler in 1744, recounted in [GFGG97, Chapter 29]. Laplace considered*

$$u(z) = \int_C dk\, K(z,k)\, f(k) \tag{1.3.3}$$

with the kernel $K(z,k)$ first equal to e^{zk} ("the Laplace kernel") and then k^z ("the Mellin kernel").

The Fokas method can be considered as the extension of this to PDEs. *Indeed, in hindsight the "moral" of the Watson transformation is that the best representation of the solution to a separable PDE is an integral in the complex k plane, which can be deformed to either of the 2 representations obtained by transforms (which is precisely the representation obtained by the Fokas method). It is therefore perhaps surprising that no-one tried to find this representation directly until the advent of the Fokas method.*

Just as Green's integral representation gives rise to a numerical method for solving these PDEs (the boundary integral method), the Fokas method can also be used to design new numerical schemes, which are presented in Chapter 6. The idea is that, for a bounded domain, the global relation is valid *for all* $k \in \mathbb{C}$. Let the domain have n sides and suppose we expand the n unknown functions (the unknown boundary value on each side) in some series such as Fourier or Chebyshev up to N terms. If the global relation is evaluated at nN points, this yields nN equations for the nN unknowns, which in principle can be solved. (Questions immediately arise about how to choose the basis and how to choose the points k.)

1.4 The Fokas method versus classical techniques

Three questions now arise regarding comparing the Fokas method to classical techniques for finding explicit expressions for the solution of BVPs:

(a) Is the Fokas method easier or harder to implement than the classical methods?

(b) How does the expression of the solution to a BVP obtained by the Fokas method compare to the expressions obtained using classical methods?

(c) Can the Fokas method solve (i.e. find an explicit expression for the solution) any BVP that cannot be solved using classical methods?

In this section we will discuss these questions in the context of three classical methods: the standard transform method, the method of images, and conformal mapping. We begin with some general remarks and then discuss each method in detail.

Question (a) is the least important, and the most subjective, question, and it is hard to give an answer to it except when comparing the Fokas method to the classical transform method.

It is difficult to provide a definitive answer to question (b), since much depends on the specific BVP in question. Indeed, a key factor in answering question (b) is whether the prescribed boundary conditions are such that their transforms appearing in the Fokas method solution can be expressed in terms of simple, easily computable, functions. If this is the case, then the solution from the Fokas method is given as a single integral in the complex k plane which can be deformed to involve exponentially decaying integrands. Otherwise, two integrations need to be performed: one to compute the transforms, and a second to compute the k-integral.

Regarding (c): given a domain for which the solutions of certain boundary value problems can be found, the Fokas method can solve more complicated boundary conditions than those that can be solved classically. However, so far the Fokas method has not been able to extend the set of domains for which explicit solutions of boundary value problem for (1.1.1) can be found.

1.4.1 Fokas vs. transforms

Since the Fokas method is closely related to the classical transform method, it is relatively straightforward to compare the two, this is done in Table 1.1.

For some very simple BVPs it is immediately clear which are the appropriate transforms, as well as which of these (if any) provides the best solution representation; thus for certain very simple BVPs it may be easier to apply the standard transform procedure instead of the new method. However, as Table 1.1 shows, in general the Fokas method requires less mathematical input, is simpler to implement, yields more useful solution formulae, and is more widely applicable than the classical transform method.

	The Fokas method	Transforms
Mathematical input	• Completeness relation for each separated ODE in whole space (independent of the domain and boundary conditions) • Green's theorem	• Completeness relation for one separated ODE dependent on domain and boundary conditions (bcs)
Implementation given a BVP	• Same steps independent of bcs • Algebraic manipulation • Unknowns vanish by Cauchy	• Different transforms for different bcs • Integration by parts • Solve ODE using 1-d Green's function
Solution	• Uniformly convergent at $\partial\Omega$ • Given as an integral, deform contour so integrand decays exponentially	• Not uniformly convergent at $\partial\Omega$ • Either an infinite sum or an integral depending on domain. If an integral, can deform contour so integrand decays exponentially
Boundary conditions	• Separable and some non-separable	• Only separable
Domains	• Separable and some non-separable	• Only separable

Table 1.1: Comparison of the Fokas method and classical transforms in 2-D

1.4.2 Fokas vs. conformal mapping

As discussed in Section 1.2.5, it is extremely hard to compete with conformal mapping as a method for solving the Dirichlet and the Neumann boundary value problems for the Poisson equation when the appropriate mapping function is known explicitly. The class of domains for which this is the case is much wider than the class of domains admissible to the classical transform method, the method of images, and the Fokas method. However, for a BVP that can be solved both by conformal mapping and by the Fokas method (such as the Dirichlet problem for the Poisson equation in a right isosceles triangle in Chapter 5), we expect that the Fokas method may be competitive when the transforms of the boundary conditions can be expressed in terms of easily computable functions. It is beyond the scope of this thesis to test this, however, a concrete comparison should be performed.

Of course, for boundary conditions other than Dirichlet and Neumann conformal mapping fails in general, and it also cannot be applied to the Helmholtz and modified Helmholtz equations.

1.4.3 Fokas vs. images

Regarding the question (a): the image method is simpler than the Fokas method in that it takes place in the physical, as opposed to spectral, space.

Regarding the question (c): the Fokas method is applicable to a wider class of BVPs than the method of images. Indeed, images fails for boundary conditions other than Dirichlet and Neumann for all but the simplest of domains, however the Fokas method can solve many BVPs with Robin and oblique Robin boundary conditions.

Regarding the question (b): there are two different type of solutions obtained by the method of images:

(i) the Green's function given as a sum of eigenfunctions (this is possible only when the domain is bounded), and

(ii) the Green's function given as a sum of fundamental solutions evaluated at the image points.

The solution given by (i) is, by construction, not uniformly convergent at $\partial\Omega$ whereas the Fokas method solution *is* uniformly convergent on $\partial\Omega$. Regarding the evaluation of (i) versus the evaluation of the Fokas solution: at first sight it appears that the expression for u given by the Fokas method as an integral involving transforms of the boundary conditions is superior to the expression for u given by (i) as an integral of a bi–infinite sum. However this sum can be computed using the FFT, so a case–by–case comparison should be carried out.

When an infinite number of images are required in the solution given by (ii) it appears the solution given by the Fokas method is superior. When only a finite number of images are required (the wedge with angle π/n, $n \geq 1$) the situation is less clear. If the boundary conditions are such that their transforms can be computed explicitly, then the solution from the Fokas method is given as a single integral over k which can be deformed to involve exponentially decaying integrands and this is in general superior to the image solution. If not, both images and the Fokas method yield the kernels against which the boundary conditions must be integrated in Green's IR (either the Green's function or its derivative evaluated on the boundary): the Fokas method gives them as an integral over k, and images gives them in terms of the fundamental solution. The question of which expression is superior boils down to how easily the relevant fundamental solution can be computed versus how easily the k-integrals in the Fokas solution can be computed. For the Poisson equation, where the fundamental solution is given as a logarithm, we expect the image solution will be easier to compute than the Fokas solution. For the Helmholtz and modified Helmholtz equations, where the fundamental solutions must be computed via an integral representation, we expect both methods to require comparable resources to compute, although, again, concrete comparisons on specific examples should be performed.

1.4.4 Dirichlet to Neumann map

If one only wants to find the unknown boundary values, the Fokas method has a clear advantage over classical methods. Indeed, with both transforms and images, to find the Dirichlet to Neumann map one must first find the solution, then evaluate it (or its derivative) on the boundary. In the Fokas method manipulating the GR (in a very similar way to the procedure for finding the solution u) leads immediately to the unknown boundary values. The expression for the Dirichlet to Neumann map then also has all the advantages listed above over the corresponding classical solution. This is illustrated in remark 4.1.4 for the half plane and §5.2.2 for the right isosceles triangle.

1.4.5 The classical techniques reformulated

The two classical techniques of images and transform methods can be reformulated, and simplified, in the framework of the Fokas method:

1.4.5.1 Images – invariances of the GR in physical space

When $\boldsymbol{x} \notin \Omega$ the delta function in the definition of E (1.2.2) is zero and hence the left hand side of (1.2.1) is zero. We supplement (1.2.1) with this additional equation to obtain

$$\int_{\partial\Omega} \left(E(\boldsymbol{\xi}, \boldsymbol{x}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial E}{\partial n}(\boldsymbol{\xi}, \boldsymbol{x}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) E(\boldsymbol{\xi}, \boldsymbol{x}) dV(\boldsymbol{\xi}) = u(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.4.1}$$

$$= 0, \quad \boldsymbol{x} \notin \Omega. \tag{1.4.2}$$

Call the first equation the *integral representation in the physical space*, and call the second equation the *global relation in the physical space*.

The method of images is equivalent to eliminating the unknown boundary values in (1.4.1) by using the *invariance* properties of (1.4.2).

For example, let Ω be the upper half plane $y > 0$ and for simplicity consider Laplace's equation ((1.1.1) with $\lambda = 0$ and $f = 0$). Then (1.4.1) and (1.4.2) become

$$\int_{-\infty}^{\infty} \left(u(\xi, 0) \frac{y}{2\pi\left[(\xi - x)^2 + y^2\right]} - \frac{1}{4\pi} \log\left[(\xi - x)^2 + y^2\right] \frac{\partial u}{\partial n}(\xi, 0) \right) d\xi = u(x, y), \quad y > 0, \tag{1.4.3}$$

$$= 0, \qquad y < 0. \tag{1.4.4}$$

If we let $y \mapsto -y$ in the second equation, the first term on the left hand side changes sign, whereas the second term remains unchanged, so (1.4.4) becomes

$$\int_{-\infty}^{\infty} \left(u(\xi, 0) \frac{-y}{2\pi\left[(\xi - x)^2 + y^2\right]} - \frac{1}{4\pi} \log\left[(\xi - x)^2 + y^2\right] \frac{\partial u}{\partial n}(\xi, 0) \right) d\xi = 0, \quad y > 0. \tag{1.4.5}$$

Both (1.4.5) and (1.4.3) are valid for $y > 0$ and subtracting (1.4.5) from (1.4.3) we eliminate the unknown Neumann boundary value $\frac{\partial u}{\partial n}(\xi, 0)$, to obtain

$$u(x, y) = \int_{-\infty}^{\infty} u(\xi, 0) \frac{y}{\pi\left[(\xi - x)^2 + y^2\right]} d\xi, \quad y > 0 \tag{1.4.6}$$

which is equivalent to (1.2.6) with $f = 0$ and with $G(\boldsymbol{\xi}, \boldsymbol{x})$ given by

$$G(\boldsymbol{\xi}, \boldsymbol{x}) = E(\xi, \eta, x, y) - E(\xi, \eta, x, -y) \tag{1.4.7}$$

which is the familiar result obtained using the method of images. In a similar way, all the boundary value problems solvable using the method of images can be solved by analysing the invariance properties of (1.4.2).

Thus the "recipe",

"Construct an integral representation for u in Ω in terms of the boundary values of u on $\partial\Omega$. Supplement this with the ***global relation****, which is an equation coupling the boundary values. By manipulating these equations, and in particular by using the invariance*

Divergence form

PHYSICAL SPACE	SPECTRAL SPACE
IR: u as integral of boundary values	IR: u as integral of transforms of b.v.s
Global relation	Global relation
Method of images: eliminate unknown boundary values	New transform method: eliminate transforms of unknown b.v.s
Boundary integral method: solve GR numerically	New numerical method: solve GR numerically

Figure 1.5: Solution methods in the physical and spectral space.

properties of the global relation, eliminate the unknown boundary values from the integral representation."

can be applied in *both* the physical and spectral spaces, where it is the method of images, and the Fokas method respectively. We also note that the linear integral equation for the unknown boundary values, on which the boundary integral method is based, was obtained by taking the limit of (1.4.1) as $\boldsymbol{x}$ tends to the boundary from inside Ω. We emphasise that this integral equation could also be obtained from (1.4.2) by taking the limit as $\boldsymbol{x}$ tends to the boundary from outside Ω. Hence, **the boundary integral method is based on the solution of the global relation in the physical space**.

This motivates Figure 1.5.

1.4.5.2 Transforms – applying the global relation in subdomain

The classical transform method can be reformulated as the following procedure:

- Split the separable domain Ω along a line of constant co-ordinate and apply the global relation in the two subdomains.
- Eliminate the transform of unknown functions via algebraic manipulation of the

GR.

- The result is that a transform of the solution is given in terms of transform of the known boundary values, which can then be inverted.

This formulation has two advantages over the usual implementation of the classical transform method:

1. One does not need to specify the appropriate transform in advance, it comes out of the GR.

2. Integration by parts and solving an ODE are replaced by algebraic manipulations.

The only drawback is that one still needs to invert a transform at the end in order to find the solution. This inversion can be obtained by the classical method, or by a method which is a spin-off of the Fokas method - a new way to inverting integrals [Fok08] Chapter 6.

These reformulations/simplifications of the classical transform and image methods are applied to boundary value problems for (1.1.1) in spherical co-ordinates in 3-dimensions in [DF08].

1.5 From Green to Lax via Fourier: a unification

This thesis is concerned with the Fokas method applied to (1.1.1). However, this method solves boundary value problems for many other linear PDEs, and also the so-called "integrable" PDEs. This section discusses this bigger picture.

1.5.1 Integrable systems in classical mechanics

There does not yet exist a universally agreed definition of what it means to be "integrable". In many contexts, and broadly speaking, the term "integrable" means "linearisable", that is, can be solved using linear methods. In the context of Hamiltonian systems of ODEs the notion of integrability was introduced by Liouville around 1850. This roughly states that if there are "enough" conserved quantities, there is a change of variables (so-called "action-angle co-ordinates" under which the flow in phase space is linear. Very few of these classic integrable systems were known, and so there was little interest in the subject until it was discovered that the notion of integrability could be extended to certain, widely applicable, PDEs.

1.5.2 The Korteweg-de Vries (KdV) equation

The KdV equation,

$$q_t + q_{xxx} + 6qq_x = 0, \tag{1.5.1}$$

was first written down by Boussinesq in several papers in the 1870s, but it is named after Korteweg and deVries who derived the same equation in 1895 in the same context as Boussinesq: the asymptotic limit of the 2-d Euler equations in the case of long waves in shallow water . It is the "simplest" PDE with both dispersion (spreading) and nonlinearity (steepening), [DJ89, §1.1] Korteweg and de Vries discovered the travelling wave solution

$$q(x,t) = \frac{p^2}{2\cosh^2[\frac{1}{2}p(x-p^2t)+c]},$$

which was proposed as an explanation of the British experimentalist J. Scott Russell's observations of a "solitary wave" in a canal in 1834.

In 1967 Kruskal and Zabusky were investigating the so-called Fermi–Pasta–Ulam problem (FPU) of coupled, vibrating, non-linear springs. They obtained the KdV equation from a simplified model, and solved it numerically. They discovered that the KdV equation had solutions consisting of several travelling waves ("solitons"), which interacted with each

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
q(x,0) \arrow[r, "\text{transform}", "x \text{ part}"'] \arrow[d, dashed, "\text{PDE}"'] & \hat{q}(k,0) \arrow[d, "\begin{array}{c}\text{time evolution}\\ t \text{ part}\end{array}"] \\
q(x,T) & \hat{q}(k,T) \arrow[l, "\text{inverse transform}"', "x \text{ part}"]
\end{tikzcd}
$$

Figure 1.6: The steps of the Inverse Scattering Transform

other almost as if the equation were linear. This sparked renewed interest in the KdV equation, and the discovery of an ingenious method of solution of the initial value problem (IVP), that is the problem of determining the solution, q, of the PDE (1.5.1) posed on $x \in (-\infty, \infty)$, $t \in [0, T]$, and subject to $q(x, 0)$ equal to some given function, [GGKM67].

This method was called the "Inverse Scattering Transform" (IST) and relied on the fact that the KdV equation could be written as the compatibility condition of two eigenvalue equations involving an auxiliary function ϕ and a parameter k, called a "Lax pair" after Peter Lax, [Lax68],

$$
\begin{aligned}
&\phi_{xx} + (q + k^2)\phi = 0, \\
&\phi_t + (2q - 4k^2)\phi_x - (q_x + \nu)\phi = 0, \quad k \in \mathbb{C}.
\end{aligned}
$$

The analysis of the first part of the Lax pair (the x-part) produces a transform pair, and the second part of the Lax pair (the t-part) then immediately gives the time-evolution of that transform, see Figure 1.6. (The method was called the "Inverse Scattering Transform" because the x-part of the Lax pair of the KdV is the 1-d time-independent Schrödinger equation. This method was discovered precisely because there was great interest in this equation which is the simplest model of scattering in 1-d.)

It was realised that this resembled the solution of the IVP for linear evolution PDEs in one space and one time variables (eg. the heat/diffusion equation $q_t = q_{xx}$) by the Fourier transform, see Figure 1.7, and also that in the linear limit the solution of the KdV equation by the IST reduced to the Fourier transform solution of the linearised equation;

$$\begin{tikzcd}
q(x,0) \arrow[r, "\text{F.T.}"] \arrow[d, dashed, "\text{PDE}"'] & \hat{q}(k,0) \arrow[d, "\text{time evolution}"] \\
q(x,T) & \hat{q}(k,T) \arrow[l, "\text{inverse F.T.}"']
\end{tikzcd}$$

Figure 1.7: The solution of linear evolution PDEs in 1 space and 1 time variables by the Fourier transform

so the IST was declared as the non-linear analogue of the Fourier transform.

In 1971 the KdV was written as an integrable (in the sense of Liouville) Hamiltonian system by Zakharov and Faddeev, justifying calling it “integrable”. At first the IST was considered a “fluke”, much like the Cole-Hopf transformation which linearises Burger’s equation by mapping it to the heat equation. However, in 1972 Zakharov and Shabat showed that the non-linear Schrödinger equation

$$iq_t + q_{xx} \pm 2|q|^2 q = 0$$

possesses a Lax pair and could also be solved by the IST, re-invigorating the study of integrable equations.

1.5.3 Lax pairs

Let $q(x,t)$ satisfy a PDE. This PDE has a Lax pair formulation if the PDE can be written in the form

$$A_t - B_x + [A,B] = 0, \tag{1.5.2}$$

where $[A,B] := AB - BA$ and both A and B are matrix functions of x, t, $q(x,t)$ and $k \in \mathbb{C}$. If (1.5.2) holds, then the PDE is the *compatibility condition* of the following Lax pair of equations:

$$\phi_x = A\phi, \tag{1.5.3a}$$

$$\phi_t = B\phi, \tag{1.5.3b}$$

involving a vector function $\phi(x,t)$. The condition $\phi_{xt} = \phi_{tx}$, which is equivalent to (1.5.2), ensures that equations (1.5.3) describe the same ϕ. Indeed, to recover ϕ from ϕ_x and ϕ_t one integrates

$$\phi(x,t) = \int^{(x,t)} \phi_{x'} dx' + \phi_{t'} dt'.$$

This integral is a well-defined function of x and t if it is independent of the path of integration, and by Green's theorem this is the case if and only if $\phi_{xt} = \phi_{tx}$.

Possessing a Lax pair is a very special property, and it is often said that the integrable PDEs form a "set of measure zero" in the "space" of PDEs. However, perhaps surprisingly, they appear in many physical applications. (A possible explanation of this is given in [Cal91]).

Crucially, the matrices A and B in (1.5.3) are functions of a complex parameter k, and this allows one to analyse the Lax pair (using Riemann–Hilbert problems, see Section 1.5.4 below) and obtain useful information about the solution u of the PDE (1.5.2).

Possessing a Lax pair can be used as a possible definition of integrability, and since the investigations into the KdV equation, many other equations/systems have come under the integrable "umbrella", such as

- integrable PDEs,
- the Painlevé ODEs,
- integrable difference equations,
- orthogonal polynomials, and
- random matrix models,

see e.g. [AC91], [Dei00], and [FIKN06].

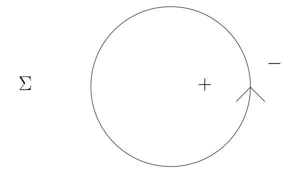


Figure 1.8: A simple example of an oriented contour $\Sigma \subset \mathbb{C}$

1.5.4 The Riemann-Hilbert (RH) problem

When the IST was introduced, the analysis of the Lax pair was conducted via the so-called Gel'fand-Levitan-Marchenko (GLM) integral equation. In 1975 Zakharov and Shabat realised this could be conducted via a classic problem of complex analysis, the Riemann-Hilbert problem, and it was realised that this had several key advantages over the GLM approach. The RH problem has 2 ingredients:

1. an oriented contour $\Sigma \subset \mathbb{C}$, Figure 1.8 shows a simple example,
2. the "jump matrix" G, a map which assigns a matrix to every point on the contour $\big(G : \Sigma \to GL(N, \mathbb{C})\big)$.

The *problem* is to find a matrix $Y(z)$ such that

- Y analytic in $\mathbb{C}/\Sigma$ (i.e. each entry of Y is a complex–differentiable function in $\mathbb{C}\backslash\Sigma$),
- $Y_+(z) = Y_-(z)G(z)$ for $z \in \Sigma$ (the "jump"), and
- $Y(z) \to I$ as $z \to \infty$,

where $Y_+(z)$ is the limit of $Y(z)$ as z tends to the contour from the + region, and $Y_-(z)$ is the limit of $Y(z)$ as z tends to the contour from the - region, see e.g. [AF03]. The technical aspects of this problem, such as the minimal smoothness restrictions on Σ, the allowable functional classes of the matrix G, and the precise sense in which the limits at infinity and on Σ are attained, are still the subject of current research (see e.g. [FIKN06] for further details).

Special cases of RH problems are the factorisation problems arising in the Wiener-Hopf technique, [Nob88], where the contour Σ is the real line.

In the IST method, the analysis of the x-part of the Lax pair is conducted by forming a RH problem in the complex variable k. This is known as "spectral analysis" of the x-part of the Lax pair.

1.5.5 The Fokas method

After the solution of the IVP for the KdV equation, arguably the most important problem was the initial-boundary value problem (IBVP) on the half line $0 < x < \infty$, $0 < t < T$.. A method for solving IBVPs for integrable PDEs *and linear* PDEs was presented in [Fok97] after a sequence of developments over several years (listed in [Fok08] page 317). A particularly important one was in 1994 when Fokas and Gel'fand formulated Lax pairs for linear evolution PDEs and showed that the spectral analysis of the x-part of the Lax pair yields the Fourier transform [FG94]; thus the IST is genuinely a *non-linear* Fourier transform.

The key idea of the new method was that both parts of the Lax Pair should be analysed *simultaneously*, as opposed to the IST where they were analysed sequentially, and the vast majority of the work was on the x-part. It was later realised that the best way to perform this simultaneous spectral analysis of the Lax Pair is through the differential form

$$\phi_x dx + \phi_t dt = A\phi \, dx + B\phi \, dt \tag{1.5.4}$$

which is exact/closed iff q satisfies the PDE. This differential form yields the global relation in a straightforward manner, for linear PDEs the global relation is the integral of (1.5.4) over the boundary of the domain.

The Fokas method can be summarised as follows. Given,

- a PDE with a Lax pair,

- a domain,

perform the simultaneous spectral analysis of the Lax pair in the domain to obtain:

1. an integral representation of the solution as an integral in $\mathbb{C}$ of transforms of the boundary values (for *linear* PDEs this is the analogue of Green's integral representation in the Fourier/transform space),
2. the global relation: an equation coupling the transforms of the boundary values (some of which are unknown).

Use the global relation *either* to find the unknown boundary values *or* to eliminate their contribution from the integral representation.

1.5.6 From Green to Lax

One of the ideas of the Fokas method is that Lax pairs - an idea which originated in the theory of nonlinear "integrable" PDEs - are equally applicable to linear PDEs. Moreover, a posteriori they should have been discovered by Green, since **for linear PDEs, Lax pairs naturally arise from the divergence form of the PDE, which is the starting point for Green's integral representation.**

In particular, for the PDE (1.1.1), the global relation (1.3.1) arises from the divergence form

$$\nabla \cdot (v\nabla u - u\nabla v) = 0, \tag{1.5.5}$$

where v is a one-parameter family of solutions of the adjoint equation (1.3.2) (obtained by separation of variables). For problems in Cartesian co-ordinates the appropriate family is $\exp(ik_1x + ik_2y)$ where $k_1^2 + k_2^2 = \lambda$. In two dimensions the divergence form (1.5.5) becomes

$$(vu_x - uv_x)_x + (vu_y - uv_y)_y = 0.$$

A Lax pair for the Helmholtz equation is then

$$\psi_x = -(vu_y - uv_y), \tag{1.5.6a}$$

$$\psi_y = vu_x - uv_x. \tag{1.5.6b}$$

Indeed, it can be verified that the condition $\psi_{xy} = \psi_{yx}$ is equivalent to the Helmholtz equation in Cartesian variables. The pair (1.5.6) can be put in the form (1.5.3), with t replaced by y, by letting

$$\phi = \begin{pmatrix} \psi \\ 1 \end{pmatrix},\ A = \begin{pmatrix} 0 & uv_y - vu_y \\ 0 & 0 \end{pmatrix},\ \text{and}\ B = \begin{pmatrix} 0 & vu_x - uv_x \\ 0 & 0 \end{pmatrix};$$

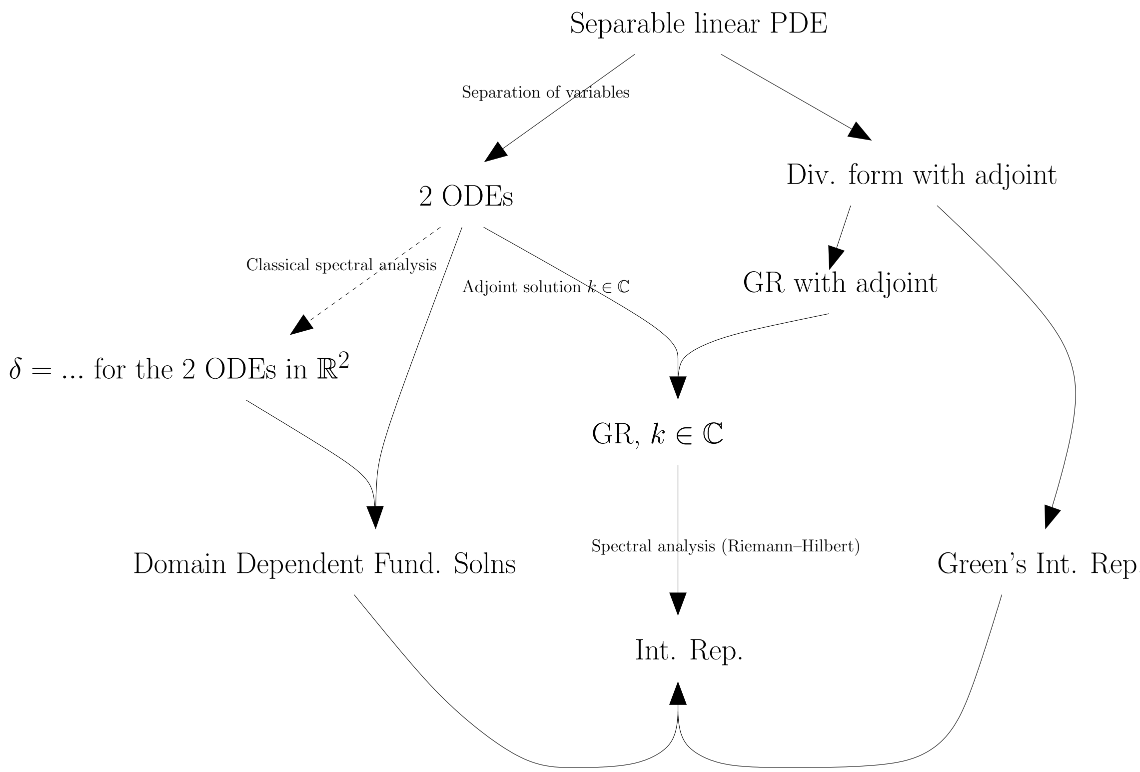
and then (1.5.3) reduces to (1.5.6).

The Fokas framework for separable linear and integrable non-linear PDEs is shown in Figure 1.5.6. This thesis includes all the steps in Figure 1.9(a) *except* for obtaining the integral representation via the spectral analysis of the global relation (equivalent to the simultaneous spectral analysis of the Lax Pair), [Fok08] Chapter 11.

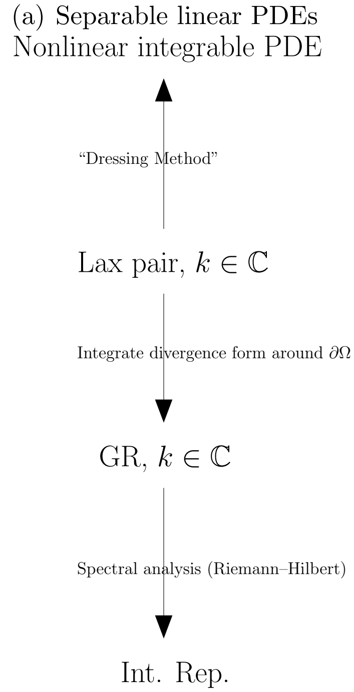
1.6 How the results of this thesis are related to previous work

This Section discusses how the analytical results in this thesis are related to previous work on the Fokas method. The numerical method of Chapter 6 is related to previous work in §6.4.

Integral representations of the solution of the *homogeneous* version of (1.1.1) in a convex polygon were given in [Fok01] via spectral analysis of the GR. A Lax pair/differential form was used which did *not* come from Green's divergence form, and so involved the Dirichlet and Neumann boundary values *and* the derivative of the Dirichlet boundary values. In addition, for Laplace only, a representation for the derivative of the solution u_z was found. (In other words, the equation $u_{\bar{z}} = 0$ was solved.) In [FZ02] these representations



(a) Separable linear PDEs



(b) Non-linear integrable PDEs

Figure 1.9: The framework for linear separable PDEs and nonlinear integrable PDEs

were re-derived from the physical space integral representation by substituting (2.1.5) into the integral representation and performing one k integration (this procedure was first performed for linear evolution PDEs by [Bre97]). In [CDF09] it was realised that convexity was *not* essential for obtaining an integral representation in the spectral space, and an integral representation of u_z for Laplace in the exterior of a convex polygon was obtained using the method of [FZ02]. Boundary value problems for the *inhomogeneous* equation

$$u_{\bar{z}} = f$$

were solved in [FP06] by first subtracting off a particular solution and then solving the homogeneous equation (in general this leads to a more complicated expression for the solution than if this trick is not employed).

Boundary value problems for $u_{\bar{z}} = 0$ and the homogeneous the modified Helmholtz equation equation were solved in [FK03],[bAF01],[bAF99],[AF05],[DF05],[FP06]. The main achievements were

- Solution for *non-separable boundary conditions*, explicitly for relatively simple ones in [FK03],[FP06], and the Dirichlet to Neumann map expressed in terms of a Riemann-Hilbert problem for more complicated ones in [FK03],[AF05].
- Solution in *non-separable domains* as an infinite series (as opposed to a bi-infinite series classically) for the equilateral triangle in [DF05] and the right isosceles triangle in [FK03] (although it was not appreciated at the time that this was an improvement on the classical).
- For boundary value problems where the solution is given as an integral this expression is *uniform convergent at the boundary* and *the contours can be deformed to give exponential decay*: these 2 achievements were not realised initially, but later emphasised in the book [Fok08].

The main achievements of this thesis are:

- Integral representations are presented for the *inhomogeneous* equation (1.1.1).
- The integral representations contain transforms of only Dirichlet and Neumann boundary values.
- An integral representation for the solution u of the Poisson equation has been obtained (as opposed to the derivative u_z).
- The solution in non-separable domains is now given as an integral (as opposed to an infinite sum in [FK03],[DF05], and a bi-infinite sum classically).
- The method has been used to solve boundary value problems in polygonal domains for the Helmholtz equation for the first time. These are more complicated than for the modified Helmholtz equation or the Poisson equation because the contours of integration now involve circular arcs and must be suitably deformed to avoid poles.
- The method has been extended from polygonal domains to domains in polar coordinates.

In addition

- The connection with Green's identity is now better understood via the domain-dependent fundamental solutions, and these have simplified the method of obtaining the novel integral representations from the physical space.
- The process for solving boundary value problems has been stated algorithmically (Chapter 1 §3).
- The invariances of the GR in polygons have been completely understood, which gives an indication of why the method has not yet been able to solve boundary value problems in more complicated domains.

Remark 1.6.1 (Shanin) *The GR for the Helmholtz equation in the interior of an equilateral triangle was first discovered by Shanin [Sha97] (later published as [Sha00]). He* *considered impedance boundary conditions, found the eigenvalues and eigenfunctions of the Laplacian in this situation, and solved the associated BVP for the Helmholtz equation, with the solution given as an infinite series.*

1.7 Summary of thesis

Chapter 2: The integral representation

- In this Chapter we use transforms (depending on the co-ordinate system) to obtain integral representations (IRs) of the fundamental solution ("domain-dependent fundamental solutions").
- These domain-dependent fundamental solutions are then substituted into Green's IR of the solution and the order of integration interchanged.
- This results in a representation of the solution as an integral of *transforms* of the Dirichlet and Neumann boundary values – the analogue of Green's IR in the transform space.

Chapter 3: The global relation

- The global relation (GR) is Green's divergence form of the equation, integrated over the domain, with particular solutions of the (homogeneous) adjoint equation replacing the fundamental solution.
- Separation of variables gives a one-parameter family of solutions to the adjoint equation depending on the parameter $k \in \mathbb{C}$ (the separation constant).
- When Ω is infinite, k must be restricted so that the integral on the boundary at infinity is zero.
- The GR is useful because it involves the transforms of the boundary values appearing in the IRs, and so gives information about the unknown transforms. In some

cases this information can be used to eliminate the unknowns from the IR and hence find the solution to the boundary value problem.

Chapter 4: Solution of boundary value problems in separable domains

- In this Chapter we solve
  - the Poisson, modified Helmholtz, and Helmholtz equations in the half plane for Dirichlet and oblique Robin boundary conditions §4.1,
  - the Helmholtz equation in the quarter plane for Dirichlet and oblique Robin boundary conditions §4.2,
  - the Helmholtz equation in the exterior of the circle with Dirichlet boundary conditions §4.3.
- The half plane is included as it is the simplest possible example of applying the new method, and only involves Step 1 of Chapter 1 §3. No boundary value problems in the half plane are solved with the new method that cannot be solved classically, however we include their solution by the new method for pedagogical reasons.
- The quarter plane is included as it is the simplest possible case where the new method solves certain boundary value problems which *cannot* be solved classically. The solution involves Steps 1 and 2 of §1.3.
- The Helmholtz equation in the exterior of the circle played a prominent role in the development of classical transforms and the Fokas method sheds new light on this classic problem. The solution involves Steps 1, 2 and 3 of Chapter 1 §3.
- For the half plane and quarter plane we use the IR and GR for polygons from §2.1 and §3.1 respectively. For the exterior of the circle we use the IR and GR in polar co-ordinates from §3.2 and §3.2 respectively.

Chapter 5: Solution of boundary value problems in a non-separable domain

- This Chapter presents the solution of the Dirichlet problem for the Poisson and modified Helmholtz equations in the interior of a right isosceles triangle.

- In this domain the Fokas method, yielding the solution as an integral, has a huge advantage over the classical solution of a bi-infinite sum of eigenfunctions.

- The solution procedure involves Steps 1-3 of §1.3.

- We also include an example of how to obtain the Dirichlet to Neumann map directly (without going via the solution) for the Dirichlet problem.

Chapter 6: A new numerical method

- In this Chapter the Dirichlet to Neumann map is solved numerically for the Laplace and modified Helmholtz equations in general convex polygons (with n sides).

- The main idea is that the global relation is valid *for all* $k \in \mathbb{C}$. Expanding the n unknown functions (the unknown boundary value on each side) in some series such as Fourier or Chebyshev up to N terms and then evaluating the global relation at a properly chosen set of nN points (collocation points in the spectral space) yields nN equations for the nN unknowns.

- Numerical experiments suggest that the method inherits the order of convergence of the basis used to expand the unknown functions; namely exponential for a polynomial basis, such as Chebyshev, and algebraic for a Fourier basis.

- However the condition number of the associated linear system is much higher for a polynomial basis than for a Fourier one.

Chapter 2

The integral representation

Summary:

- In this Chapter we use transforms (depending on the co-ordinate system) to obtain integral representations (IRs) of the fundamental solution ("domain-dependent fundamental solutions").

- These domain-dependent fundamental solutions are then substituted into Green's IR of the solution and the order of integration interchanged.

- This results in a representation of the solution as an integral of *transforms* of the Dirichlet and Neumann boundary values – the analogue of Green's IR in the transform space.

This chapter is concerned with obtaining the new IRs – the analogues of Green's IR in the transform space. Section 2.1 presents these for the solution of (1.1.1) in polygonal domains, section 2.2 presents them for domains in polar co-ordinates.

2.1 Polygons

Notations We identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $z = x+iy$ be the physical variable, and $z' = \xi+i\eta$ the "dummy variable" of integration. We will consider u both as a function of (x, y), and as a function of $(z, \bar{z})$ interchangeably. The chain rule implies

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so (1.1.1) becomes

$$\frac{\partial^2}{\partial z \partial \bar{z}} u + \frac{\lambda}{4} u = -\frac{f}{4}.$$

For the Helmholtz equation , $\lambda = 4\beta^2$, $\beta \in \mathbb{R}^+$, and for the modified Helmholtz equation, $\lambda = -4\beta^2$, $\beta \in \mathbb{R}^+$. The subscripts T and N will denote tangential and normal respectively.

Let $\Omega^{(i)}$ be the interior of a polygon in $\mathbb{R}^2$, and $\Omega^{(e)}$ the exterior. Let $\partial\Omega$ denote the boundary of the polygon, oriented *anticlockwise*, where the vertices of the polygon $z_1, z_2, ..., z_n$ are labelled anti-clockwise. Let S_j be the side (z_j, z_{j+1}) and let $\alpha_j = \arg(z_{j+1} - z_j)$ be the angle of S_j. Each side divides $\mathbb{R}^2$ into two open regions: $S_j^{(i)}, S_j^{(e)}$. If $z' \in S_j$ then

$$z \in S_j^{(i)} \iff \alpha_j < \arg(z - z') < \alpha_j + \pi, \tag{2.1.1a}$$

$$z \in S_j^{(e)} \iff \pi + \alpha_j < \arg(z - z') < \alpha_j + 2\pi, \tag{2.1.1b}$$

see Figure 2.1.

2.1.1 Co-ordinate system dependent fundamental solutions

In cartesian co-ordinates, (1.2.2) is

$$E_{\xi\xi} + E_{\eta\eta} + \lambda E = -\delta(\xi - x)\delta(\eta - y). \tag{2.1.2}$$

The differential operator

$$-\frac{d^2}{d\xi^2} u = \lambda u$$

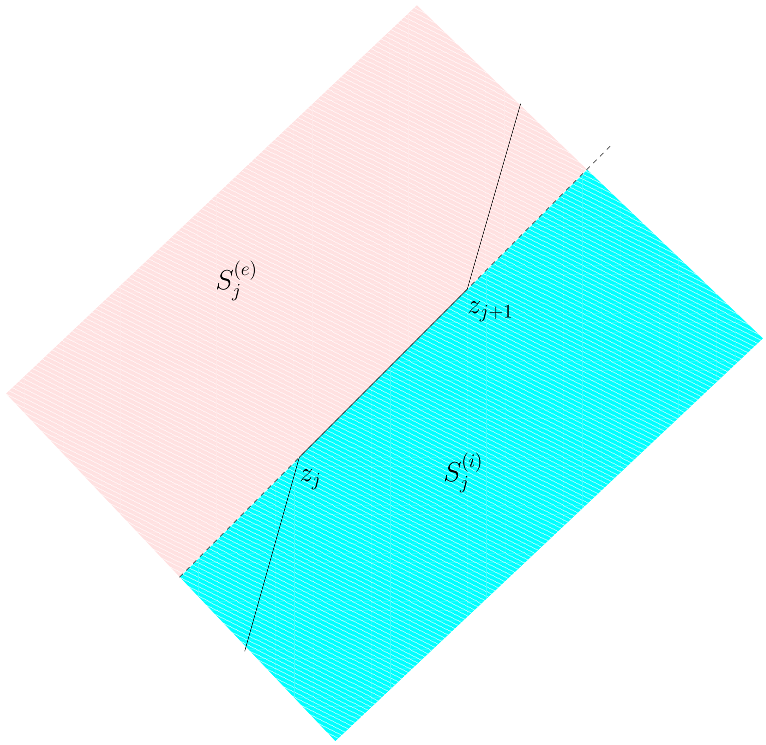


Figure 2.1: Part of the polygon showing the side S_j and regions $S_j^{(e)}$,$S_j^{(i)}$

on $(-\infty, \infty)$ possesses the associated completeness relation

$$\delta(\xi - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_1(x-\xi)}dk_1, \tag{2.1.3}$$

i.e. the Fourier transform [Sta67, Chapter 4]. Since the differential operator in η is the same as the one in ξ the completeness relation for the η co-ordinate is

$$\delta(\eta - y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_2(y-\eta)}dk_2. \tag{2.1.4}$$

These two completeness relations give rise to the following integral representation of E:

$$E(\xi, \eta, x, y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} dk_1 dk_2 \frac{e^{ik_1(x-\xi)+ik_2(y-\eta)}}{k_1^2 + k_2^2 - \lambda}, \tag{2.1.5}$$

where the contours of integration must be suitably deformed to avoid the poles of the integrand. (Strictly, the contours of (2.1.3) and (2.1.4) should be deformed before using them to obtain (2.1.5). See Remark 2.1.5 at the end of this section.)

Proposition 2.1.1 (Integral representation of E for the modified Helmholtz equation) *Given $\theta \in (-\pi, \pi)$, for the modified Helmholtz equation, the fundamental*

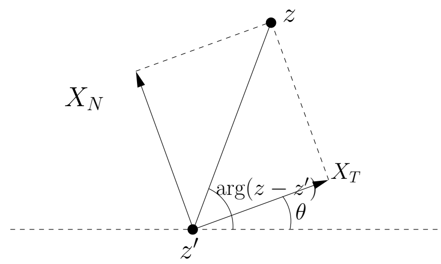


Figure 2.2: The points z, z' and the local co-ordinate system (X_T, X_N)

solution E is given by

$$E(z', z) = \frac{1}{4\pi}\int_0^\infty \frac{dk}{k}\exp\left(i\beta\left(k(z-z')e^{-i\theta} - \frac{1}{k}(\overline{z-z'})e^{i\theta}\right)\right), \qquad (2.1.6)$$

where $\theta \le \arg(z-z') \le \theta+\pi$. (Note that if $\arg(z-z') = \theta$ or $\theta+\pi$, then the integral in (2.1.6) *is not absolutely convergent.)*

Proof Starting with (2.1.5) where $\lambda = -4\beta^2$, define X_T, X_N by

$$z - z' = (x-\xi) + i(y-\eta) = (X_T + iX_N)e^{i\theta}, \qquad (2.1.7)$$

so that, if $\theta \le \arg(z-z') \le \theta+\pi$, then $X_N = \mathrm{Im}\big((z-z')e^{-i\theta}\big) \ge 0$, see Figure 2.2.

In a similar way, rotate the (k_1, k_2) plane by letting

$$k_1 + ik_2 = (k_T + ik_N)e^{i\theta},$$

so that

$$k_1(x-\xi) + k_2(y-\eta) = k_T X_T + k_N X_N, \qquad (2.1.8)$$
$${k_1}^2 + {k_2}^2 = {k_T}^2 + {k_N}^2$$

and (2.1.5) becomes

$$E(\xi,\eta,x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} dk_T dk_N \frac{e^{ik_T X_T + ik_N X_N}}{k_T^2 + k_N^2 + 4\beta^2}. \qquad (2.1.9)$$

Now perform the k_N integral by closing the contour in the upper half plane (as $X_N \ge 0$) to obtain

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ik_T X_T - \sqrt{k_T^2+4\beta^2}X_N}}{\sqrt{k_T^2+4\beta^2}} dk_T. \qquad (2.1.10)$$

The substitution $k_T = \beta(l - 1/l)$ transforms (2.1.10) to

$$E(\xi, \eta, x, y) = \frac{1}{4\pi} \int_0^\infty \frac{dl}{l} \exp\left(i\beta\left(l - \frac{1}{l}\right) X_T - \beta\left(l + \frac{1}{l}\right) X_N \right), \tag{2.1.11}$$

which becomes (2.1.6) after using (2.1.7) and relabelling l as k. The right hand side of (2.1.6) defines a function of $z - z'$ in a half plane. To show that this expression defines a function of $z - z'$ in the whole plane (except zero), we must show that the right hand sides of (2.1.6) for different values of θ are equal for $z - z'$ in the common domain of definition. This is achieved by rotating the contour, using Cauchy's theorem and the analyticity of the integrand in k. □

The main differences between the Helmholtz equation and the modified Helmholtz equation are that

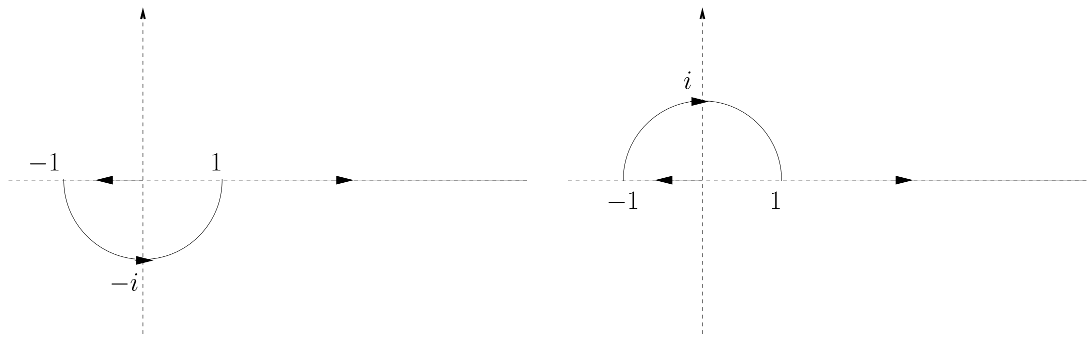
1. there are *two* fundamental solutions, and
2. the contour of integrations for both the fundamental solutions contain circular arcs as well as rays in the complex plane.

These differences follow from the fact that, for $\lambda > 0$, the integral representation of E (2.1.5) has poles on the contour; thus it is *not* well-defined. There are two different choices of contour that resolve this ambiguity, and then these two choices yield two fundamental solutions.

Proposition 2.1.2 (Integral representation of E_{out} and E_{in} for the Helmholtz equation) *Given $\theta \in (-\pi, \pi)$, for the Helmholtz equation, the two fundamental solutions are given by*

$$E_{out}(z', z) = \frac{i}{4} H_0^{(1)}(2\beta|z - z'|) = \frac{1}{4\pi} \int_{L_{out}} \frac{dk}{k} \exp\left(i\beta\left(k(z - z')e^{-i\theta} + \frac{1}{k}(\overline{z - z'})e^{i\theta} \right)\right), \tag{2.1.12a}$$

$$E_{in}(z', z) = -\frac{i}{4} H_0^{(2)}(2\beta|z - z'|) = \frac{1}{4\pi} \int_{L_{in}} \frac{dk}{k} \exp\left(i\beta\left(k(z - z')e^{-i\theta} + \frac{1}{k}(\overline{z - z'})e^{i\theta} \right)\right), \tag{2.1.12b}$$



(a) The contour L_{out} of integration in the k plane for $H_0^{(1)}$, i.e. outgoing

(b) The contour L_{in} of integration in the k plane for $H_0^{(2)}$, i.e. incoming

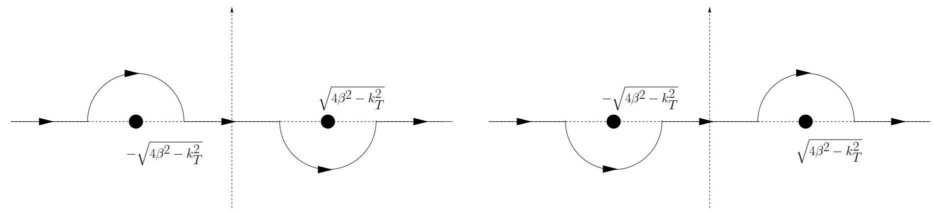
Figure 2.3: The contours L_{out} and L_{in}.

where $\theta \leq \arg(z - z') \leq \theta + \pi$, *and the contours* L_{out} *and* L_{in} *are shown in Figure 2.3. (Note that if* $\arg(z - z') = \theta$ *or* $\theta + \pi$, *then the integrals in* (2.1.12) *are not absolutely convergent.)*

Proof For the Helmholtz equation the fundamental solution is given by (2.1.5) with $\lambda = 4\beta^2$, $\beta \in \mathbb{R}$. As before, first perform the k_N integral. For $|k_T| \geq 2\beta$ the poles are on the imaginary axis (like for the modified Helmholtz equation) at $k_N = \pm i\sqrt{k_T^2 - 4\beta^2}$. For $|k_T| \leq 2\beta$ the poles are on the real axis, at $k_N = \pm\sqrt{4\beta^2 - k_T^2}$, and the k_N integral is not well-defined unless the path of integration about the poles is specified. The two paths around the poles yielding non-zero answers are given in Figure 2.4. The two choices differ in their asymptotic behaviour at infinity, which correspond to outgoing or incoming waves respectively (with the assumed time dependence $e^{-i\omega t}$). A priori we do not know which choice leads to which fundamental solution, but once both expressions have been obtained this can be determined by computing their asymptotics (by the method of steepest descent). Perform the k_N integration to obtain

$$E(\xi, \eta, x, y) = \frac{1}{4\pi} \int_{|k_T|>2\beta} \frac{e^{ik_T X_T - \sqrt{k_T^2 - 4\beta^2}|X_N|}}{\sqrt{k_T^2 - 4\beta^2}} dk_T \pm \frac{i}{4\pi} \int_{-2\beta}^{2\beta} \frac{e^{ik_T X_T \pm i\sqrt{4\beta^2 - k_T^2}|X_N|}}{\sqrt{4\beta^2 - k_T^2}} dk_T \tag{2.1.13}$$

with the top sign corresponding to $H_0^{(1)}$ and the bottom $H_0^{(2)}$. As for the modified Helmholtz equation, a change of variables can be used to eliminate the square roots. In this case this change of variables is $k_T = \beta\,(l + 1/l)$. This is motivated by the fact that



(a) $H_0^{(1)}$ i.e. Outgoing (b) $H_0^{(2)}$ i.e. Incoming

Figure 2.4: The contours for the k_T integration in the fundamental solution to obtain $H_0^{(1)}$ and $H_0^{(2)}$

if $l = e^{i\phi}$ then $k_T = 2\beta\cos\phi$, and then $\sqrt{4\beta^2 - k_T^2} = \pm 2\beta\sin\phi = \mp i\beta(l - 1/l)$. Due to the square roots there are several choices for the range of integration in l (but all these choices lead to equivalent answers). In order to get the same integrand for both integrals in (2.1.13) we choose $l \in (1, \infty)$ and $l \in (0, -1)$ in the first integral, $\phi \in (-\pi, 0)$ in the second for outgoing and $\phi \in (0, \pi)$ in the second for incoming. A few lines of calculation, as well as relabelling l as k, yields (2.1.12). The proof that (2.1.12) defines a function of $z - z'$ in the whole plane follows by contour deformation in a similar way to that for the modified Helmholtz equation. $\square$

For Poisson, this algorithm of constructing representations of the fundamental solution results in a representation that is *formal*, namely it involves divergent integrals. This is a consequence of the fact that the fundamental solution for Poisson, $-\frac{1}{2\pi}\log|\boldsymbol{\xi} - \boldsymbol{x}|$, does not decay at infinity, and so its Fourier transform is not well defined in a classical sense. Nevertheless, when the representation of the fundamental solution is substituted into Green's IR, the resulting IR for u *is* well defined.

Proposition 2.1.3 (Integral representation of E for the Poisson equation) *For Poisson's equation, a formal representation of the fundamental solution is given by*

$$E(z', z) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\left(\int_{i\varepsilon}^{\infty}\frac{dk}{k}\exp\left(ike^{-i\theta}(z - z')\right) + \int_{-i\varepsilon}^{\infty}\frac{dk}{k}\exp\left(-ike^{i\theta}\overline{(z - z')}\right)\right), \tag{2.1.14}$$

$$\theta \le \arg(z - z') \le \theta + \pi.$$

(Note that if $\arg(z - z') = \theta$ *or* $\theta + \pi$ *then the integrals in* (2.1.14) *are not absolutely convergent.)*

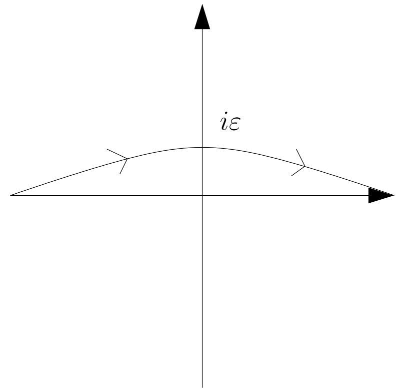


Figure 2.5: The deformed contour of integration in the k_T plane.

Proof Start with (2.1.5) with $\lambda = 0$ and perform the same change of variables, (2.1.7) and (2.1.8), as for the modified Helmholtz equation to obtain (2.1.9) with $\beta = 0$. Deform the k_T contour above or below the singularity at 0 so it passes through $i\varepsilon$, see Figure 2.5. Perform the k_N integral by closing the contour in the upper half plane to enclose the pole at $k_N = ik_T$ for $\Re k_T > 0$ and $k_N = -ik_T$ for $\Re k_T < 0$, to obtain

$$E(\xi, \eta, x, y) = \frac{1}{4\pi}\left(\int_{i\varepsilon}^{\infty} dk_T\, \frac{e^{ik_T X_T - k_T X_N}}{k_T} + \int_{-\infty}^{i\varepsilon} dk_T\, \frac{e^{ik_T X_T + k_T X_N}}{-k_T}\right). \tag{2.1.15}$$

Now let $k_T \mapsto -k_T$ in the second integral and use (2.1.7) to obtain the right hand side of (2.1.14) without the $\varepsilon \to 0$ limit (after relabelling k_T as k).

The rotation of contours that shows (2.1.6) and (2.1.12) are well defined functions of $z - z'$ in the whole plane requires that the contours start from zero and hence fails for (2.1.15). Taking the $\varepsilon \to 0$ limit allows this argument to proceed for (2.1.15); however the integrals do not converge in this limit. □ □

Remark 2.1.4 (Derivation via one transform) *The IRs of E have been obtained by taking two transforms to obtain* (2.1.5)*, then computing one integral. Alternatively, they can be obtained by taking one transform, then solving one ODE. For example, after rotating co-ordinates using* (2.1.7)*, equation* (2.1.2) *with $\lambda = -4\beta^2$ becomes*

$$E_{X_T X_T} + E_{X_N X_N} - 4\beta^2 E = -\delta(X_T)\delta(X_N).$$

Taking the Fourier transform in X_T yields

$$\widehat{E}_{X_N X_N} - (4\beta^2 + k_T^2)\widehat{E} = -e^{ik_T X_T}\delta(X_N),$$

which can be solved using a 1-d Green's function to give

$$\widehat{E} = \frac{e^{ik_T X_T - \sqrt{k_T^2+4\beta^2}|X_N|}}{2\sqrt{k_T^2+4\beta^2}}.$$

Then the inverse Fourier transform yields (2.1.10)*.*

Remark 2.1.5 (The Malgrange-Ehrenpreis theorem) *The Malgrange-Ehrenpreis theorem states that every linear partial differential operator with constant coefficients has a fundamental solution. The first proofs of this were presented independently by Malgrange and Ehrenpreis in 1953 and were* non-constructive*, using the Hahn Banach theorem (see, e.g. [Rud91, Theorem 8.5, page 195 in the 1974 edition]. In 1955 an* explicit *formula for the fundamental solution was obtained by Hörmander. This was obtained using the Fourier transform, like in* (2.1.5)*. The hardest part of the proof was specifying a contour which avoided the singularities of the integrand. This was achieved by a construction later called "Hörmander's staircase" (e.g. [FJ98, Theorem 10.4.1 page 139]. [OW97] provides a good survey of this area.*

2.1.2 Domain dependent fundamental solutions

The definitions of $S_j^{(i/e)}$, (2.1.1), and the integral representations of E given by Propositions 2.1.1, 2.1.3 and 2.1.2 immediately imply the following:

Proposition 2.1.6 *Consider $\Omega^{(i)}$. Given $z' \in S_j$*

- *for the modified Helmholtz equation*

$$E(z', z) = \frac{1}{4\pi}\int_{l_j(z)} \frac{dk}{k} \exp\left(i\beta\left(k(z-z') - \frac{\overline{(z-z')}}{k}\right)\right), \tag{2.1.16}$$

where

$$l_j(z) = \begin{cases} l_j^{(i)} : \{k \in \mathbb{C} : \arg k = -\alpha_j\}, & \textit{when } z \in S_j^{(i)}, \\ l_j^{(e)} : \{k \in \mathbb{C} : \arg k = \pi - \alpha_j\}, & \textit{when } z \in S_j^{(e)}, \\ \textit{either } l_j^{(i)} \textit{ or } l_j^{(e)}, & \textit{when } z \in \overline{S_j^{(e)}} \cap \overline{S_j^{(i)}} \setminus \partial\Omega. \end{cases}$$

- *for the Helmholtz equation*

$$E_{out}(z',z)=\frac{1}{4\pi}\int_{L_{out\,j}(z)}\frac{dk}{k}\exp\left(i\beta\left(k(z-z')+\frac{\overline{(z-z')}}{k}\right)\right), \tag{2.1.17}$$

where

$$L_{out\,j}(z)=\begin{cases} L^{(i)}_{out\,j}:\{k\in\mathbb{C}:k=le^{-i\alpha_j},l\in L_{out}\}, & \textit{when } z\in S^{(i)}_j,\\ L^{(e)}_{out\,j}:\{k\in\mathbb{C}:k=le^{i(\pi-\alpha_j)},l\in L_{out}\}, & \textit{when } z\in S^{(e)}_j,\\ \textit{either } L^{(i)}_{out\,j} \textit{ or } L^{(e)}_{out\,j}, & \textit{when } z\in \overline{S^{(e)}_j}\cap\overline{S^{(i)}_j}\backslash\partial\Omega. \end{cases}$$

- *for the Poisson equation*

$$E(z',z)=\lim_{\varepsilon\to 0}\frac{1}{4\pi}\left(\int_{l_{j\varepsilon}(z)}\frac{dk}{k}\exp\left(ik(z-z')\right)+\int_{\overline{l_{j\varepsilon}(z)}}\frac{dk}{k}\exp\left(-ik\overline{(z-z')}\right)\right), \tag{2.1.18}$$

where

$$l_{j\varepsilon}(z)=\begin{cases} l^{(i)}_{j\varepsilon}:\{k\in\mathbb{C}:k=le^{-i\alpha_j},l\in(i\varepsilon,\infty)\}, & \textit{when } z\in S^{(i)}_j,\\ l^{(e)}_{j\varepsilon}:\{k\in\mathbb{C}:k=le^{i(\pi-\alpha_j)},l\in(i\varepsilon,\infty)\},, & \textit{when } z\in S^{(e)}_j,\\ \textit{either } l^{(i)}_{j\varepsilon} \textit{ or } l^{(e)}_{j\varepsilon}, & \textit{when } z\in \overline{S^{(e)}_j}\cap\overline{S^{(i)}_j}\backslash\partial\Omega. \end{cases}$$

Proof For $z'\in S_j$, (2.1.1a) implies that when $z\in S^{(i)}_j$ (2.1.6) holds with $\theta=\alpha_j$. Let $k\mapsto ke^{i\alpha_j}$ to get (2.1.16). Similarly when $z\in S^{(e)}_j$ (2.1.6) holds with $\theta=\alpha_j+\pi$ and let $k\mapsto ke^{i(\alpha_j+\pi)}$ to get (2.1.16). When $z\in\overline{S^{(i)}_j}\cap\overline{S^{(e)}_j}$ the transformation $k\mapsto -1/k$ shows that (2.1.16) with $l_j(z)=l^{(i)}_j$ and (2.1.16) $l_j(z)=l^{(e)}_j$ are equal. The representations (2.1.17) and (2.1.18) follow in a similar way. For the Helmholtz equation when $z\in\overline{S^{(i)}_j}\cap\overline{S^{(e)}_j}$ the transformation $k\mapsto 1/k$ shows that the two expressions are equal. For the Poisson equation, when $z\in\overline{S^{(i)}_j}\cap\overline{S^{(e)}_j}$ the two expressions are equal after changing the sign of ε in one of them (no change of variable is needed as for the modified Helmholtz equation). $\square$

2.1.3 Green's integral representations

In this section we derive Green's IR for the equation (1.1.1). The IRs of the Fokas method will be obtained from these by substituting in the domain-dependent fundamental solutions of the previous section.

Theorem 2.1.7 (Green's integral representation) *Let $u^{(i)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(i)}$, and let $u^{(e)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(e)}$ with the following boundary conditions at infinity:*

- $\lambda = 0$ *(Poisson), $u^{(e)} \to 0$ as $r \to \infty$.*
- $\lambda = -4\beta^2$ *(Modified Helmholtz), $u^{(e)} \to 0$ as $r \to \infty$.*
- $\lambda = 4\beta^2$ *Helmholtz,*

$$\sqrt{r}\left(\frac{\partial u^{(e)}}{\partial r} - 2i\beta u^{(e)}\right) \to 0 \text{ as } r \to \infty \tag{2.1.19}$$

where $r = \sqrt{x^2 + y^2}$. With assumed time dependence $e^{-i\omega t}$ the boundary condition at infinity for Helmholtz corresponds to outgoing *waves (see the discussion at the beginning of §1.1). Assume also that f has compact support (which implies that the integral $\int_{\Omega^{(e)}} f\, E\, d\xi\, d\eta$ is well defined). Then*

$$u^{(i)}(z,\bar{z}) = \sum_{j=1}^{n} \int_{z_j}^{z_{j+1}} E\left(u_\xi^{(i)} d\eta - u_\eta^{(i)} d\xi\right) - u^{(i)}\left(E_\xi d\eta - E_\eta d\xi\right) + \int_{\Omega^{(i)}} f\, E\, d\xi d\eta, \tag{2.1.20a}$$

$$= -i\sum_{j=1}^{n} \int_{z_j}^{z_{j+1}} E\left(u_{z'}^{(i)} dz' - u_{\bar{z}'}^{(i)} d\bar{z}'\right) - u^{(i)}\left(E_{z'} dz' - E_{\bar{z}'} d\bar{z}'\right) + \int_{\Omega^{(i)}} f\, E\, d\xi d\eta, \tag{2.1.20b}$$

for $z \in \Omega^{(i)}$ and

$$u^{(e)}(z,\bar{z}) = -\sum_{j=1}^{n} \int_{z_j}^{z_{j+1}} E\left(u_\xi^{(e)} d\eta - u_\eta^{(e)} d\xi\right) - u^{(e)}\left(E_\xi d\eta - E_\eta d\xi\right) + \int_{\Omega^{(e)}} f\, E\, d\xi d\eta, \tag{2.1.21a}$$

$$= i\sum_{j=1}^{n} \int_{z_j}^{z_{j+1}} E\left(u_{z'}^{(e)} dz' - u_{\bar{z}'}^{(e)} d\bar{z}'\right) - u^{(e)}\left(E_{z'} dz' - E_{\bar{z}'} d\bar{z}'\right) + \int_{\Omega^{(e)}} f\, E\, d\xi d\eta, \tag{2.1.21b}$$

for $z \in \Omega^{(e)}$.

Proof (standard) In two dimensions the equations satisfied by u and E are

$$u_{\xi\xi} + u_{\eta\eta} + \lambda u = -f,$$

$$E_{\xi\xi} + E_{\eta\eta} + \lambda E = -\delta(\xi - x)\delta(\eta - y).$$

Multiplying the first by E, the second by u and then subtracting yields the following divergence form

$$\Big(Eu_\xi - uE_\xi\Big)_\xi - \Big(uE_\eta - Eu_\eta\Big)_\eta = u\,\delta - f\,E. \tag{2.1.22}$$

Integral representations for u can now be obtained using (2.1.22) and Green's theorem in the plane which states that

$$\int_{\partial A} F_1 dx + F_2 dy = \iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \tag{2.1.23}$$

where the orientation of ∂A is anti-clockwise.

Interior: apply (2.1.23) for $A = \Omega^{(i)}$, $u = u^{(i)}$ to obtain

$$\int_{\partial\Omega} E\left(u_\xi^{(i)} d\eta - u_\eta^{(i)} d\xi\right) - u^{(i)}\left(E_\xi d\eta - E_\eta d\xi\right) + \int_{\Omega^{(i)}} f\,E\,d\xi d\eta = \begin{cases} u^{(i)}(x,y), & (x,y) \in \Omega^{(i)}, \\ 0, & (x,y) \in \Omega^{(e)}. \end{cases} \tag{2.1.24}$$

where the orientation of $\partial\Omega$ is anti-clockwise.

Exterior: apply (2.1.23) with $u = u^{(e)}$ and A the region bounded internally by $\partial\Omega$ and externally by a large circle of radius R. Then the assumed boundary conditions at infinity imply that the integral over the large circle tends to zero as $R \to \infty$, see e.g. [Sta68, §6.5, page 124, §7.14, page 297], [Ble84, §6.4 page 180]. Thus,

$$-\int_{\partial\Omega} E\left(u_\xi^{(e)} d\eta - u_\eta^{(e)} d\xi\right) - u^{(e)}\left(E_\xi d\eta - E_\eta d\xi\right) + \int_{\Omega^{(e)}} f\,E\,d\xi d\eta = \begin{cases} 0, & (x,y) \in \Omega^{(i)}, \\ u^{(e)}(x,y), & (x,y) \in \Omega^{(e)}, \end{cases} \tag{2.1.25}$$

where the orientation of $\partial\Omega$ is still anti-clockwise. The equations (2.1.20) and (2.1.21) now follow from (2.1.24a) and (2.1.25b) using

$$\int_{\partial\Omega} = \sum_{j=1}^{n} \int_{z_j}^{z_{j+1}} \tag{2.1.26}$$

and

$$\left(E_\xi d\eta - E_\eta d\xi\right) = -i\left(E_{z'} dz' - E_{\bar{z}'} d\bar{z}'\right) = \frac{\partial E}{\partial n}\,ds \tag{2.1.27}$$

where n is the outward normal, and ds the arc length along the $\partial\Omega$ which is oriented anticlockwise (so outward normal points to the right). $\square$

Remark 2.1.8 • *If the polygon is* open *(i.e. extends to infinity), then $u^{(i)}$ must obey the same conditions at infinity as $u^{(e)}$.*

- *By expansion in polar co-ordinates (see e.g. [Sta68, pages 128, 296],*
  - *for $\lambda = 0$ (Poisson) : if $u^{(e)} \to 0$ as $r \to \infty$ then in fact $u^{(e)} = \mathcal{O}\left(\frac{1}{r}\right)$ as $r \to \infty$.*
  - *for $\lambda = -4\beta^2$ (Modified Helmholtz): if $u^{(e)} \to 0$ as $r \to \infty$ then in fact $u^{(e)} = \mathcal{O}\left(\frac{e^{-2\beta r}}{r^{1/2}}\right)$ as $r \to \infty$.*
  - *for $\lambda = 4\beta^2$ (Helmholtz): if $u^{(e)}$ satisfies the radiation condition* (2.1.19) *then $u^{(e)} \sim g(\phi)e^{2i\beta r}/r^{1/2}$ as $r \to \infty$, where ϕ is the polar angle.*
- *The boundary conditions at infinity assumed in the theorem appear in many problems of physical significance. However, other boundary conditions (which lead to a non-zero contribution from the integral over the large circle) also appear in applications, see [Sta68, §6.8]. The IR can be suitably modified to apply to these more general boundary conditions.*

Remark 2.1.9 (Rigorous considerations) *[CWL07] provides a good, user-friendly, overview of the formulation in appropriate function spaces of the PDE* (1.1.1) *posed in polygonal domains.*

2.1.4 The novel integral representations

The novel IRs are obtained by substituting the domain dependent fundamental solutions into Green's IRs. Two important things to note are the following:

1. The contours of the new IRs depend on which half-plane z is in relative to each side of the polygon (for side j these half-planes are $S_j^{(i)}$ and $S_j^{(e)}$ defined by (2.1.1)). If the polygon Ω is *convex*, for the interior z is always in one half plane ($S_j^{(i)}$), so the contours in the IR are then *independent* of z.

2. In order to obtain a spectral representation of the forcing term in Green's IR using (2.1.6) we require that for $z' \in \Omega^{(i)}$ and $z \in \Omega^{(i)}$, $\theta \leq \arg(z - z') \leq \theta + \pi$ for some θ. This is *impossible* unless we split the domain $\Omega^{(i)}$ into two regions by a line through z, where we are free to choose the angle of the split (see Figure 2.6). This means that the transforms of the forcing term depend on z, which in some ways is unsatisfactory (but unavoidable). The same result is obtained by the two other methods for deriving these representations (spectral analysis of the global relation and applying the global relation in a subdomain) and this is consistent with the situation for *evolution* PDEs where the transforms of the forcing term appearing in the integral representations depend on t [Fok08]; in this case, however, it is expected by causality and Duhamel's principle (e.g. [Eva98]).

2.1.4.1 The modified Helmholtz equation

Proposition 2.1.10 (Integral representation of the solution of the modified Helmholtz equation) *Let $\lambda = -4\beta^2$. Let $u^{(i)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(i)}$ and let $u^{(e)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(e)}$. Suppose that $u^{(i)}$ and $u^{(e)}$ have the integral representations* (2.1.20) *and* (2.1.21) *respectively. Then $u^{(i)}$ and $u^{(e)}$ have the* alternative *representations*

$$u^{(i)}(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j(z)}\frac{dk}{k}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\widehat{u^{(i)}}_j(k) + F^{(i)}(z,\bar{z}), \quad z \in \Omega^{(i)}, \tag{2.1.28}$$

$$u^{(e)}(z,\bar{z}) = -\frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j(z)}\frac{dk}{k}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\widehat{u^{(e)}}_j(k) + F^{(e)}(z,\bar{z}), \quad z \in \Omega^{(e)}, \tag{2.1.29}$$

where $l_j(z)$, $j = 1, ..., n$ are rays in the complex k plane, oriented from 0 *to ∞, and defined by*

$$l_j(z) = \begin{cases} l_j^{(i)} : \{k \in \mathbb{C} : \arg k = -\alpha_j\}, & \textit{when } z \in S_j^{(i)}, \\ l_j^{(e)} : \{k \in \mathbb{C} : \arg k = \pi - \alpha_j\}, & \textit{when } z \in S_j^{(e)}, \\ \textit{either } l_j^{(i)} \textit{ or } l_j^{(e)}, & \textit{when } z \in \overline{S_j^{(e)}} \cap \overline{S_j^{(i)}} \backslash \partial\Omega \end{cases} \tag{2.1.30}$$

where $S_j^{(i/e)}$ *are defined by* (2.1.1)*. The transforms of the boundary values of* $u^{(i/e)}$ *on the side* j*,* $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$*, are given by*

$$\widehat{u^{(i/e)}}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'-\frac{\bar{z}'}{k}\right)}[(u_{z'}^{(i/e)} + i\beta k u^{(i/e)})dz' - (u_{\bar{z}'}^{(i/e)} + \frac{\beta}{ik}u^{(i/e)})d\bar{z}'], \tag{2.1.31}$$

$$= \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'+\frac{\bar{z}'}{k}\right)} \left[i\frac{\partial u^{(i/e)}}{\partial n}(z'(s)) + i\beta\left(k\frac{dz'}{ds} + \frac{1}{k}\frac{d\bar{z}'}{ds}\right)u^{(i/e)}(z'(s))\right]ds, \tag{2.1.32}$$

where $j = 1,..,n$*,* $z_{n+1} = z_1$*,* $z' = z'(s) \in S_j$*, and* n *is the* outward-pointing *normal to the polygon (in* both *interior and exterior cases). The forcing term is given by*

$$F^{(i/e)}(z,\bar{z}) = \iint_{\Omega^{(i/e)}} f(\xi,\eta)\, E(z',z)\, d\xi d\eta \tag{2.1.33}$$

where E *is the fundamental solution for the modified Helmholtz equation. A spectral representation of this term is given by*

$$F^{(i/e)}(z,\bar{z}) = \frac{1}{4\pi}\left(\int_{l_L}\frac{dk}{k}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\widehat{f}_L^{(i/e)}(k) + \int_{l_R}\frac{dk}{k}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\widehat{f}_R^{(i/e)}(k)\right) \tag{2.1.34}$$

where

$$\widehat{f}_{L/R}^{(i/e)}(k) = \iint_{\Omega_{L/R}^{(i/e)}(z)} e^{-i\beta\left(kz'-\frac{\bar{z}'}{k}\right)} f(\xi,\eta)d\xi d\eta, \tag{2.1.35}$$

and where

$$\Omega_L^{(i/e)}(z) = \{z' : z' \in \Omega^{(i/e)}, \theta_F < \arg(z'-z) < \theta_F + \pi\}, \tag{2.1.36a}$$

$$\Omega_R^{(i/e)}(z) = \{z' : z' \in \Omega^{(i/e)}, \theta_F - \pi < \arg(z'-z) < \theta_F\}. \tag{2.1.36b}$$

(see Figures 2.6 and 2.7). The angle $\theta_F \in [0,2\pi)$ *is* arbitrary*, and the rays* l_L *and* l_R *are defined by*

$$l_L : \{k \in \mathbb{C} : \arg k = \pi - \theta_F\}, \tag{2.1.37a}$$

$$l_R : \{k \in \mathbb{C} : \arg k = -\theta_F\}. \tag{2.1.37b}$$

Remark 2.1.11 (Convex polygons) *If* $\Omega = \Omega^{(i)}$*, and* $\Omega^{(i)}$ *is convex, then*

$$z \in \Omega^{(i)} \iff z \in S_j^{(i)},\ \forall j,\ j = 1,...,n, \tag{2.1.38}$$

so

$$l_j(z) = l_j^{(i)},\ \forall j,\ j = 1,...,n, \tag{2.1.39}$$

so that, in this case, the contours of integration are independent *of* z*.*

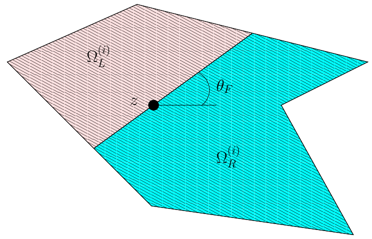


Figure 2.6: The regions $\Omega_L^{(i)}, \Omega_R^{(i)}$

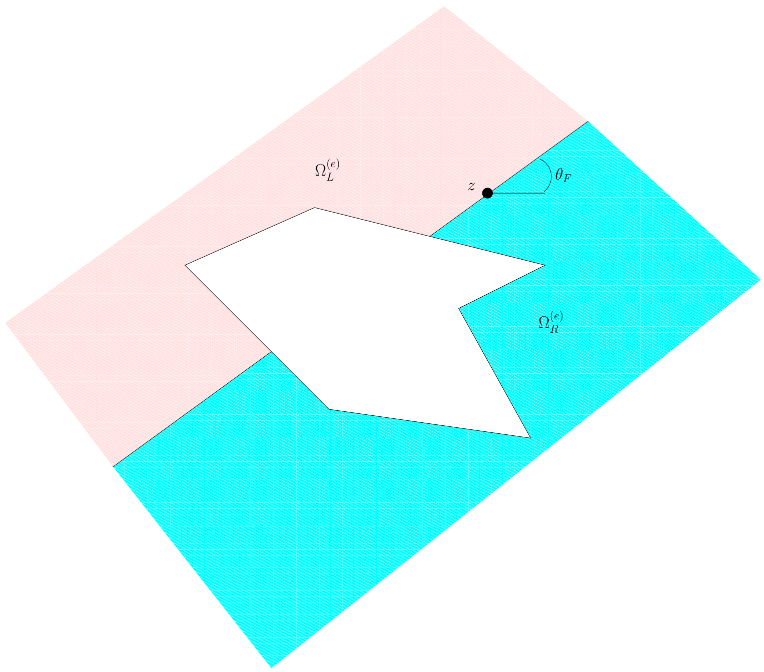


Figure 2.7: The regions $\Omega_L^{(e)}, \Omega_R^{(e)}$

Remark 2.1.12 (Return to physical space) *If the order of the k and z' integrals is interchanged in* (2.1.28) *and* (2.1.29)*, and the k integration performed, then these IRs become Green's IRs* (2.1.20) *and* (2.1.21) *as expected.*

Proof of Proposition 2.1.10 Consider first the representation for $u^{(i)}$, (2.1.28). Substituting (2.1.16) into the first term on the right hand side of (2.1.20b) (the boundary integral) and interchanging the order of integration, this term becomes the first term on the right hand side of (2.1.28).

In order to obtain a spectral representation of the forcing term using (2.1.6) we require that for $z' \in \Omega^{(i)}$ and $z \in \Omega^{(i)}$, $\theta \leq \arg(z - z') \leq \theta + \pi$ for some θ. This is impossible unless we split the domain $\Omega^{(i)}$ into two regions by a line through z, where we are free to choose the angle of the split θ_F, see Figure 2.6 and (2.1.36). Once this split is made, the forcing terms (2.1.34) follow by substituting (2.1.6) with $\theta = \pi + \theta_F$ and $\theta = \theta_F$ into the integrals over $\Omega_L^{(i)}$ and $\Omega_R^{(i)}$ respectively and then interchanging the order of integration.

The integral representation for $u^{(e)}$ follows from (2.1.21b) in an identical way. $\square$

2.1.4.2 Helmholtz

For the Helmholtz equation there are two fundamental solutions. When the domain is $\Omega^{(e)}$ we can choose one of these fundamental solutions using the requirement that we require *outgoing* waves. However, when the domain is $\Omega^{(i)}$, the domain is bounded and "outgoing" and "incoming" are meaningless, so we have two different fundamental solutions which then apparently lead to two different $u^{(i)}$. If $4\beta^2$ is not an eigenvalue, the solution is *unique* (by the Fredholm alternative), so how do we reconcile this apparent contradiction? Reassuringly, substituting the integral representation of *either* fundamental solution into Green's integral representation yields the same result for $\Omega^{(i)}$.

Proposition 2.1.13 (Integral representation of the solution of the Helmholtz equation) *Let $\lambda = 4\beta^2$. Let $u^{(i)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(i)}$, and $u^{(e)}$ be a solution of* (1.1.1) *in the domain $\Omega^{(e)}$ satisfying the radiation condition at infinity* (2.1.19). *Suppose that $u^{(i)}$ and $u^{(e)}$ have the integral representations* (2.1.20) *and* (2.1.21). *Then $u^{(i)}$ and $u^{(e)}$ have the* alternative *representations*

$$u^{(i)}(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{L_{out\,j}(z)}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\widehat{u^{(i)}}_j(k) + F^{(i)}(z,\bar{z}), \quad z\in\Omega^{(i)}, \tag{2.1.40}$$

$$u^{(e)}(z,\bar{z}) = -\frac{1}{4\pi i}\sum_{j=1}^{n}\int_{L_{out\,j}(z)}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\widehat{u^{(e)}}_j(k) + F^{(e)}(z,\bar{z}), \quad z\in\Omega^{(e)}, \tag{2.1.41}$$

where $L_{out\,j}(z)$, $j=1,...,n$ are rays in the complex k plane, oriented from 0 to ∞, and defined by

$$L_{out\,j}(z) = \begin{cases} L^{(i)}_{out\,j} : \{k\in\mathbb{C} : k = le^{-i\alpha_j}, l\in L_{out}\}, & \text{when } z\in S_j^{(i)},\\ L^{(e)}_{out\,j} : \{k\in\mathbb{C} : k = le^{i(\pi-\alpha_j)}, l\in L_{out}\}, & \text{when } z\in S_j^{(e)},\\ \text{either } L^{(i)}_{out\,j} \text{ or } L^{(e)}_{out\,j}, & \text{when } z\in \overline{S_j^{(e)}}\cap\overline{S_j^{(i)}}\backslash\partial\Omega \end{cases} \tag{2.1.42}$$

where the contour L_{out} is shown in Figure 2.3 and where $S_j^{(i/e)}$ are defined by (2.1.1). The transforms of the boundary values of $u^{(i/e)}$ on the side j, $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$ are given by

$$\widehat{u^{(i/e)}}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'+\frac{\bar{z}'}{k}\right)}[(u_{z'}^{(i/e)} + i\beta k u^{(i/e)})dz' - (u_{\bar{z}'}^{(i/e)} - \frac{\beta}{ik}u^{(i/e)})d\bar{z}'], \qquad (2.1.43)$$

$$= \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'-\frac{\bar{z}'}{k}\right)} \left[i\frac{\partial u^{(i/e)}}{\partial n}(s) + i\beta\left(k\frac{dz'}{ds} - \frac{1}{k}\frac{d\bar{z}'}{ds}\right)u^{(i/e)}(s)\right] ds, \qquad (2.1.44)$$

where $j = 1, .., n$, $z_{n+1} = z_1$, $z' = z'(s) \in S_j$, and n is the outward *pointing normal to the polygon (in* both *interior and exterior cases). The forcing term is given by*

$$F^{(i/e)}(z, \bar{z}) = \iint_{\Omega^{(i/e)}} f(\xi, \eta)\, E_{out}(z', z)\, d\xi d\eta \qquad (2.1.45)$$

where E_{out} is the outgoing fundamental solution for the Helmholtz equation (2.1.12a). A spectral representation of this term is given by

$$F^{(i/e)}(z, \bar{z}) = \frac{1}{4\pi}\left(\int_{L_{out\,L}} \frac{dk}{k} e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)} \widehat{f}_L^{(i/e)}(k) + \int_{L_{out\,R}} \frac{dk}{k} e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)} \widehat{f}_R^{(i/e)}(k)\right) \qquad (2.1.46)$$

where

$$\widehat{f}_{L/R}^{(i/e)}(k) = \iint_{\Omega_{L/R}^{(i/e)}(z)} e^{-i\beta\left(kz'+\frac{\bar{z}'}{k}\right)} f(\xi, \eta) d\xi d\eta, \qquad (2.1.47)$$

and $L_{out\,L}$ and $L_{out\,R}$ are defined by

$$L_{out\,L} = \{k \in \mathbb{C} : k = se^{i(\pi-\theta_F)}, s \in L_{out}\}, \qquad (2.1.48)$$

$$L_{out\,R} = \{k \in \mathbb{C} : k = se^{-i\theta_F}, s \in L_{out}\}. \qquad (2.1.49)$$

Proof This follows in *exactly* the same way as for the modified Helmholtz equation, using of course (2.1.17) instead of (2.1.16). If E_{in} is used in (2.1.20) and (2.1.21) instead of E_{out} then we obtain (2.1.40) and (2.1.41) with L_{out} replaced by L_{in}. For $u^{(e)}$ these solutions are genuinely different (the two $u^{(e)}$s satisfy different radiation conditions at infinity); but, as noted at the beginning of this section $u^{(i)}$ is unique (when $4\beta^2$ is not an eigenvalue). To show that the representation (2.1.40) and the corresponding representation with L_{out} replaced by L_{in} are equivalent, it is necessary to use the *global relation* for the Helmholtz equation in this domain,

$$\sum_{j=1}^{n} \widehat{u^{(i)}}_j(k) + i\left(\widehat{f^{(i)}}_L(k) + \widehat{f^{(i)}}_R(k)\right) = 0, \quad k \in \mathbb{C}. \qquad (2.1.50)$$

This equation is derived in part c) of Proposition 3.1.2, §3.1, below. The two representations differ by

$$\frac{1}{4\pi i}\oint_{\{|k|=1\}}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\left(\sum_{j=1}^{n}\widehat{u^{(i)}}_j(k)+i\left(\widehat{f^{(i)}}_L(k)+\widehat{f^{(i)}}_R(k)\right)\right).$$

Indeed: adding the circle that is oriented anticlockwise to L_{out} transforms L_{out} to L_{in}. Similar considerations apply to the L_{outj}, L_{outL} and L_{outR}, which are just L_{out} rotated by certain angles. Due to the global relation (2.1.50), the above integral vanishes since its integrand is zero. For $u^{(e)}$ there does not exist a corresponding global relation of the form (2.1.50) and this is consistent with the fact that the two solutions are genuinely different.

□

2.1.4.3 Poisson

The representation of the fundamental solution for Poisson (2.1.18) is formal, since the integrals do not converge in the limit of $\varepsilon \to 0$. However, after the domain-dependent fundamental solution (2.1.18) is substituted into Green's IR, the $\varepsilon \to 0$ limit *does* exist because the solutions $u^{(i)}$ and $u^{(e)}$ satisfies the following consistency conditions:

$$\int_{\partial\Omega}\frac{\partial u^{(i)}}{\partial n}ds := \int_{\partial\Omega}u_\xi^{(i)}d\eta - u_\eta^{(i)}d\xi = -\int_{\Omega^{(i)}}fd\xi d\eta, \tag{2.1.51a}$$

$$\int_{\partial\Omega}\frac{\partial u^{(e)}}{\partial n}ds := \int_{\partial\Omega}u_\xi^{(e)}d\eta - u_\eta^{(e)}d\xi = \int_{\Omega^{(e)}}fd\xi d\eta. \tag{2.1.51b}$$

These equations can be obtained by integrating (1.1.1) over $\Omega^{(i/e)}$ and applying Green's theorem (2.1.23) (remember that $\partial\Omega$ is oriented anticlockwise in both cases). These conditions imply that $k = 0$ is a removable singularity.

Proposition 2.1.14 (Integral representation of the solution of Poisson's equation) *Let* $\lambda = 0$. *Let* $u^{(i)}$ *be a solution of* (1.1.1) *in the domain* $\Omega^{(i)}$ *and let* $u^{(e)}$ *be a solution of* (1.1.1) *in the domain* $\Omega^{(e)}$. *Suppose that* $u^{(i)}$ *and* $u^{(e)}$ *have the integral representations* (2.1.20) *and* (2.1.21) *respectively. Then* $u^{(i)}$ *and* $u^{(e)}$ *have the* alternative *representations*

$$u^{(i)}(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j(z)}\frac{dk}{k}e^{ikz}\widehat{u^{(i)}}_j(k) - \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{\overline{l_j}(z)}\frac{dk}{k}e^{-ik\bar{z}}\widetilde{u^{(i)}}_j(k) + F^{(i)}(z,\bar{z}), \tag{2.1.52}$$

for $z \in \Omega^{(i)}$, *and*

$$u^{(e)}(z,\bar z) = -\frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j(z)}\frac{dk}{k}e^{ikz}\widehat{u^{(e)}}_j(k) + \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{\overline{l_j(z)}}\frac{dk}{k}e^{-ik\bar z}\widetilde{u^{(e)}}_j(k) + F^{(e)}(z,\bar z), \tag{2.1.53}$$

for $z \in \Omega^{(e)}$ *respectively. The transforms of the boundary values of* $u^{(i/e)}$ *on the side* j, $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$ *and* $\{\widetilde{u^{(i/e)}}_j(k)\}_1^n$ *are defined by*

$$\widehat{u^{(i/e)}}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz'}\left[\left(u^{(i/e)}_{z'} + iku^{(i/e)}\right)dz' - u^{(i/e)}_{\bar{z'}}d\bar{z'}\right], \quad j=1,..,n, \quad z_{n+1}=z_1, \tag{2.1.54}$$

$$= \int_{z_j}^{z_{j+1}} e^{-ikz'}\left[i\frac{\partial u^{(i/e)}}{\partial n}(s) + ik\frac{dz'}{ds}u^{(i/e)}(s)\right]ds, \quad z'=z'(s)\in S_j$$

and by

$$\widetilde{u^{(i/e)}}_j(k) = \int_{z_j}^{z_{j+1}} e^{ik\bar z}\left[(u^{(i/e)}_{\bar z} - iku^{(i/e)})d\bar z - u^{(i/e)}_z dz\right], \quad j=1,..,n, \quad z_{n+1}=z_1, \tag{2.1.55}$$

$$= \int_{z_j}^{z_{j+1}} e^{ik\bar{z'}}\left[-i\frac{\partial u^{(i/e)}}{\partial n}(s) - ik\frac{d\bar{z'}}{ds}u^{(i/e)}(s)\right]ds, \quad z'=z'(s)\in S_j$$

The contours $l_j(z)$, $j=1,...,n$ *are the same as in Proposition 2.1.10, i.e. given by* (2.1.30), *and* $\overline{l_j(z)}$, $j=1,...,n$, *are the complex conjugates of* $l_j(z)$, $j=1,...,n$, *that is, the rays in the complex* k*-plane oriented toward infinity defined by*

$$\overline{l_j(z)} = \begin{cases} \overline{l_j^{(i)}} : \{k\in\mathbb{C} : \arg k = \alpha_j\}, & \textit{when } z\in S_j^{(i)}, \\ \overline{l_j^{(e)}} : \{k\in\mathbb{C} : \arg k = \pi+\alpha_j\}, & \textit{when } z\in S_j^{(e)}, \\ \textit{either } \overline{l_j^{(i)}} \textit{ or } \overline{l_j^{(e)}}, & \textit{when } z\in \overline{S_j^{(e)}}\cap\overline{S_j^{(i)}}\backslash\partial\Omega \end{cases} \tag{2.1.56}$$

where $S_j^{(i/e)}$ *are defined by* (2.1.1).

The forcing term is given by

$$F^{(i/e)}(z,\bar z) = \iint_{\Omega^{(i/e)}} f(\xi,\eta)\,E(z',z)\,d\xi d\eta \tag{2.1.57}$$

where E *is the fundamental solution for the Poisson equation. A spectral representation of this term is*

$$F(z,\bar z) = \frac{1}{4\pi}\left(\int_{l_L}\frac{dk}{k}e^{ikz}\widehat{f}_L^{(i/e)}(k) + \int_{\overline{l_L}}\frac{dk}{k}e^{-ik\bar z}\tilde{f}_L^{(i/e)}(k) + \int_{l_R}\frac{dk}{k}e^{ikz}\widehat{f}_R^{(i/e)}(k) + \int_{\overline{l_R}}\frac{dk}{k}e^{-ik\bar z}\tilde{f}_R^{(i/e)}(k)\right) \tag{2.1.58}$$

where

$$\widehat{f}^{(i/e)}_{L/R}(k)=\iint_{\Omega^{(i/e)}_{L/R}} e^{-ikz'}f(\xi,\eta)d\xi d\eta,\quad \widetilde{f}^{(i/e)}_{L/R}(k)=\iint_{\Omega^{(i/e)}_{L/R}} e^{ik\bar{z}'}f(\xi,\eta)d\xi d\eta, \tag{2.1.59}$$

with $\Omega^{(i/e)}_{L/R}$ *given by* (2.1.36), l_L, l_R *given by* (2.1.37), *and with* $\bar{l}_L$ *and* $\bar{l}_R$ *given by the complex conjugates of* l_L *and* l_R *respectively, i.e.* $\bar{l}_L$ *is the ray from* 0 *to* ∞ *with* $\arg k = -\pi+\theta_F$ *and similarly for* $\bar{l}_R$*.*

Remark 2.1.15 (Removable singularity at $k=0$ and consistency conditions) *In the integral representations* (2.1.52) *and* (2.1.53) *there appear to be poles at* $k=0$ *which would mean that the integrals are not well-defined. However, the consistency conditions* (2.1.51) *imply that*

$$-i\sum_{j=1}^{n}\widehat{u^{(i)}_j}(0)=i\sum_{j=1}^{n}\widetilde{u^{(i)}_j}(0)=-\left(\widehat{f^{(i)}_L}(0)+\widehat{f^{(i)}_R}(0)\right)=-\left(\widetilde{f^{(i)}_L}(0)+\widetilde{f^{(i)}_R}(0)\right), \tag{2.1.60}$$

$$-i\sum_{j=1}^{n}\widehat{u^{(e)}_j}(0)=i\sum_{j=1}^{n}\widetilde{u^{(e)}_j}(0)=\left(\widehat{f^{(e)}_L}(0)+\widehat{f^{(e)}_R}(0)\right)=\left(\widetilde{f^{(e)}_L}(0)+\widetilde{f^{(e)}_R}(0)\right), \tag{2.1.61}$$

and due to these conditions, when the integrals are grouped together, the singularity at $k=0$ *is removable.*

Remark 2.1.16 (Short-cut when solving BVPs for the Poisson equation) *If the boundary values of* $u^{(i/e)}$ *are real, the second term of* (2.1.52) *is the complex conjugate of the first term, and similarly for* (2.1.52)*. If the forcing* f *is real then the second term of* (2.1.58) *is the complex conjugate of the first, and the fourth the complex conjugate of the third. (Thus, if both the boundary values and the forcing are real then* u *is real.) This provides a useful shortcut when solving boundary value problems with this integral representation - we can* assume *that* u *is real, work with half the terms, and then recover the other terms by complex conjugation. The final answer is then valid even if* u *is not real.*

Proof of Proposition 2.1.14 This is identical to the proof of prop. 2.1.10 using (2.1.18) instead of (2.1.16), and also letting $\varepsilon \to 0$ at the end. This limit exists due to remark 2.1.15. $\square$

2.2 Polar co-ordinates

Notations Let (r, θ) be the physical variable, and (ρ, ϕ) be the "dummy variable" of integration.

We will consider only the Poisson equation, $\lambda = 0$, and the Helmholtz equation, $\lambda = \beta^2$. (Note that this is different to $\lambda = 4\beta^2$ used for the Helmholtz equation in polygonal domains in the previous section.)

2.2.1 Co-ordinate system dependent fundamental solutions

In polar co-ordinates (1.2.2) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E}{\partial\phi^2} + \beta^2 E = -\frac{\delta(\rho - r)\delta(\phi - \theta)}{\rho}, \quad 0 < r, \rho < \infty, \quad \theta, \phi \ 2\pi \text{ periodic.} \tag{2.2.1}$$

For reasons which will be explained later (see Remark 2.2.4), consider the *non-periodic* fundamental solution, denoted by E_s, defined by

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_s}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_s}{\partial\phi^2} + \beta^2 E_s = -\frac{\delta(\rho - r)\delta(\phi - \theta)}{\rho}, \quad 0 < r, \rho < \infty, \quad -\infty < \theta, \phi < \infty. \tag{2.2.2}$$

The outgoing radiation condition for the Helmholtz equation ($\beta \neq 0$) is

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - i\beta u\right) \to 0 \text{ as } r \to \infty. \tag{2.2.3}$$

Proposition 2.2.1 (Integral representation of E_s for the Helmholtz equation) *For the the Helmholtz equation equation, the outgoing non-periodic fundamental solution E_s can be expressed in terms of radial eigenfunctions (*the radial representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} \frac{i}{4}\Bigg(\int_0^{i\infty} dk\, e^{\varepsilon k^2} H_k^{(1)}(\beta\rho) J_k(\beta r) e^{ik|\theta - \phi|} \tag{2.2.4}$$

$$+ \int_0^{-i\infty} dk\, e^{\varepsilon k^2} H_k^{(1)}(\beta\rho) J_k(\beta r) e^{-ik|\theta - \phi|} \Bigg). \tag{2.2.5}$$

*Alternatively, it can be expressed in terms of angular eigenfunctions (*the angular representation*) in the form*

$$E_s(\rho,\phi;r,\theta) = \frac{i}{4}\left(\int_0^\infty dk\, H_k^{(1)}(\beta r_>)J_k(\beta r_<)e^{ik(\theta-\phi)} + \int_0^\infty dk\, H_k^{(1)}(\beta r_>)J_k(\beta r_<)e^{-ik(\theta-\phi)}\right), \tag{2.2.6}$$

where $r_> = \max(r,\rho)$*,* $r_< = \min(r,\rho)$ *and* $-\infty < \theta, \phi < \infty$*.*

Proof The differential operator

$$-\frac{d^2}{d\theta^2}u = \lambda u \tag{2.2.7}$$

on $(-\infty,\infty)$ possesses the completeness relation

$$\delta(\theta-\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_2(\theta-\phi)}dk_2, \tag{2.2.8}$$

i.e. the Fourier transform([Sta67, Chapter4]). The completeness relation associated with the operator

$$-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) - \beta^2\rho u = \lambda\frac{u}{\rho} \tag{2.2.9}$$

on $0 < \rho < \infty$, with the additional condition that the eigenfunctions satisfy the *outgoing* radiation condition, is

$$\rho\delta(r-\rho) = \lim_{\varepsilon\to 0}\frac{1}{2}\int_{-i\infty}^{i\infty} dk_1\, k_1\, e^{\varepsilon k_1^2}\, H_{k_1}^{(1)}(\beta r_1)J_{k_1}(\beta r_2), \tag{2.2.10}$$

where either $r_1 = r, r_2 = \rho$ or vice versa, see Remark 2.2.2. The completeness relations (2.2.8) and (2.2.10) give rise to the following integral representation of E:

$$E_s(\rho,\phi;r,\theta) = \lim_{\varepsilon\to 0} -\frac{1}{4\pi}\int_{-i\infty}^{i\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \frac{e^{ik_2(\theta-\phi)}\, k_1\, e^{\varepsilon k_1^2}\, H_{k_1}^{(1)}(\beta r_1)J_{k_1}(\beta r_2)}{k_1^2 - k_2^2}. \tag{2.2.11}$$

Choose $r_1 = \rho$ and $r_2 = r$, so that E_s satisfies the outgoing radiation condition in ρ. (In fact, r and ρ can be interchanged in the right hand side of (2.2.5) using an identity involving integrals of Bessel functions, equation (4.3.12) below.) If $\theta > \phi$, close the k_2 integral in the upper half k_2 plane, enclosing the pole at $k_2 = k_1$ for $\Im k_1 \geq 0$ and at $k_2 = -k_1$ for $\Im k_1 \leq 0$. Relabel k_1 as k to obtain (2.2.5) for $\theta > \phi$. Similarly, if $\theta < \phi$, close in the lower half k_2 plane to obtain (2.2.5) for $\theta < \phi$.

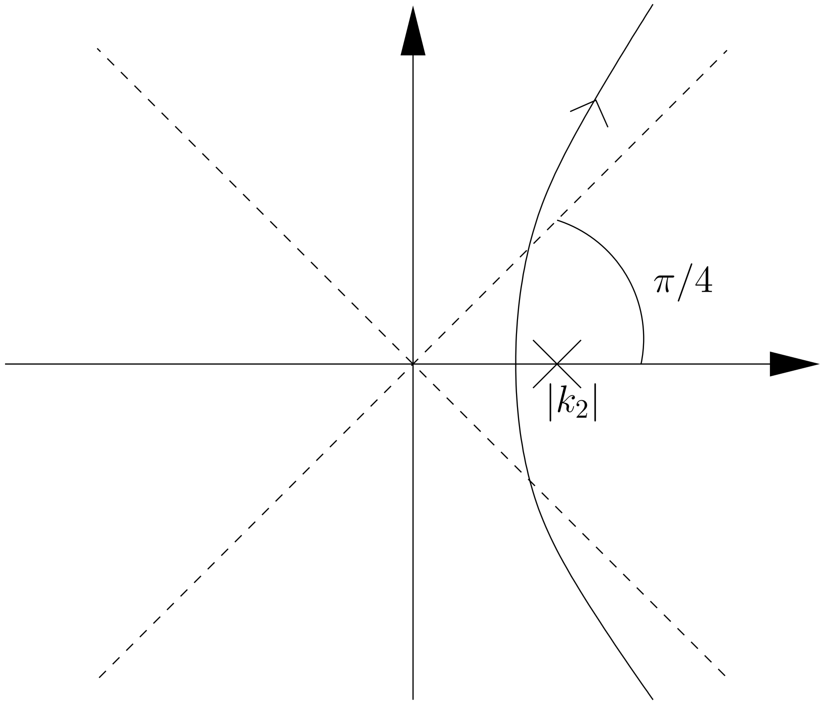


Figure 2.8: The deformed contour in the k_1 plane.

Evaluating the k_1 integral requires knowledge of the asymptotics of the product of the Bessel functions as $k \to \infty$. For x, y fixed, the following formulae are valid:

$$kJ_k(y)H_k^{(1)}(x) \sim -\frac{i}{\pi}\left(\frac{y}{x}\right)^k, \qquad |k| \to \infty,\ -\frac{\pi}{2} < \arg k < \frac{\pi}{2}, \tag{2.2.12}$$

$$kJ_k(y)H_k^{(1)}(x) \sim -2i\frac{\sin k\pi}{\pi}e^{-i\pi k}\left(\frac{-2k}{ex}\right)^{-k}\left(\frac{-2k}{ey}\right)^{-k}, \qquad |k| \to \infty,\ \frac{\pi}{2} < \arg k < \frac{3\pi}{2} \tag{2.2.13}$$

(these can be established using standard results about the asymptotics of Bessel functions of large order, see e.g. [AS65, §9.3], [Wat66]). Hence the product $kJ_k(y)H_k^{(1)}(x)$ is bounded at infinity only when $\Re k > 0$ and $x > y$.

For $r_1 > r_2$ deform the k_1 contour to some contour in the right half plane, enclosing the pole at $k_1 = |k_2|$, with $\frac{\pi}{4} < \arg k_1 < \frac{\pi}{2}$ as $|k| \to \infty$ for $\Im k_1 > 0$ and $-\frac{\pi}{2} < \arg k_1 < -\frac{\pi}{4}$ as $|k| \to \infty$ for $\Im k_1 < 0$, see Figure 2.8. Now the k_1 integral converges absolutely even with $\varepsilon = 0$, so by the dominated convergence theorem ε can be set to zero in the integrand.

After calculating the residue, this results in

$$E_s(\rho,\phi;r,\theta) = \frac{i}{4}\int_{-\infty}^{\infty} dk_2\, H^{(1)}_{|k_2|}(\beta r_>)J_{|k_2|}(\beta r_<)e^{ik_2(\theta-\phi)}.$$

Relabel k_2 as k and let $k \mapsto -k$ for $k < 0$ to obtain (2.2.5). The proof that (2.2.5) and (2.2.6) define the same function follows by deforming the integrals on $(0, i\infty)$ and $(0, -i\infty)$ to $(0, \infty)$ using (2.2.12) and the fact, mentioned earlier, that r and ρ can be interchanged in the right hand side of (2.2.5). □

Remark 2.2.2 (The Kontorovich-Lebedev transform) *Spectral analysis of* (2.2.9) *yields*

$$g(k) = \int_0^{\infty} dy f(y) H_k^{(1/2)}(y), \tag{2.2.14}$$

$$xf(x) = \pm\frac{1}{2}\int_{-i\infty}^{i\infty} dk\, k\, J_k(x) g(k), \tag{2.2.15}$$

with $H_k^{(1)}$ *and the plus sign in equation* (2.2.15) *if the eigenfunctions satisfy the* outgoing *radiation condition* (2.2.3) *and* $H_k^{(2)}$ *and the minus sign if the eigenfunctions satisfy the* incoming *radiation condition (i.e.* (2.2.3) *with the sign of the second term changed). D.S. Jones produced a counterexample of the very "nice" function* e^{-ay}, $\Re a > 0$, *for which the inversion integral* (2.2.15) *diverged, [Jon80]. Essentially, the reason for this is that the product of* H_k *and* J_k *is unbounded on* $i\mathbb{R}$. *However, most of the solutions to BVPs obtained by using* (2.2.15) *are correct because the contour is deformed (albeit illegally) and the resulting expression converges. Jones proposed the inversion integral*

$$xf(x) = \pm\lim_{\varepsilon\to 0}\frac{1}{2}\int_{-i\infty}^{i\infty} dk\, e^{\varepsilon k^2} k\, J_k(x) g(k), \tag{2.2.16}$$

and proved the validity of the resulting transform pair rigorously. The $e^{\varepsilon k^2}$ *term in the new inversion formula justifies rigorously the contour deformation previously used, after which* ε *can be set to zero.*

Reciprocity in r *and* ρ *in* (2.2.10) *follows from properties of the delta function on the left hand side. To prove it using only the right hand side, the following identity allows the swapping of the arguments of the* H_k *and* J_k*:*

$$\int_{-i\infty}^{i\infty} dk\, J_k(x) H_k^{(1)}(y) Q(k) = \int_{-i\infty}^{i\infty} dk\, J_k(y) H_k^{(1)}(x) Q(k) \tag{2.2.17}$$

where $Q(k) = -Q(-k)$. To prove this identity, expand the H_k as a linear combination of J_k and J_{-k} (from its definition), and then let $k \mapsto -k$ in the term involving J_{-k}. This identity is used when solving problems using the Kontorovich-Lebedev transform [Jon80], [Jon86, §9.19, page 587], and we will use a similar identity when solving a BVP for the Helmholtz equation later in §4.3.

Proposition 2.2.3 (Integral representation of E_s for Poisson) *For the the Poisson equation equation the non-periodic fundamental solution E_s can be expressed formally either in terms of radial eigenfunctions (*the radial representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(\int_{\varepsilon}^{i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{ik|\theta-\phi|} + \int_{\varepsilon}^{-i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{-ik|\theta-\phi|} \right), \tag{2.2.18}$$

*or in terms of angular eigenfunctions (*the angular representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(\int_{i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_>}{r_<}\right)^{-k} e^{ik(\theta-\phi)} + \int_{-i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_>}{r_<}\right)^{-k} e^{-ik(\theta-\phi)} \right), \tag{2.2.19}$$

where $r_> = \max(r, \rho)$, $r_< = \min(r, \rho)$, $-\infty < \theta, \phi < \infty$.

Proof The operator in θ is the same as for the Helmholtz equation, namely (2.2.7), and so the appropriate completeness relation is (2.2.8). The differential operator

$$-\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = \lambda \frac{u}{\rho}$$

on $0 < \rho < \infty$, possesses the completeness relation

$$\rho\, \delta(r - \rho) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{\rho}{r}\right)^{k_1} dk_1, \tag{2.2.20}$$

i.e. the Mellin transform([Sta67, Chapter4, p.308]). After rewriting (2.2.2) in the form

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial E_s}{\partial \rho}\right) + \frac{\partial^2 E_s}{\partial \phi^2} = -\rho\, \delta(r - \rho) \delta(\theta - \phi),$$

the above two completeness relations give rise to the following integral representation of E:

$$E_s(\rho, \phi; r, \theta) = -\frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \frac{e^{ik_2(\theta-\phi)}}{k_1^2 - k_2^2} \left(\frac{\rho}{r}\right)^{k_1}. \tag{2.2.21}$$

If $\theta > \phi$, deform the k_1 integral off $i\mathbb{R}$ near the origin so that it passes through $\varepsilon \neq 0$. Then close the k_2 integral in the upper half k_2 plane, enclosing the pole at $k_2 = k_1$ for

$\Im k_1 \geq 0$ and at $k_2 = -k_1$ for $\Im k_1 \leq 0$. Relabel k_1 as k to obtain (2.2.18), without the limit, for $\theta > \phi$. Similarly, if $\theta < \phi$, close in the lower half k_2 plane to obtain (2.2.18), without the limit, for $\theta < \phi$.

If $\rho > r$, deform the k_2 integral off $\mathbb{R}$ near the origin so that it passes through $i\varepsilon$, $\varepsilon \neq 0$. Then close the k_1 integral in the left half k_1 plane enclosing the pole at $k_1 = -k_2$ for $\Re k_2 \geq 0$ and at $k_1 = k_2$ for $\Re k_2 \leq 0$. Let $k_2 \mapsto -k_2$ for $\Re k_1 \leq 0$ and relabel k_2 as k to obtain (2.2.19) for $r > \rho$. Similarly, if $\rho < r$, close in the right half k_1 plane to obtain (2.2.19) for $\rho < r$ without the limit. The proof that (2.2.18) and (2.2.19) define the same function follows by contour deformation in a similar manner to the analogous proof for the Helmholtz equation, except that the differences become zero in the limit $\varepsilon \to 0$. $\square$

Remark 2.2.4 (Periodicity) *For polar co-ordinates, periodicity in the angle co-ordinate is more natural. The operator* (2.2.7) *on* $(0, 2\pi)$ *with* θ *periodic has the associated completeness relation*

$$\delta(\theta - \phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}. \tag{2.2.22}$$

Using this, the analogues of (2.2.21) *and* (2.2.11) *are*

$$E(\rho, \phi; r, \theta) = -\frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} dk_1 \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\phi)}}{k_1^2 - n^2} \left(\frac{\rho}{r}\right)^{k_1} \tag{2.2.23}$$

and

$$E(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int_{-i\infty}^{i\infty} dk_1 \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\phi)}\, e^{\varepsilon k_1^2}\, k_1\, H_{k_1}^{(1)}(\beta r_1) J_{k_1}(\beta r_2)}{k_1^2 - n^2} \tag{2.2.24}$$

and as before the k_1 *integrals can be computed to yield the angular expansions*

$$E(\rho, \phi; r, \theta) = -\frac{\log r_>}{2\pi} + \frac{1}{4\pi} \sum_{n \neq 0} e^{in(\theta-\phi)} \frac{1}{|n|} \left(\frac{r_<}{r_>}\right)^{|n|} \tag{2.2.25}$$

and

$$E(\rho, \phi; r, \theta) = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(\beta r_>) J_n(\beta r_<) e^{in(\theta-\phi)} \tag{2.2.26}$$

which are the analogues of (2.2.19) *and* (2.2.5). *However, the sum in* n cannot *be computed and so there do* not *exist analogues of the representations* (2.2.18), (2.2.5). *Another* *way to see this fact is to try and obtain the "radial" representations by taking the appropriate transform in ρ of* (2.2.1) *(see Remark 2.1.4): For Poisson the Mellin transform in ρ of* (2.2.1) *yields*

$$\frac{d^2\widetilde{E}}{d\phi^2}(k,\phi) + k^2\widetilde{E}(k,\phi) = -\delta(\phi-\theta)r^k.$$

For solutions periodic in ϕ we need $k \in \mathbb{Z}$ (instead of $k \in \mathbb{R}$), but under this restriction the Mellin transform cannot *be inverted.*

The fact that there do not exist radial representations under periodicity implies that there do not exist IRs in the transform space with periodicity in the angular variable for any domains other than the interior and exterior of the circle. The so-called "hybrid method" of [DF08] for solving boundary value problems in the interior and exterior of the sphere, uses the analogue of (2.2.26) *in 3-d to obtain an IR in the transform space (the construction of the GR and the solution of boundary value problems in 3-d follow steps similar to the method in 2-d described in this thesis).*

Remark 2.2.5 (Derivation of both co-ordinate dependent fundamental solutions using only one completeness relation) *In the proof of Proposition 2.2.1 it was shown that* (2.2.5) *and* (2.2.6) *can be obtained from each other by contour deformation. Combining this with Remark 2.1.4 shows that both* (2.2.5) *and* (2.2.6) *can actually be obtained using either one of the completeness relations* (2.2.8) *or* (2.2.14)*, although this approach is less algorithmic than the approach in the proof of Proposition 2.2.1.*

2.2.2 Green's integral representation

In this thesis we only consider one domain in polar co-ordinates, the exterior of a circle, and so for simplicity we only formulate Green's integral representation and the new integral representation in this domain. Since we use the non-periodic fundamental solution E_s we must consider the exterior of a circle as the domain

$$D = \{a < r < \infty, \quad 0 < \theta < \alpha\}, \tag{2.2.27}$$

see Gigure 2.9, with $\alpha = 2\pi$ and prescribe boundary conditions so that the sides $\theta = 0$ and $\theta = 2\pi$ are the same.

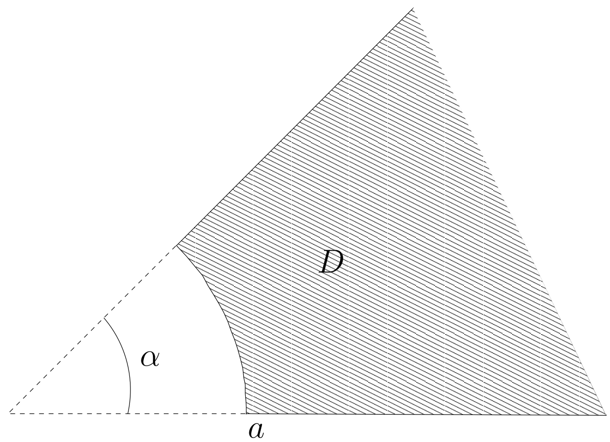


Figure 2.9: The domain D (shaded).

Theorem 2.2.6 (Green's Integral Representations) *Let u be the solution of* (1.1.1) *for $\Omega = D$ where D is given by* (2.2.27) *see Figure 2.9, with the boundary conditions at infinity:*

- $\lambda = 0$ *(Poisson), $u \to 0$ as $r \to \infty$.*
- $\lambda = \beta^2$ *(Helmholtz), u satisfies* (2.2.3).

Assume also that f has compact support (so that $\int_D f(\rho,\phi)\,\rho\,d\rho\,d\phi$ is well defined). Then u admits the following integral representation:

$$\begin{aligned} u(r,\theta) = & \int_a^\infty \frac{d\rho}{\rho}\left(u\frac{\partial E_s}{\partial\phi} - E_s\frac{\partial u}{\partial\phi}\right)(\rho,0) - \int_a^\infty \frac{d\rho}{\rho}\left(u\frac{\partial E_s}{\partial\phi} - E_s\frac{\partial u}{\partial\phi}\right)(\rho,\alpha) \\ & -a\int_0^\alpha d\phi\left(E_s\frac{\partial u}{\partial\rho} - u\frac{\partial E_s}{\partial\rho}\right)(a,\phi) + \iint_D f(\rho,\phi)E_s(\rho,\phi;r,\theta)\,\rho\,d\rho\,d\phi. \end{aligned} \tag{2.2.28}$$

Proof (Standard) In polar co-ordinates the equations satisfied by u and E_s are

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} + \beta^2 u = -f(\rho,\phi) \tag{2.2.29}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_s}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 E_s}{\partial\phi^2} + \beta^2 E_s = -\frac{\delta(\rho-r)\delta(\phi-\theta)}{\rho}$$

Multiple the first by ρE_s, the second by ρu and subtract to obtain the divergence form of the PDE:

$$\left(\rho\frac{\partial u}{\partial\rho}E_s - \rho\frac{\partial E_s}{\partial\rho}u\right)_\rho + \left(\frac{1}{\rho}\frac{\partial u}{\partial\phi}E_s - \frac{1}{\rho}\frac{\partial E_s}{\partial\phi}u\right)_\phi = u(\rho,\phi)\,\delta(\rho-r)\delta(\phi-\theta) - f\,\rho\,E_s. \tag{2.2.30}$$

Integrate over D and use Green's theorem (2.1.23) to obtain the integral representation

$$u(r,\theta) = \int_{\partial D} \rho\left(E_s\frac{\partial u}{\partial \rho} - u\frac{\partial E_s}{\partial \rho}\right) d\phi + \frac{1}{\rho}\left(u\frac{\partial E_s}{\partial \phi} - E_s\frac{\partial u}{\partial \phi}\right) d\rho + \iint_D f(\rho,\phi)E_s(\rho,\phi;r,\theta)\,\rho\,d\rho\,d\phi. \tag{2.2.31}$$

Parametrise the sides of D by $\{\phi = \alpha, \infty > \rho > a\}$, $\{\rho = a, \alpha > \phi > 0\}$, and $\{\phi = 0, 0 < \rho < \infty\}$ to yield (2.2.28). □

2.2.3 The novel integral representation

The novel integral representation follows from Green's integral representation (2.2.28) using the representations of the fundamental solutions of propostions 2.2.3 and 2.2.1.

Proposition 2.2.7 *Let u be the solution of* (1.1.1) *for $\Omega = D$ where D is given by* (2.2.27)*. Suppose that u has the integral representation* (2.2.28)*. Then u has the* alternative *representations*

- $\lambda = 0$ *(Poisson),*

$$\begin{aligned} u(r,\theta) = \frac{1}{4\pi i}\Bigg(&\int_0^{i\infty} dk\, r^{-k}e^{ik\theta}\left[D_0(k) + \frac{1}{ik}N_0(k)\right] \\ &- \int_0^{-i\infty} dk\, r^{-k}e^{-ik\theta}\left[D_0(k) - \frac{1}{ik}N_0(k)\right] \\ &\int_0^{i\infty} dk\, r^{-k}e^{-ik\theta}e^{ik\alpha}\left[D_\alpha(k) - \frac{1}{ik}N_\alpha(k)\right] \\ &- \int_0^{-i\infty} dk\, r^{-k}e^{ik\theta}e^{-ik\alpha}\left[D_\alpha(k) + \frac{1}{ik}N_\alpha(k)\right]\Bigg) \\ -\frac{1}{4\pi}\Bigg(&\int_0^{\infty} dk e^{ik\theta}\left(\frac{r}{a}\right)^{-k}\left[\frac{a}{k}N(-ik) - D(-ik)\right] \\ &\int_0^{\infty} dk e^{-ik\theta}\left(\frac{r}{a}\right)^{-k}\left[\frac{a}{k}N(ik) - D(ik)\right]\Bigg) \\ &+ \iint_\Omega d\rho\, d\phi\, \rho\, f(\rho,\phi)E_s(\rho,\phi;r,\theta) \end{aligned} \tag{2.2.32}$$

where

$$D_\chi(k) = \int_a^\infty \frac{d\rho}{\rho}\rho^k\, u(\rho,\chi), \quad N_\chi(k) = \int_a^\infty \frac{d\rho}{\rho}\rho^k\, u_\theta(\rho,\chi), \quad \chi = 0 \text{ or } \alpha, \tag{2.2.33}$$

and

$$D(\pm ik) = \int_0^\alpha d\phi\, e^{\pm ik\phi}\, u(a,\phi), \quad N(\pm ik) = \int_0^\alpha d\phi\, e^{\pm ik\phi}\, u_r(a,\phi), \tag{2.2.34}$$

- $\lambda = \beta^2$ *(Helmholtz),*

$$\begin{aligned} u(r,\theta) = \lim_{\varepsilon\to 0} \frac{i}{4} \Bigg(& \int_0^{i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) e^{ik\theta} \left[-ikD_0(k) - N_0(k)\right] \\ & + \int_0^{-i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) e^{-ik\theta} \left[ikD_0(k) - N_0(k)\right] \\ & - \int_0^{i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) e^{-ik\theta} e^{ik\alpha} \left[ikD_\alpha(k) - N_\alpha(k)\right] \\ & - \int_0^{-i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) e^{ik\theta} e^{-ik\alpha} \left[-ikD_\alpha(k) - N_\alpha(k)\right] \Bigg) \\ -\frac{ia}{4} \Bigg(& \int_0^\infty dk e^{ik\theta} H_k^{(1)}(\beta r) \Big[J_k(\beta a) N(-ik) - \beta J_k'(\beta a) D(-ik)\Big] \\ & + \int_0^\infty dk e^{-ik\theta} H_k^{(1)}(\beta r) \Big[J_k(\beta a) N(ik) - \beta J_k'(\beta a) D(ik)\Big] \Bigg) \\ & + \iint_\Omega d\rho\, d\phi\, \rho\, f(\rho,\phi) E_s(\rho,\phi; r,\theta) \end{aligned} \tag{2.2.35}$$

where

$$D_\chi(k) = \int_a^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho)\, u(\rho,\chi), \quad N_\chi(k) = \int_a^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho)\, u_\theta(\rho,\chi), \quad \chi = 0 \text{ or } \alpha, \tag{2.2.36}$$

and $D(\pm ik), N(\pm ik)$ *are given by* (2.2.34),

and where E_s *is the non-periodic fundamental solution for Poisson and the Helmholtz equation respectively. A spectral representation of the forcing term can be obtained by using either the radial* (2.2.18),(2.2.5) *or the angular* (2.2.19),(2.2.6), *representations of* E_s *for the Poisson equation and the Helmholtz equation respectively.*

Proof On $\{\phi = \alpha, \infty > \rho > a\}$, use the radial representations of E_s (2.2.18) and (2.2.5) with $\theta < \phi$ for the Poisson equation and the Helmholtz equation respectively. On $\{\rho = a, \alpha > \phi > 0\}$ use the angular representations of E_s (2.2.19) and (2.2.6) with $r > a$ for the Poisson equation and the Helmholtz equation respectively. On $\{\phi = 0, 0 < \rho < \infty\}$, use the radial representations of E_s (2.2.18) and (2.2.5) with $\theta > \phi$ for the Poisson equation and the Helmholtz equation respectively. For the Poisson equation the singularity at

$k=0$ on the contours of integration is *removable* (after using a spectral representation of the forcing term). Like the cartesian case, this is due to the consistency condition

$$\int_{\partial D} r\frac{\partial u}{\partial r}d\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}dr + \iint_D f(r,\theta)rdrd\theta = 0$$

which written in terms of the transform of the boundary values is

$$-N_0(0) + N_\alpha(0) - aN(0) = -\iint_D d\rho\, d\phi\, \rho\, f(\rho,\phi),$$

this is the GR for this problem, (3.2.6), evaulated at $k=0$. □

2.3 Relation to classical results

The idea of obtaining integral representations of the fundamental solution E is certainly not new. However, the representations presented here contain certain novel features which are stated in this section, and contrasted with the classical representations.

2.3.1 Cartesian

The integral representations of the different fundamental solution, (2.1.6),(2.1.14), and (2.1.12) differ from the classical in two novel ways:

- rotation to half-planes,
- change of variables to eliminate square roots,

which we now explain.

Rotation to half-planes Starting with (2.1.5), one of the following operations is usually performed:

1. Rotate the k_1, k_2 co-ordinates so k_2 lies in direction of $(x-\xi, y-\eta)$; for the modified Helmholtz equation this yields

$$E(\xi,\eta,x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} dk_1 dk_2 \frac{e^{ik_2\sqrt{(x-\xi)^2+(y-\eta)^2}}}{k_1^2+k_2^2+4\beta^2}.$$

   Computing the k_2 integral, this becomes

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk_1 \frac{e^{-\sqrt{k_1^2+4\beta^2}\sqrt{(x-\xi)^2+(y-\eta)^2}}}{\sqrt{k_1^2+4\beta^2}}. \qquad (2.3.1)$$

   Such integral representations appear in [Sta68, p.56-57 and 279] and [AS65, equations 9.6.23, 9.6.24, p. 376]

2. Compute the k_2 integral without first performing a rotation; for the modified Helmholtz equation this yields

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ik_1(x-\xi)-\sqrt{k_1^2+4\beta^2}|y-\eta|}}{\sqrt{k_1^2+4\beta^2}} dk_1,$$

   which is (2.1.10) with $\theta = 0$ in (2.1.7). This appears in [AF03, p.298]; the analogue for Helmholtz, which is (2.1.13) with $\theta = 0$ in (2.1.7), appears as equation (5.1.22) of [Duf01, p. 278] (actually, the equation (5.1.22) has a error in the sign of the square root in the exponent, but the equation (5.1.21), from which it is obtained, is correct.)

Change of variables to eliminate square roots The transformations $k_T = \beta(l-1/l)$ for the modified Helmholtz equation and $k_T = \beta(l+1/l)$ for the Helmholtz equation, take $\sqrt{4\beta^2+k_T^2}$ and $\sqrt{k_T^2-4\beta^2}$ into $\beta(l+1/l)$ and $\beta(l-1/l)$ respectively (modulo a sign depending on the range of l). These transformations eliminate the square roots at the cost of introducing a pole at $l=\infty$. These transformations have been used earlier but only in polar co-ordinates, in particular using them in the two analogues of (2.3.1) for the Helmholtz equation yields the IRs for $H_0^{(1)}$ and $H_0^{(2)}$ involving the contours of integration L_{out} and L_{in} respectively, see [Wat66, §6.21p. 179]. It is surprising that apparently these transformations have *not* been used before in the solution of the Helmholtz equation in cartesian co-ordinates. Indeed, [OHLM03, §5.8.3 page 191] states that “in a half plane transform methods for the Helmholtz equation equation ... are cursed by the presence of branch points in the transform plane".

It is important to note that it is not possible to obtain the integral representations of propositions 2.1.10-2.1.14 using the representations of E of the form (2.3.1), without first transforming (2.3.1) into the representations of propositions 2.1.1-2.1.3. This is because the integrand of (2.3.1) cannot be written as a function of (x, y) multiplied by a function of (ξ, η), and thus it is not possible to interchange the physical and spectral integrals when (2.3.1) is substituted into Green's integral representation.

2.3.2 Polar

The novel idea in section §2.2.1 is **since a radial representation is impossible under periodicity, consider the non-periodic fundamental solution, E_s, instead of E.** As discussed in the introduction §1.2.3, Sommerfeld was led to consider E_s for different reasons, and the angular representation for E_s for the Helmholtz equation (2.2.6) appears in [Sta68, p. 268]. However, perhaps due to the unfamiliarity with the Kontorovich-Lebedev transform, the radial representation (2.2.5) does not appear to be known.

The periodic angular representations (2.2.25),(2.2.26) and their analogues in 3d are well known, e.g. [MF53, vol 1 p.827], and the authors of [MF53] are also aware of the non-availability of a radial representation (vol. 1 p.829).

Chapter 3

The global relation

Summary:

- The global relation (GR) is Green's divergence form of the equation, integrated over the domain, with particular solutions of the (homogeneous) adjoint equation replacing the fundamental solution.

- Separation of variables gives a one-parameter family of solutions to the adjoint equation depending on the parameter $k \in \mathbb{C}$ (the separation constant).

- When Ω is infinite, k must be restricted so that the integral on the boundary at infinity is zero.

- The GR is useful because it involves the transforms of the boundary values appearing in the IRs, and so gives information about the unknown transforms. In some cases this information can be used to eliminate the unknowns from the IR and hence find the solution to the boundary value problem.

This chapter is concerned with the second ingredient of the Fokas method: the global relation. In §3.1 we consider polygonal domains and §3.2 we consider domains in polar co-ordinates.

3.1 Polygons

First consider bounded domains.

3.1.1 The GR for bounded domains

Proposition 3.1.1 *Let u be a solution of* (1.1.1) *for Ω some bounded domain D. Let v be any solution of the adjoint of* (1.1.1), *that is,*

$$\Delta v(\boldsymbol{x}) + \lambda v(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D \tag{3.1.1}$$

Then

$$\int_{\partial D} u\,(v_\eta d\xi - v_\xi d\eta) - v\,(u_\eta d\xi - u_\xi d\eta) = -\int_D f\,v\,d\xi d\eta; \tag{3.1.2}$$

or

$$\int_{\partial D} u\,(v_{z'}dz' - v_{\bar{z}'}d\bar{z}') - v\,(u_{z'}dz' - u_{\bar{z}'}d\bar{z}') = i\int_D f\,v\,d\xi d\eta \tag{3.1.3}$$

in complex co-ordinates.

Proof In two dimensions the equations satisfied by u and E are

$$u_{\xi\xi} + u_{\eta\eta} + \lambda u = -f,$$

$$v_{\xi\xi} + v_{\eta\eta} + \lambda v = 0.$$

Multiply the first by v, the second by u and then subtract to obtain the following divergence form

$$\Big(vu_\xi - uv_\xi\Big)_\xi - \Big(uv_\eta - vu_\eta\Big)_\eta = -f\,v. \tag{3.1.4}$$

Now apply (2.1.23) with $F_2 = vu_\xi - uv_\xi$, $F_1 = uv_\eta - vu_\eta$ to obtain (3.1.2). Use (2.1.27) to convert this to (3.1.3). $\square$ $\square$

Proposition 3.1.2 (Global relation for $\Omega^{(i)}$, where $\Omega^{(i)}$ is bounded.) *a)* ***Modified Helmholtz****: Let $u^{(i)}$ be a solution of* (1.1.1) *for $\lambda = -4\beta^2$, $\beta \in \mathbb{R}$, and $\Omega = \Omega^{(i)}$ where $\Omega^{(i)}$ is* bounded. *Then the transforms of the boundary values of u, $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$, and the transform of the inhomogeneous term f, $\widehat{f}^{(i)}(k)$ satisfy the following relation*

$$\sum_{j=1}^{n} \widehat{u^{(i)}}_j(k) + i\widehat{f^{(i)}}(k) = 0, \quad k \in \mathbb{C}, \tag{3.1.5}$$

where $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$ *are defined by* (2.1.31) *and* $\widehat{f}^{(i)}(k)$ *is defined by*

$$\widehat{f}^{(i)}(k) = \iint_{\Omega^{(i)}} e^{-i\beta\left(kz' - \frac{\bar{z}'}{k}\right)} f(\xi,\eta)d\xi d\eta, \tag{3.1.6}$$

(so $\widehat{f}^{(i)}(k) = \widehat{f}_L^{(i)}(k) + \widehat{f}_R^{(i)}(k)$ *where* $\widehat{f}_{L/R}^{(i)}(k)$ *are defined by* (2.1.35)*).*

b) ***Poisson****: Let* $u^{(i)}$ *be a solution of* (1.1.1) *for* $\lambda = 0$ *and* $\Omega = \Omega^{(i)}$ *where* $\Omega^{(i)}$ *is* bounded*. Then the transforms of the boundary values of* u*,* $\{\widehat{u^{(i)}}_j(k)\}_1^n$*,* $\widetilde{u^{(i)}}_j(k)$ *and the transform of the inhomogeneous term* f*,* $\widehat{f}_{L/R}^{(i)}(k)$*,* $\widetilde{f}_{L/R}^{(i/e)}(k)$ *satisfy the following relations*

$$\sum_{j=1}^{n} \widehat{u^{(i)}}_j(k) + i\widehat{f^{(i)}}(k) = 0, \tag{3.1.7a}$$

$$\sum_{j=1}^{n} \widetilde{u^{(i)}}_j(k) - i\widetilde{f^{(i)}}(k) = 0, \quad k \in \mathbb{C}, \tag{3.1.7b}$$

where $\{\widehat{u^{(i)}}_j(k)\}_1^n$*,* $\{\widetilde{u^{(i)}}_j(k)\}_1^n$*,* $\widetilde{f}_{L/R}^{(i)}(k)$ *are defined by* (2.1.54) *and* (2.1.55) *respectively, and* $\widehat{f^{(i)}}(k), \widetilde{f}^{(i)}(k)$ *are defined by*

$$\widehat{f}^{(i)}(k) = \iint_{\Omega^{(i)}} e^{-ikz'} f(\xi,\eta)d\xi d\eta, \quad \widetilde{f}^{(i)}(k) = \iint_{\Omega^{(i)}} e^{ik\bar{z}'} f(\xi,\eta)d\xi d\eta \tag{3.1.8}$$

(so $\widehat{f}^{(i)}(k) = \widehat{f}_L^{(i)}(k) + \widehat{f}_R^{(i)}(k)$ *where* $\widehat{f}_{L/R}^{(i)}(k)$ *are defined by* (2.1.59) *and similarly for* $\widetilde{f}^{(i)}(k)$*).*

c) ***Helmholtz****: Let* $u^{(i)}$ *be a solution of* (1.1.1) *for* $\lambda = 4\beta^2$*,* $\beta \in \mathbb{R}$*, and* $\Omega = \Omega^{(i)}$ *where* $\Omega^{(i)}$ *is* bounded*. Then the transforms of the boundary values of* u*,* $\{\widehat{u^{(i)}}_j(k)\}_1^n$*, and the transforms of the inhomogeneous term* f*,* $\widehat{f}_{L/R}^{(i)}(k)$ *satisfy the following relation*

$$\sum_{j=1}^{n} \widehat{u^{(i)}}_j(k) + i\widehat{f^{(i)}}(k) = 0, \quad k \in \mathbb{C}, \tag{3.1.9}$$

where $\{\widehat{u^{(i/e)}}_j(k)\}_1^n$ *are defined by* (2.1.43) *and* $\widehat{f}^{(i)}(k)$ *is defined by*

$$\widehat{f}^{(i)}(k) = \iint_{\Omega^{(i)}} e^{-i\beta\left(kz' + \frac{\bar{z}'}{k}\right)} f(\xi,\eta)d\xi d\eta, \tag{3.1.10}$$

(so $\widehat{f}^{(i)}(k) = \widehat{f}_L^{(i)}(k) + \widehat{f}_R^{(i)}(k)$ where $\widehat{f}_{L/R}^{(i)}(k)$ are defined by (2.1.47)).

Prop. 3.1.2 follows from prop. 3.1.1 and the following lemma:

Lemma 3.1.3 (Adjoint solutions) *Particular solutions of the adjoint equation* (3.1.1) *are given by* (3.1.11).

$$\lambda = 0, \quad v = \text{ either } e^{-ikz} \text{ or } e^{ik\bar{z}}, \tag{3.1.11a}$$

$$\lambda = -4\beta^2, \quad v = e^{-i\beta\left(kz' - \frac{\bar{z}'}{k}\right)}, \tag{3.1.11b}$$

$$\lambda = 4\beta^2, \quad v = e^{-i\beta\left(kz' + \frac{\bar{z}'}{k}\right)}, \tag{3.1.11c}$$

Proof By separation of variables, the exponential

$$v = e^{m_1\xi + m_2\eta} \tag{3.1.12}$$

satisfies (3.1.1) if

$$m_1^2 + m_2^2 + \lambda = 0. \tag{3.1.13}$$

For $\lambda = 0$ a natural way to parametrize this 1-parameter family of solutions is by $m_1 = \pm ik$, $m_2 = \pm k$ which leads to the solutions

$$e^{-ik\xi + k\eta}, \quad e^{ik\xi + k\eta}, \tag{3.1.14}$$

and two more obtained by letting $k \to -k$. In complex co-ordinates these become (3.1.11a).

For $\lambda = -4\beta^2$ a natural parametrization of (3.1.13) is $m_1 = \pm 2\beta \sin\phi$, $m_1 = \pm 2\beta\cos\phi$; then let $k = e^{i\phi}$ to obtain

$$m_1 = \mp i\beta\left(k - \frac{1}{k}\right), \quad m_2 = \pm\beta\left(k + \frac{1}{k}\right). \tag{3.1.15}$$

This parametrisation leads to four particular solutions: (3.1.11)b and three obtained from (3.1.11b) using the transformations $k \to -k$ and $k \to 1/k$.

In a similar way, for $\lambda = 4\beta^2$ a natural parametrization of (3.1.13) is $m_1 = \pm 2i\beta\sin\phi$, $m_2 = \pm 2i\beta\cos\phi$ which leads to the particular solution (3.1.11c), and three more obtained from (3.1.11c) using the transformations $k \to -k$ and $k \to 1/k$.

$\square$

Remark 3.1.4 (Why there are two global relations for Poisson but only one for modified Helmholtz and Helmholtz) *In the particular adjoint solution v for the modified Helmholtz equation, let $k \to k/\beta$ to obtain*

$$v = e^{-ikz' + i\frac{\beta^2 \bar{z}'}{k}}, \tag{3.1.16}$$

which reduces to e^{-ikz}, i.e. the first v for Poisson, when $\beta = 0$. (3.1.16) *also contains the second v for Poisson: first let $k \to \beta^2/k$, and then let $\beta = 0$ to obtain $e^{ik\bar{z}}$, the second v for Poisson. Thus the 'information' contained in the two adjoint solutions for Poisson is contained in one adjoint solution for the modified Helmholtz equation.* (3.1.16)

3.1.2 The GR for unbounded domains

If the domain is unbounded, k must be restricted so that the integral at the boundary at infinity is zero. Unbounded domains contain at least one of an infinite arc, considered in lemma 3.1.5 or a semi-infinite strip, lemma 3.1.9.

If the domain is *not* convex at infinity (in the chordal metric of the Riemann sphere, then the restriction on k is too great and (3.1.5), (3.1.7), (3.1.9) are only valid for $k = 0$ (see corollary 3.1.7). In this case, the best one can do is have (3.1.2) in subdomains of Ω which are convex at infinity.

Lemma 3.1.5 (Infinite arc) *Let*

$$I(R) = \int_{C_R(\theta_1,\theta_2)} u\left(v_\eta d\xi - v_\xi d\eta\right) - v\left(u_\eta d\xi - u_\xi d\eta\right) \tag{3.1.17}$$

where

$$C_R(\theta_1, \theta_2) = \left\{ z = z_s + Re^{i\theta} : \theta_1 \le \theta \le \theta_2, z_s \in \mathbb{C} \right\}, \tag{3.1.18}$$

and u satisfies (1.1.1). *Then*

a) if $\lambda = 0$, $u(r, \theta) \to 0$ as $r \to \infty$ and $v = e^{-ikz}$ then

$$I(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K_{P1}(\theta_1, \theta_2)$$

where

$$K_{P1}(\theta_1, \theta_2) = \left\{ k \in \mathbb{C} : \pi - \theta_1 \le \arg k \le 2\pi - \theta_2, |k| \ge 0 \right\}; \tag{3.1.19}$$

and if $v = e^{ik\bar{z}}$ *then*

$$I(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K_{P2}(\theta_1, \theta_2)$$

where

$$K_{P2}(\theta_1, \theta_2) = \Big\{ k \in \mathbb{C} : \theta_2 \le \arg k \le \theta_1 + \pi, |k| \ge 0 \Big\}. \tag{3.1.20}$$

b) if $\lambda = -4\beta^2$, $u(r, \theta) \to 0$ *as* $r \to \infty$ *and* $v = e^{-i\beta\left(kz' - \frac{\bar{z}'}{k}\right)}$ *then*

$$I(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K_{MH}(\theta_1, \theta_2)$$

where

$$K_{MH}(\theta_1, \theta_2) = \Big\{ k \in \mathbb{C} : \pi - \theta_1 \le \arg k \le 2\pi - \theta_2, |k| > 0 \Big\}. \tag{3.1.21}$$

c) if $\lambda = 4\beta^2$, $u(r, \theta)$ *satisfies the radiation condition*

$$u = \mathcal{O}\left(\frac{1}{r^{1/2}}\right), \quad \frac{\partial u}{\partial r} - 2i\beta u = \mathcal{O}\left(\frac{1}{r^{3/2}}\right), \text{ as } r \to \infty, \tag{3.1.22}$$

and $v = e^{-i\beta\left(kz' + \frac{\bar{z}'}{k}\right)}$ *then*

$$I(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K_H(\theta_1, \theta_2)$$

where

$$K_H(\theta_1, \theta_2) = \Big\{ |k| > 1\,, \pi - \theta_1 \le \arg k \le 2\pi - \theta_2 \Big\} \cup \Big\{ |k| < 1\,, -\theta_1 \le \arg k \le \pi - \theta_2 \Big\} \tag{3.1.23}$$

$$\cup \Big\{ |k| = 1\,, -\theta_1 < \arg k < 2\pi - \theta_2 \Big\},$$

see Figure 3.1.

Remark 3.1.6 *For Helmholtz, if* $u^{(i)}$ *satisfies the incoming radiation condition, i.e.* (2.1.19) *with the sign of the second term changed, then* K_H *must be changed so that*

$$K_H \cap \{|k| = 1\} = \{\theta_2 < \arg k < 2\pi + \theta_1\}.$$

Corollary 3.1.7 *If* $\theta_2 - \theta_1 > \pi$ *then* $I(R)$ *does not tend to zero for any* k *with* $|k| > 0$.

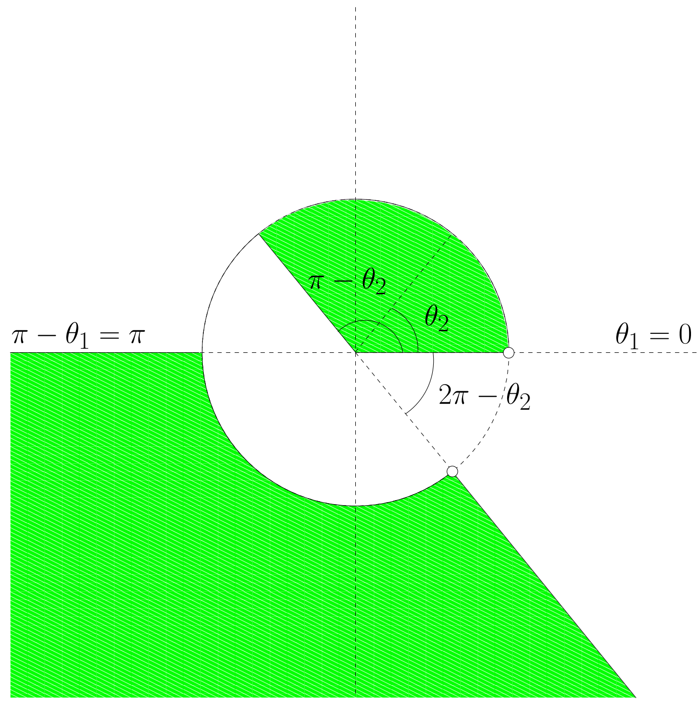


Figure 3.1: The region in the complex k plane where the integral over $C_R(\theta_1, \theta_2)$ tends to zero as $k \to \infty$ for the Helmholtz global relation when $\theta_1 = 0$.

Remark 3.1.8 (Radiation conditions for the Helmholtz equation) *As discussed in §2.1.3, BVPs involving the Helmholtz equation in unbounded domains require a radiation condition to be well-posed. For domains that are the exterior of a bounded obstacle (e.g. $\Omega^{(i)}$) it is a standard result that the problem is well-posed under the condition* (2.1.19). *The situation is more complicated in wedge domains, especially when Robin boundary conditions are imposed (see e.g. [BLG08, §2.4-§2.7]). The BVPs for the Helmholtz equation in a wedge domain considered in Chapter 4 are well-posed under the radiation conditions* (3.1.22) *and so in this Section we only obtain the global relation under these conditions.*

Proof The main ingredient of the proof is the argument of the well-known *Jordan's lemma*, see e.g. [AF03, p.222, Lemma 4.2.2], which relies on the fact that

$$\int_0^{\pi/2} e^{-R\sin\phi} d\phi = \mathcal{O}\left(\frac{1}{R}\right), \text{ as } R \to \infty \tag{3.1.24}$$

which follows from the inequality

$$\sin\phi \geq \frac{2\phi}{\pi}, \quad 0 \leq \phi \leq \pi. \tag{3.1.25}$$

For case c) a more delicate argument is required: for $|k| = 1$ the integral must be estimated using *the method of stationary phase*, see e.g. [AF03] §6.3.

a) first consider the case when $v = e^{-ikz}$. In complex co-ordinates

$$I(R) = \int_{C_R(\theta_1,\theta_2)} e^{-ikz'} \left[(u_{z'} + iku)\, dz' - u_{\bar{z}'} d\bar{z}' \right].$$

Without loss of generality $z_s = 0$. Parametrise $C_R(\theta_1, \theta_2)$ by $z = Re^{i\theta}$, $\theta_1 \le \theta \le \theta_2$, change variables to $\phi = \arg k + \theta - \pi$ and take the modulus to obtain

$$|I(R)| \le R\, M(R) \int_{\theta_1 + \arg k - \pi}^{\theta_2 + \arg k - \pi} e^{-|k| R \sin\phi} d\phi, \tag{3.1.26}$$

where $M(R)$ depends on u and its derivatives and so tends to zero as $R \to \infty$. This integral tends to zero as $R \to \infty$ for $\pi - \theta_1 \le \arg k \le 2\pi - \theta_2$, $|k| > 0$, that is, $k \in K_{P1}(\theta_1, \theta_2)$. Indeed split the range into $[\theta_1 + \arg k - \pi, \pi/2]$, $[\pi/2, \theta_2 + \arg k - \pi]$, use (3.1.25) in the first integral and again in the second integral after first making a change of variable $\phi' = \pi - \phi$. The two integrals are then $\mathcal{O}\left(R^{-1}\right)$ by (3.1.24) and so

$$|I(R)| \le c M(R),$$

where c is a constant, and hence $I(R) \to 0$ as $R \to \infty$. If the range of the integral in (3.1.26) is *not* in $[0, \pi]$ then $\sin\phi$ is negative in part of the range and this means that the integral tends to infinity as $R \to \infty$. When $k = 0$, $I(R) = \int_{C_R} \frac{\partial u}{\partial n} dS$ which tends to zero if $u = \mathcal{O}\left(R^{-\varepsilon}\right)$ as $R \to \infty$.

For $v = e^{ik\bar{z}}$ an almost identical argument results in the analogue of (3.1.26) being

$$|I(R)| \le R M(R) \int_{\arg k - \theta_1}^{\arg k - \theta_2} e^{-|k| R \sin\phi} d\phi,$$

and proceeding as before yields $I(R) \to 0$ as $R \to \infty$ for $\theta_2 \le \arg k \le \pi + \theta_1$, $|k| > 0$, that is, $k \in K_{P2}(\theta_1, \theta_2)$.

b) The analogue of (3.1.26) is now

$$|I(R)| \le R M(R) \int_{\theta_1 + \arg k - \pi}^{\theta_2 + \arg k - \pi} e^{-\left(|k| + \frac{1}{|k|}\right) R \sin\phi} d\phi.$$

For $|k| > 0$, $\left(|k| + \frac{1}{|k|}\right) > 0$, so the situation is identical to the first Poisson case; and $I(R) \to 0$ as $R \to \infty$ for $k \in K_{P1}(\theta_1, \theta_2) = K_{MH}(\theta_1, \theta_2)$.

c) The analogue of (3.1.26) is now

$$|I(R)| \leq RM(R) \int_{\theta_1 + \arg k - \pi}^{\theta_2 + \arg k - \pi} e^{-\left(|k| - \frac{1}{|k|}\right) R \sin \phi} d\phi. \tag{3.1.27}$$

For $|k| > 1$, $\left(|k| - \frac{1}{|k|}\right) > 0$, and the situation is identical to that for the modified Helmholtz equation, and the integral tends to zero for $\theta_2 \leq \arg k \leq \pi + \theta_1$. However, for $|k| < 1$, $\left(|k| - \frac{1}{|k|}\right) < 0$ and so now the integral tends to zero when its range is in $[-\pi, 0]$, which occurs when $-\theta_1 \leq \arg k \leq \pi - \theta_2$ using a similar argument as before. For $|k| = 1$ the exponent is zero and (3.1.27) cannot be used to determine for what range of $\arg k$ the integral tends to zero.

The definition of $I(R)$, (3.1.17), and (2.1.27) imply that

$$I(R) = \int_{\theta_1}^{\theta_2} \left(u \left(\frac{\partial v}{\partial R} - 2i\beta v \right) - v \left(\frac{\partial u}{\partial R} - 2i\beta u \right) \right) R \, d\theta, \tag{3.1.28}$$

where the term $2i\beta uv$ is both added and subtracted in the integrand. The particular solution v given by (3.1.11c) becomes $e^{-2i\beta R \cos(\theta + \arg k)}$ which is $\mathcal{O}(1)$ as $R \to \infty$. Using the radiation condition (3.1.22), the limit of the integral of the second term in the integrand of (3.1.28) is zero. Denote the rest of (3.1.28) by I_1, substitute in the expression for v, and change variables $\phi = \theta + \arg k$, to obtain

$$I_1(R) = -2i\beta R \int_{\theta_1 + \arg k}^{\theta_2 + \arg k} u \left(\cos \phi + 1 \right) e^{-2i\beta R \cos \phi} d\phi. \tag{3.1.29}$$

This integral has points of stationary phase when $\sin \phi = 0$, that is, at $\phi = n\pi$, $n \in \mathbb{Z}$. If none of these lies in the range of the integral then the integral is $\mathcal{O}\left(R^{-3/2}\right)$ as $R \to \infty$ by integration by parts (remember that $u = \mathcal{O}\left(R^{-1/2}\right)$), and so $I_1(R) = \mathcal{O}\left(R^{-1/2}\right)$ and tends to zero as $R \to \infty$. Suppose that ϕ_s is a stationary point in $[\theta_1 + \arg k, \theta_2 + \arg k]$, then, by the usual stationary phase calculation

$$\int_{\theta_1 + \arg k}^{\theta_2 + \arg k} u \left(\cos \phi + 1 \right) e^{-2i\beta R \cos \phi} d\phi \sim \frac{c\, u(R, \phi) \left(\cos \phi_s + 1 \right)}{R^{1/2}} + \mathcal{O}\left(\frac{1}{R^{3/2}} \right),$$

where c is a constant. Thus $I_1(R) = \mathcal{O}(1)$ as $R \to \infty$ (remember again that $u = \mathcal{O}\left(R^{-1/2}\right)$) *unless* $\cos\phi_s = -1$. This occurs when $\phi_s = (2n+1)\pi$, $n \in \mathbb{Z}$, in which case $I_1(R) = \mathcal{O}(R^{-1/2})$ as $R \to \infty$.

In summary, if the range of integration contains one of the stationary points $2n\pi, n \in \mathbb{Z}$ then $I_1(R) = \mathcal{O}(1)$ as $R \to \infty$, otherwise $I_1(R) = \mathcal{O}(R^{-1/2})$ as $R \to \infty$. Thus, $I_1(R) \to 0$ as $R \to \infty$ if and only if the range is contained in $(0, 2\pi)$, that is, $-\theta_1 < \arg k < 2\pi - \theta_2$. Note that both 0 and 2π are *not* allowed to be an endpoints of the range of integration, which results in strict inequalities for $\arg k$. $\square$

The second way a domain can be unbounded is to contain a semi-infinite strip:

Lemma 3.1.9 (Semi-infinite strip) *Let*

$$I^s(R) = \int_{C^s_R(\psi)} u\left(v_\eta d\xi - v_\xi d\eta\right) - v\left(u_\eta d\xi - u_\xi d\eta\right) \tag{3.1.30}$$

where

$$C^s_R(\psi) = \left\{ z = (R + is)e^{i\psi} : a < s < b,\ \psi \in (0, 2\pi) \right\}, \tag{3.1.31}$$

and u satisfies (1.1.1). *('s' standing for 'strip'.) Then*

a) if $\lambda = 0$, $u(r,\theta) \to 0$ as $r \to \infty$ and $v = e^{-ikz}$ then

$$I^s(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K^s_{P1}(\psi)$$

where

$$K^s_{P1}(\psi) = \left\{ k \in \mathbb{C} : \pi - \psi \le \arg k \le 2\pi - \psi \right\}; \tag{3.1.32}$$

and if $v = e^{ik\bar{z}}$ then

$$I^s(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K^s_{P2}(\psi)$$

where

$$K^s_{P2}(\psi) = \left\{ k \in \mathbb{C} : \psi \le \arg k \le \pi - \psi \right\}. \tag{3.1.33}$$

b) if $\lambda = -4\beta^2$, $u(r,\theta) \to 0$ as $r \to \infty$ and $v = e^{-i\beta\left(kz' - \frac{\bar{z}'}{k}\right)}$ then

$$I^s(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K^s_{MH}(\psi)$$

where

$$K^s_{MH}(\psi) = \Big\{ k \in \mathbb{C} : \pi - \psi \leq \arg k \leq 2\pi - \psi \Big\}. \tag{3.1.34}$$

c) if $\lambda = 4\beta^2$*,* $u(r,\theta)$ *satisfies* (2.1.19) *as* $r \to \infty$ *and* $v = e^{-i\beta\left(kz' + \frac{\bar{z}'}{k}\right)}$ *then*

$$I^s(R) \to 0 \text{ as } R \to \infty, \iff \text{ for } k \in K^s_H(\psi)$$

where

$$K^s_H(\psi) = \Big\{ |k| > 1\,, \pi - \psi \leq \arg k \leq -\psi \Big\} \cup \Big\{ |k| < 1\,, -\psi \leq \arg k \leq -\psi + \pi \Big\} \tag{3.1.35}$$
$$\cup \Big\{ |k| = 1 \Big\}.$$

Proof Since the range of integration is finite, the proof for the semi-infinite strip is easier than for the arc – the exponentials need only be bounded over the range of integration for $I^s(R)$ to go to zero because of the decay of u.

a) when $v = e^{-ikz}$

$$\begin{aligned} |I(R)| &\leq M(R) \left| \int_a^b e^{-ik(R+is)e^{i\psi}} ds \right|, \\ &\leq c\, M(R) e^{-ikRe^{i\psi}}, \end{aligned}$$

where c is a constant and $M(R)$ depends on u and its derivatives and so tends to zero as $R \to \infty$.

For $v = e^{ik\bar{z}}$ the exponential which needs to be bounded is

$$e^{ikRe^{-i\psi}}.$$

b) Proceeds exactly as in a) - as $k \to 0$ the k^{-1} term in the exponent behaves in the same way as the k term behaves as $k \to \infty$.

c) The exponential which needs to be bounded is now

$$e^{-i\beta R\left(ke^{i\psi} + \frac{1}{ke^{i\psi}}\right)}.$$

$\square$

3.1.3 The invariances of the global relation

Summary:

- In order to solve BVPs by eliminating the transforms of the unknown BVs from the IR, the global relation (3.1.5)/(3.1.7)/(3.1.9) must be supplemented with equations derived from it by certain transformations.
- In this subsection a proposition is proved (prop. 3.1.11) which gives an algorithmic process, given a domain, to determine these transformations.
- As stated earlier, this method has not yet been used to solve any BVPs in domains with more than four sides. The result of this section perhaps gives an indication of why this is the case: for regular polygons with n sides the number of equations resulting from this process is $\mathcal{O}(n)$ but the number of unknowns is $\mathcal{O}(n^2)$.

First consider the Poisson equation. Let $\Omega^{(i)}$ be either bounded or unbounded but convex at ∞ so both (3.1.7a) and (3.1.7b) hold, each for some non-zero $k \in \mathbb{C}$ (by corollary 3.1.7).

Parametrise each side by

$$z' = z_j + sh_j, \quad h_j = \frac{(z_{j+1} - z_j)}{|z_{j+1} - z_j|}, \quad s \in (0,1), \tag{3.1.36}$$

so that (3.1.7a) and (3.1.7b) become

$$\sum_{j=1}^{n} e^{-ikz_j}\left(N_j(-ike^{i\alpha_j}) + kh_j D_j(-ike^{i\alpha_j})\right) = -\widehat{f^{(i)}}(k), \tag{3.1.37a}$$

$$\sum_{j=1}^{n} e^{ik\bar{z}_j}\left(N_j(ike^{-i\alpha_j}) + k\bar{h}_j D_j(ike^{-i\alpha_j})\right) = -\widetilde{f^{(i)}}(k), \tag{3.1.37b}$$

where

$$N_j(k) = \int_0^{|z_{j+1}-z_j|} e^{ks}\,\frac{\partial u^{(i)}}{\partial n}(z'(s))ds, \quad D_j(k) = \int_0^{|z_{j+1}-z_j|} e^{ks}\,u^{(i)}(z'(s))ds. \tag{3.1.38}$$

If $z_p = \infty$, write $\int_{z_p}^{z_{p+1}} = -\int_{z_{p+1}}^{z_p}$ and let $z' = z_{p+1} + se^{i(\pi+\alpha_j)}$, $s \in (0,\infty)$, to obtain

$$\widehat{u}_p(k) = e^{-ikz_{p+1}}\left(N_p(ike^{i\alpha_P}) - kh_p D_p(ike^{i\alpha_p})\right). \tag{3.1.39}$$

$$\widetilde{u}_p(k) = e^{ikz_{\bar{p+1}}}\left(N_p(-ike^{-i\alpha_P}) - k\bar{h}_p D_p(-ike^{-i\alpha_p})\right). \tag{3.1.40}$$

All the results in the rest of this section are unaffected by this change of sign in the argument of the N_p, D_p. (If both $z_j, z_{j+1} = \infty$ then without loss of generality the domain is either the upper half plane or the infinite strip. Treat these on a case by case basis.)

If u is real then (3.1.37b) can be obtained from (3.1.37a) by taking the complex conjugate, and then letting $k \mapsto \bar{k}$. We will refer to this prodecure as "Schwartz conjugation". If u is complex (3.1.37b) cannot be obtained in this way (but this equation still holds by Proposition 3.1.2). Because of this fact, in the rest of the thesis we will call (3.1.37a) the global relation (GR) and (3.1.37b) the Schwartz conjugate (SC).

The structure of the equations (3.1.37) is unchanged when one considers the modified Helmholtz equation and the Helmholtz equation. Indeed, for the modified Helmholtz equation using the above parametrisation in (3.1.5) and the equation obtained from it by $k \mapsto 1/k$ yields the pair of equations

$$\sum_{j=1}^{n} e^{-i\beta\left(kz_j - \frac{\bar{z_j}}{k}\right)}\left(N_j(-ike^{i\alpha_j}) + \beta\left(kh_j + \frac{\bar{h_j}}{k}\right)D_j(-ike^{i\alpha_j})\right) = -\widehat{f}(k), \tag{3.1.41a}$$

$$\sum_{j=1}^{n} e^{i\beta\left(k\bar{z}_j - \frac{z_j}{k}\right)}\left(N_j(ike^{-i\alpha_j}) + \beta\left(k\bar{h}_j + \frac{h_j}{k}\right)D_j(ike^{-i\alpha_j})\right) = -\widetilde{f}(1/k), \tag{3.1.41b}$$

where

$$N_j(k) = \int_0^{|z_{j+1}-z_j|} e^{\beta\left(k+\frac{1}{k}\right)s}\,\frac{\partial u^{(i)}}{\partial n}(z'(s))ds, \quad D_j(k) = \int_0^{|z_{j+1}-z_j|} e^{\beta\left(k+\frac{1}{k}\right)s}\,u^{(i)}(z'(s))ds, \tag{3.1.42}$$

and

$$\widetilde{f}(k) = \widehat{f}(1/k). \tag{3.1.43}$$

Again, (3.1.41)b can be obtained from (3.1.41)a by Schwartz conjugation. For Helmholtz the only difference to the modified Helmholtz equation is that the signs of $1/k$ terms are changed, so the second equation is obtained by $k \mapsto -1/k$, and hence in this case $\widetilde{f}(k) = \widehat{f}(-1/k)$.

For either the Dirichlet or the Neumann problem one of D_j and N_j is known and the other unknown. Without loss of generality consider the Dirichlet problem. Then D_j,

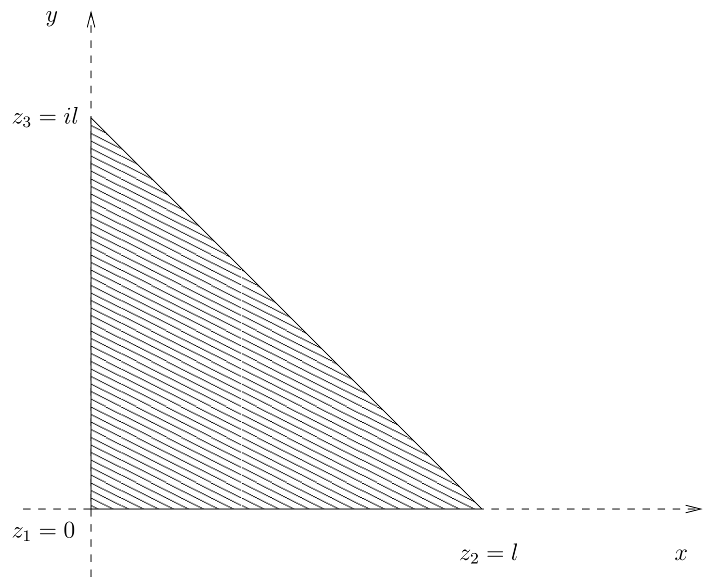


Figure 3.2: The right isosceles triangle.

$\widehat{f^{(i)}}, \widetilde{f^{(i)}}$ are known and N_j are unknown.

Consider side p. Applying the transformations

$$k \mapsto -ke^{-2i\alpha_p}, \tag{3.1.44a}$$

$$k \mapsto -ke^{2i\alpha_p}, \tag{3.1.44b}$$

in (3.1.37)a and (3.1.37)b respectively means the argument of N_p and D_p in (3.1.37)a becomes the argument of N_p, D_p in (3.1.37)b and visa versa.

It is helpful to consider a concrete example:

Example 3.1.10 (The invariances of the GR for the right isosceles triangle) *Consider the right isosceles triangle, Figure 3.2, where the angles of the sides are $\alpha_1 = 0$, $\alpha_2 = 3\pi/4$, $\alpha_3 = -\pi/2$. Parametrise sides 1 and 2 using* (3.1.36). *For side 3, instead of parametrising $z' = i(l-s)$, $0 \le s \le l$ (from z_3 to z_1) it is more convenient to parametrise $z' = is$, $0 \le s \le l$ (from z_1 to z_3, as in* (3.1.39)*).*

Thus, (3.1.37) *become*

$$N_1(-ik) + e^{-ikl}N_2(e^{i\pi/4}k) + N_3(k) = G(k), \tag{3.1.45a}$$

$$N_1(ik) + e^{ikl}N_2(e^{-i\pi/4}k) + N_3(k) = \overline{G(\bar{k})}, \tag{3.1.45b}$$

where $G(k)$ is the known function given by

$$G(k) = -kD_1(-ik) + ke^{-i\pi/4}e^{-ikl}D_2(e^{i\pi/4}k) + ikD_3(k) + \widehat{f}(k).$$

(3.1.45) *are two equations for five unknown functions* $N_1(-ik)$, $N_1(ik)$, $N_2(e^{i\pi/4}k)$, $N_2(e^{-i\pi/4}k)$, $N_3(k)$.

The pair $\{N_2(e^{i\pi/4}k), N_2(e^{-i\pi/4}k)\}$ *in* (3.1.45) *is invariant under the transformations* $\{k \mapsto -ik, k \mapsto ik\}$ *and applying these to (3.1.45a), (3.1.45b) respectively we obtain*

$$N_1(-k) + e^{-kl}N_2(e^{-i\pi/4}k) + N_3(-ik) = G(-ik), \tag{3.1.46a}$$

$$N_1(-k) + e^{-kl}N_2(e^{i\pi/4}k) + N_3(ik) = \overline{G(-i\bar{k})}, \tag{3.1.46b}$$

The pair $\{N_1(-ik), N_1(ik)\}$ *in* (3.1.45) *is invariant under the transformation* $k \mapsto -k$ *and so we obtain another pair of equations by letting* $k \mapsto -k$ *in* (3.1.45)*:*

$$N_1(ik) + e^{ikl}N_2(e^{-i\pi/4}k) + N_3(-k) = G(-k), \tag{3.1.47a}$$

$$N_1(-ik) + e^{-ikl}N_2(-e^{-i\pi/4}k) + N_3(-k) = \overline{G(-\bar{k})}. \tag{3.1.47b}$$

We now repeat this process with (3.1.46)*: the pair* $\{N_2(e^{-i\pi/4}k), N_2(e^{i\pi/4}k)\}$ *is invariant under the transformations* $\{k \mapsto ik, k \mapsto -ik\}$*, but these bring us back to* (3.1.45)*. The pair* $\{N_3(-ik), N_3(ik)\}$ *is invariant under the transformation* $k \mapsto -k$ *and so we obtain another pair of equations*

$$N_1(k) + e^{kl}N_2(-e^{-i\pi/4}k) + N_3(ik) = G(ik), \tag{3.1.48a}$$

$$N_1(k) + e^{kl}N_2(-e^{i\pi/4}k) + N_3(-ik) = \overline{G(i\bar{k})}. \tag{3.1.48b}$$

One can check that applying the process to (3.1.47) *and* (3.1.48) *yields no more new pairs of equations. Thus, we end up with* ***eight equations*** (3.1.45), (3.1.46), (3.1.47), (3.1.48) *for* ***twelve unknowns****:* $N_j(\pm ik), N_j(\pm k), j = 1, 3$ *and* $N_2(\pm e^{i\pi/4}k)$, $N_2(\pm e^{-i\pi/4}k)$.

We now consider the general case:

Proposition 3.1.11 *Given a polygon* $\Omega^{(i)}$*, either bounded or unbounded but convex at* ∞*, let* I *denote the pairs of transformations applied to the GR* (3.1.37a) *and SC* (3.1.37b) *(including the identity* $k \mapsto k$*). Let* R_β *be the rotation by angle* β*. Without loss of generality, orientate the polygon such that* $\alpha_1 = \pi/2$ *or* $3\pi/2$*, so* $R_{\pi-2\alpha_1} = R_0$*, i.e. the identity. Then*

$$I \cong \left\langle R_{\pi-2\alpha_j}, j = 1, ..., n \right\rangle \tag{3.1.49}$$

by the mapping

$$\begin{pmatrix} k \mapsto ke^{i\phi} \\ k \mapsto ke^{-i\phi} \end{pmatrix} \mapsto R_\phi \tag{3.1.50}$$

That is: I is isomorphic (as a group) to the group of rotations generated by $R_{\pi-2\alpha_j}, j = 1, ..., n$.

Corollary 3.1.12

$$|I| < \infty \iff \left\langle R_{\pi-2\alpha_j}, j = 1, ..., n \right\rangle \text{ is cyclic.} \tag{3.1.51}$$

$$\iff \alpha_j \in \pi\mathbb{Q},\ j = 1, ..., n, \tag{3.1.52}$$

Corollary 3.1.13 *If $|I| < \infty$, let the generator of $\left\langle R_{\pi-2\alpha_j}, j = 1, ..., n \right\rangle$ be $R_{2\pi/m}$. Then I consists of*

$$\begin{pmatrix} k \mapsto ke^{2\pi i l/m} \\ k \mapsto ke^{-2\pi i l/m} \end{pmatrix}, l = 1, ..., m, \tag{3.1.53}$$

where the first acts on the GR and the second acts on the SC. This results in a system of $2m$ (m pairs) of equations involving nm unknowns.

Proof of Proposition 3.1.11 Since the second of each pair of transformations can be obtained from the first by Schwartz conjugation, only consider the first of these transformations acting on the GR. Call the GR "state (0)" and "state (p)" the equations resulting when the transformation (3.1.44a) is applied to the GR. Call "state (pq)" the equations resulting when the transformation (3.1.44a) with p replaced by q is applied to (p), and so on. To condense notation let $r_p = R_{\pi-2\alpha_p}$.

Define $N_j(-ike^{i\alpha_j})$ to have "angle" α_j, then the transformation (3.1.44) rotates angle α_p in (3.1.37a) by $\pi - 2\alpha_p$ to become $\pi - \alpha_p$ in (3.1.37b).

By direct computation the "angles" of the functions N_j in the GR in states (p), (pq),and (pqs) are given by:

$$\text{state } (p) \qquad \text{angles: } \alpha_j + (\pi - 2\alpha_p), j = 1, ..., n, \tag{3.1.54}$$

$$\text{state } (pq) \qquad \text{angles: } \alpha_j - (\pi - 2\alpha_p) + (\pi - 2\alpha_q), j = 1, ..., n, \tag{3.1.55}$$

$$\text{state } (pqs) \qquad \text{angles: } \alpha_j + (\pi - 2\alpha_p) - (\pi - 2\alpha_q) + (\pi - 2\alpha_s), j = 1, ..., n. \tag{3.1.56}$$

and so it is clear that the action of these transformations on the GR is isomorphic to the rotation group containing words of the form $r_p,\ r_p^{-1}r_q,\ r_pr_q^{-1}r_s,\ r_p^{-1}r_qr_s^{-1}r_t$ and so on. Choose $q=1$ in the word $r_pr_q^{-1}r_s$ to see that the element r_pr_q is in the group for any p,q, and hence the group is that generated by the $r_j, j=1,...,n$. □ □

Proof of Corollary 3.1.12

Let $G=\left\langle R_{\pi-2\alpha_j}, j=1,...,n\right\rangle$. First we prove that $|G|<\infty$ implies $\alpha\in\pi\mathbb{Q}, j=1,...,n$. Assume $|G|<\infty$. Let V_2 be a two-dimensional vector space. Then every finite subgroup of $SO(V_2)$ (rotations) is cyclic, [NST02, Theorem 15.3, p.174]. Applying this to our example, implies that $G=\left\langle R_\phi\right\rangle$ for some ϕ. So there exists an $n\in\mathbb{N}$ such that $n\phi\equiv 0$ (where $\equiv$ denotes mod 2π). So $\phi=\frac{2\pi k}{n}$ for some $k\in\mathbb{N}$. Then for every $\alpha_j, j=1,...,n$, there is a $t_j\in\mathbb{N}$ such that $R_{\pi-2\alpha_j}=R_{t_j\phi}\iff\pi-2\alpha_j\equiv t\phi\iff\alpha_j=\frac{\pi(n-kt-2sn)}{2n}$ for some $k_j,s_j\in\mathbb{N}$; that is $\alpha_j\in\mathbb{Q}, j=1,...,n$.

Next we prove that $\alpha_j\in\pi\mathbb{Q}, j=1,...,n$ implies $|G|<\infty$. If $\alpha_p\in\pi\mathbb{Q}$ then $\alpha_p=\pi\frac{r_p}{s_p}$ for some $r_p\in\mathbb{Z}$, $s_p\in\mathbb{N}$. We claim there exists $\phi\in\pi\mathbb{Q}$ such that $\forall\alpha_j,\exists t_j\in\mathbb{Z}$ such that $\pi-2\alpha_j\equiv t_j\phi$. If this is true then $G=\left\langle R_\phi\right\rangle$ and we are done. A simple calculation shows that

$$\phi=\frac{\pi}{\prod_j s_j},\quad t_p=\left(\prod_{j\neq p}s_j\right)(s_p-2r_p) \tag{3.1.57}$$

do the job. □

Proof of Corollary 3.1.13 Supplementing the GR with the m transformations (3.1.53)a (one being the identity) yields a set of m equations involving nm unknowns. Similarly, supplementing the SC with the m transformations (3.1.53)b yields a set m equations involving nm unknowns. Since I includes the transformations (3.1.44) and is cyclic, the nm unknowns in each set of equations are the same. To see that the two sets of m equations are disjoint, note that the second set can be obtained from the first by Schwartz conjugation. If an equation were in both sets then it would imply the conjugation transformation could be obtained as a composition of rotations, which is false. □

Sides	Generator	Equations	Unknowns
n odd	$\frac{2\pi}{n}$	$2n$	n^2
n even	$\frac{2\pi}{n/2}$	n	$\frac{n^2}{2}$

Table 3.1: Regular polygons: number of equations and unknowns

Sides	Equations	Unknowns
3	6	9
4	4	8
5	10	25
6	6	18

Table 3.2: Regular polygons: examples

Example 3.1.14 (Regular polygons) *Consider the regular polygon with n sides. Then* $\alpha_j = \pi/2 + (j-1)\frac{2\pi}{n}$ *and* $\pi - 2\alpha_j = -(j-1)\frac{4\pi}{n}$. *Hence the group* $\langle R_{\pi-2\alpha_j}, j = 1, ..., n \rangle$ *is generated by* $R_{2\pi/m}$ *where* $m = n$ *for* n *odd (since* $\left(\frac{n+1}{2}\right)\frac{4\pi}{n} \equiv \frac{2\pi}{n}$*) and* $m = n/2$ *for* n *even. Table 3.1 displays the numbers of equations and unknowns given by Cor. 3.1.13, and Table 3.2 displays some specific examples.*

Remark 3.1.15 *Example 3.1.14 perhaps gives an indication of why no BVP in a domain with more than 4 sides has been solved using this new method: for regular polygons with* n *sides the number of equations resulting from this process is* $\mathcal{O}(n)$ *but the number of unknowns is* $\mathcal{O}(n^2)$. *For example, the hexagon has the same number of equations as the equilateral triangle but twice as many unknowns.*

Remark 3.1.16 (Separable domains) *For separable domains the* α_j *are multiples of* $\pi/2$, $I \cong < R_0, R_\pi >$ *so the only transformation is* $k \mapsto -k$ *which results in 4 equations – the GR, SC and two more obtained from using* $k \mapsto -k$. *These are exactly the equations obtained from using separation of variables to find particular solutions of the adjoint (see Lemma 3.1.11 and Remark 3.1.4).*

3.2 Polar co-ordinates

In this thesis we only consider one domain in polar co-ordinates, the exterior of a circle, and so for simplicity we only formulate Green's integral representation and the new integral representation in the domain D, see Figure 2.9.

Proposition 3.2.1 (Global relation in polar co-ordinates) *Let u be a solution of* (1.1.1) *for Ω some domain D. If D is unbounded assume u satisfies the boundary conditions at infinity:*

- $\lambda = 0$ *(Poisson), $u \to 0$ as $r \to \infty$.*
- $\lambda = \beta^2$ *(Helmholtz), u satisfies* (2.2.3)

Let v be any solution of the adjoint of (1.1.1),(3.1.1)*. Then*

$$\int_{\partial D} r\left(v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r}\right) d\theta + \frac{1}{r}\left(u\frac{\partial v}{\partial \theta} - v\frac{\partial u}{\partial \theta}\right) dr + \iint_D f(r,\theta)v(r,\theta;r,\theta)rdrd\theta = 0 \quad (3.2.1)$$

Proof In polar co-ordinates the adjoint equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \beta^2 v = 0 \quad (3.2.2)$$

and the divergence form (2.2.30) holds with E_s replaced by v and without the first term on the right hand side. Integrate over D and use Green's theorem (2.1.23) to obtain the (3.2.1). □

Lemma 3.2.2 (Adjoint solutions) *There are four particular solutions of the adjoint equation obtain by separation of variables in polar co-ordinates, two given by*

$$\beta = 0, \quad v = e^{\pm ik\theta}r^k, \quad (3.2.3a)$$

$$\beta \neq 0, \quad v = e^{\pm ik\theta}\chi_k(\beta r), \quad (3.2.3b)$$

and two more obtained by $k \mapsto -k$, where $\chi_k(r)$ denotes a solution of the Bessel equation of order k (i.e. a linear combination of the Bessel functions J_k and Y_k). (If v is required to be periodic in θ then $k \in \mathbb{Z}$.)

Proof Letting $v(\rho,\theta,k)=\Theta(\theta;k)R(\rho;k)$, it follows that Θ, R satisfy the ODEs

$$\frac{\partial^2\Theta}{\partial\theta^2}+k^2\Theta=0, \tag{3.2.4}$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+(\beta^2r^2-k^2)R=0, \quad k\in\mathbb{C}. \tag{3.2.5}$$

$\square$

Remark 3.2.3 (The restrictions on k in the adjoint solutions) *Depending on whether the domain contains the origin or is unbounded, some of the particular solutions for v are disallowed. Indeed, first consider the Poisson equation. At infinity, assume $u(r,\theta)=\mathcal{O}(r^{-\varepsilon})$ as $r\to\infty$ where $\varepsilon>0$; actually $u(r,\theta)=\mathcal{O}(r^{-\pi/\gamma})$ as $r\to\infty$, where γ is the angle of the wedge the domain makes at infinity [Jon86]. Then,*

$$\int_{C_R} r\left(v\frac{\partial u}{\partial r}-u\frac{\partial v}{\partial r}\right)d\theta+\frac{1}{r}\left(u\frac{\partial v}{\partial\theta}-v\frac{\partial u}{\partial\theta}\right)dr\to 0 \text{ as } R\to\infty \iff \Re k<\varepsilon,$$

with v given by (3.2.3a). At the origin, assume $u(r,\theta)=\mathcal{O}(r^{\varepsilon})$ as $r\to 0$ where $\varepsilon>0$; actually $u(r,\theta)=\mathcal{O}(r^{\pi/\gamma})$ as $r\to 0$, where γ is the angle of the wedge the domain makes at 0 [Jon86]. Then, for the integral in the global relation to exist, we require $\Re k>-\varepsilon$.

For Helmholtz, $\int_{C_R}\to 0$ as $R\to\infty$ iff u and v satisfy the radiation condition (2.2.3). Therefore if the domain is unbounded, v must be either $e^{\pm ik\theta}H_k^{(1)}(\beta r)$, or a similar expression with $k\mapsto -k$. At the origin, assume $u(r,\theta)=\mathcal{O}(r^{\varepsilon})$ as $r\to 0$ where $\varepsilon>0$; actually $u(r,\theta)=\mathcal{O}(r^{\pi/\gamma})$ as $r\to 0$ where γ is the wedge angle [Jon86]. Then, for the integral to exist, $B_k(\beta r)$ must be bounded as $r\to 0$. Recall that $J_k(\beta r)$ is bounded as $r\to 0$ for $\Re k\geq 0$ and $H_k^{(1)}(\beta r)$ is bounded as $r\to 0$ for $\Re k=0$.

Proposition 3.2.4 (Global relation for domain D defined by (2.2.27)) *Let u be the solution of (1.1.1) for $\Omega=D$ where D is given by (2.2.27) see Figure 2.9. Then*

- $\lambda=0$ *(Poisson),*

$$\pm ikD_0(k)-N_0(k)-e^{\pm ik\alpha}\left[\pm ikD_\alpha(k)-N_\alpha(k)\right]-a^k\left[aN(\pm ik)-kD(\pm ik)\right]$$
$$=-\iint_D d\rho\, d\phi\, f(\rho,\phi)\rho^{k+1}e^{\pm ik\phi}, \quad \Re k<\frac{\pi}{\alpha}. \tag{3.2.6}$$

where $D(\pm ik), N(\pm ik)$ are given by (2.2.34),$D_\chi(k), N_\chi(k)$, $\chi=0$ or α are given by (2.2.33).

- $\lambda = \beta^2$ *(Helmholtz)*,

$$\pm ikD_0(k) - N_0(k) - e^{\pm ik\alpha}\left[\pm ikD_\alpha(k) - N_\alpha(k)\right] - aH_k^{(1)}(\beta a)N(\pm ik) + aH_k^{(1)'}(\beta a)D(\pm ik)$$
$$= -\iint_D d\rho\, d\phi\, \rho\, f(\rho,\phi)H_k^{(1)}(\beta\rho)e^{\pm ik\phi}, \quad k \in \mathbb{C} \tag{3.2.7}$$

where $D(\pm ik), N(\pm ik)$ *are given by* (2.2.34),$D_\chi(k), N_\chi(k)$, $\chi = 0$ *or* α *are given by* (2.2.36).

Proof Substitute (3.2.3a) and (3.2.3b) where $B_k = H_k^{(1)}$ into (3.2.1) for the Poisson equation and the Helmholtz equation respectively (remark 3.2.3 tells us that for the Helmholtz equation v must satisfy the radiation condition). Parametrise the sides of D by $\{\phi = \alpha, \infty > \rho > a\}$, $\{\rho = a, \alpha > \phi > 0\}$, and $\{\phi = 0, 0 < \rho < \infty\}$. The regions of validity are given by remark 3.2.3. $\square$

Chapter 4

Solution of boundary value problems in separable domains

Summary:

- In this Chapter we solve
  - the Poisson, modified Helmholtz, and Helmholtz equations in the half plane for Dirichlet and oblique Robin boundary conditions §4.1,
  - the Helmholtz equation in the quarter plane for Dirichlet and oblique Robin boundary conditions §4.2,
  - the Helmholtz equation in the exterior of the circle with Dirichlet boundary conditions §4.3.
- The half plane is included as it is the simplest possible example of applying the new method, and only involves Step 1 of Chapter 1 §3. No boundary value problems in the half plane are solved with the new method that cannot be solved classically, however we include their solution by the new method for pedagogical reasons.
- The quarter plane is included as it is the simplest possible case where the new method solves certain boundary value problems which *cannot* be solved classically. The solution involves Steps 1 and 2 of §1.3.

- The Helmholtz equation in the exterior of the circle played a prominent role in the development of classical transforms and the Fokas method sheds new light on this classic problem. The solution involves Steps 1, 2 and 3 of Chapter 1 §3.

- For the half plane and quarter plane we use the IR and GR for polygons from §2.1 and §3.1 respectively. For the exterior of the circle we use the IR and GR in polar co-ordinates from §3.2 and §3.2 respectively.

Remark 4.0.5 (Rigorous considerations) *In this chapter we obtain expressions for the solutions of several boundary value problems using the Fokas method. These are given as integrals in the k plane involving certain transforms of the given boundary conditions and forcing.*

As with any expression of a solution to a differential equation, the rigorous proof that it is indeed the solution proceeds as follows:

1. Define *u by the expression. In our case we must give appropriate function spaces for the boundary conditions and forcing such that their transforms exist for appropriate $k \in \mathbb{C}$ (but we will not do this here).*
2. *Prove that u satisfies the PDE* (1.1.1).
3. *Prove that u satisfies the boundary conditions.*
4. *Once uniqueness is established by other PDE techniques, u is then the solution.*

Regarding Step 2: our expression for the solution consists of

- *the forcing term which satisfies* (1.1.1) *by construction, and*

- *integrals where the dependence on the physical co-ordinates, $(z, \bar{z})$ or (r, θ), is contained within eigenfunctions of the homogeneous problem (e.g. $e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}$, $r^{-k}e^{ik\theta}$). Therefore to prove these terms satisfy the homogeneous equation we only need to justify interchanging the operations of differentation and integration (but we will not do this here).*

For each boundary value problem we solve we shall *do Step 3. The fact that the Fokas method yields solutions which are uniformly convergent at the boundary means this is, in* *principle, straightforward. We emphasise that the solutions obtained by classical transforms are* not *uniformly convergent at the boundary, which makes Step 3 difficult.*

4.1 The half plane

In this section we use the integral representations in propositions 2.1.10, 2.1.13, 2.1.14, applied to the polygon defined by $z_1 = -\infty$, $z_2 = \infty$. In this case, and this case alone, the interior and exterior of the polygon are essentially the same domain, so w.l.o.g. we use the integral representations for the interior (2.1.28),(2.1.52),(2.1.40).

Since $\Omega = \Omega^{(i)}$ is convex, the contours of integration do not depend on the position of z (see remark 2.1.11). For the modified Helmholtz equation and the Poisson equation $l_1 = (0, \infty)$, and for the Helmholtz equation $l_1 = L_{out}$ shown in Figure 2.3(a).

4.1.1 Dirichlet boundary conditions

Proposition 4.1.1 *Let the complex-valued function $u(x, y)$ satisfy* (1.1.1) *in the upper half plane*

$$\Omega = \{-\infty < x < \infty,\, 0 < y < \infty\}, \tag{4.1.1}$$

with the condition that $u \to 0$ at infinity, and the Dirichlet boundary conditions

$$u(x, 0) = d(x), \quad -\infty < x < \infty, \tag{4.1.2}$$

where d has sufficient decay at infinity (e.g. $d \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$). Then the solution $u(x, y)$ is given by

1. *if $\lambda = -4\beta^2$ (the modified Helmholtz equation)*

$$u(z, \bar{z}) = \frac{1}{4\pi} \int_0^\infty \frac{dk}{k} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} \left(2\beta \left(k + \frac{1}{k}\right) D(-ik) - \widetilde{f}(-k)\right) + F(z, \bar{z}), \tag{4.1.3}$$

where

$$D(k) = \int_{-\infty}^{\infty} ds\, e^{\beta\left(k + \frac{1}{k}\right)s}\, d(s),$$

and

$$\widetilde{f}(k) = \widehat{f}(1/k),$$

where $\widehat{f}(k)$ *is defined by* (3.1.6), *and* $F(z,\bar{z})$ *is given by* (2.1.33).

2. if $\lambda = 0$ *(the Poisson equation)*

$$u(z,\bar{z}) = \frac{1}{4\pi}\int_0^\infty \frac{dk}{k} e^{ikz}\left(2kD(-ik) - \widetilde{f}(-k)\right) + \frac{1}{4\pi}\int_0^\infty \frac{dk}{k} e^{-ik\bar{z}}\left(2kD(ik) - \widehat{f}(-k)\right) + F(z,\bar{z}) \tag{4.1.4}$$

where

$$D(k) = \int_{-\infty}^{\infty} ds\, e^{ks}\, d(s), \tag{4.1.5}$$

and $\widehat{f}(k), \widetilde{f}(k)$ *are given by* (3.1.8) *and* $F(z,\bar{z})$ *is given by* (2.1.57).

3. if $\lambda = 4\beta^2$ *(the Helmholtz equation) and* u *satisfies the radiation condition* (3.1.22) *for* $0 < \arg z < \pi$, *then*

$$u(z,\bar{z}) = \frac{1}{4\pi}\int_{L_{out}} \frac{dk}{k} e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\left(2\beta\left(k-\frac{1}{k}\right)D(-ik) - \widetilde{f}(-k)\right) + F(z,\bar{z}), \tag{4.1.6}$$

where

$$D(k) = \int_{-\infty}^{\infty} ds\, e^{\beta\left(k-\frac{1}{k}\right)s}\, d(s),$$

and

$$\widetilde{f}(k) = \widehat{f}(-1/k),$$

where $\widehat{f}(k)$ *is defined by* (3.1.10) *and* $F(z,\bar{z})$ *is given by* (2.1.45).

Proof

1. Parametrise $\partial\Omega$ by $z = s, -\infty < s < \infty$, so that (2.1.31) becomes

$$\widehat{u}_1(k) = iN(-ik) + i\beta\left(k+\frac{1}{k}\right)D(-ik), \tag{4.1.7}$$

where

$$N(k) = \int_{-\infty}^{\infty} ds\, e^{\beta\left(k+\frac{1}{k}\right)s}\left(-u_y(s,0)\right).$$

The GR (3.1.5) is then

$$N(-ik) + \beta\left(k+\frac{1}{k}\right)D(-ik) = -\widehat{f}(k), \quad k \in \mathbb{R}^-, \tag{4.1.8}$$

where the region of validity is dictated by lemma 3.1.5 part b) as the domain contains the infinite arc $0 \leq \theta \leq \pi$. Let $k \mapsto -1/k$ (or equivalently take the Schwartz conjugate) to give the second equation

$$N(ik) + \beta \left(k + \frac{1}{k}\right) D(ik) = -\widetilde{f}(k), \quad k \in \mathbb{R}^-. \tag{4.1.9}$$

The pair $\{N(-ik), N(ik)\}$ is invariant under $k \mapsto -k$, and applying this transformation to (4.1.8) and (4.1.9) gives

$$N(ik) - \beta \left(k + \frac{1}{k}\right) D(ik) = -\widehat{f}(-k), \quad k \in \mathbb{R}^+, \tag{4.1.10}$$

$$N(-ik) - \beta \left(k + \frac{1}{k}\right) D(-ik) = -\widetilde{f}(-k), \quad k \in \mathbb{R}^+, \tag{4.1.11}$$

The spectral function $\widehat{u}_1(k)$ contains the unknown function $N(-ik)$, and is integrated over $(0, \infty)$ in the IR (2.1.28). However (4.1.11) gives $N(-ik)$ on $\mathbb{R}^+$ in terms of the known transforms $D(-ik), \widetilde{f}(-k)$. Thus

$$\widehat{u}_1(k) = 2\beta \left(k + \frac{1}{k}\right) D(-ik) - \widetilde{f}(-k).$$

Substituting this into (2.1.28) gives (4.1.3).

2. Parametrise $\partial\Omega$ by $z = s, -\infty < s < \infty$, so that (2.1.54),(2.1.55) become

$$\widehat{u}_1(k) = iN(-ik) + ikD(-ik),$$
$$\widetilde{u}_1(k) = -iN(ik) - ikD(ik),$$

where

$$N(k) = \int_{-\infty}^{\infty} ds\, e^{ks} \left(-u_y(s,0)\right).$$

The two GRs (3.1.7) are

$$N(-ik) + kD(-ik) = -\widehat{f}(k), \quad k \in \mathbb{R}^-, \tag{4.1.12}$$

$$N(ik) + kD(ik) = -\widetilde{f}(k), \quad k \in \mathbb{R}^-, \tag{4.1.13}$$

where the regions of validity are dictated by lemma 3.1.5 part (a). Letting $k \mapsto -k$ in these two equations gives

$$N(ik) - kD(ik) = -\widehat{f}(-k), \quad k \in \mathbb{R}^+, \tag{4.1.14}$$

$$N(-ik) - kD(-ik) = -\widetilde{f}(-k), \quad k \in \mathbb{R}^+. \tag{4.1.15}$$

The IR (2.1.52) contains the unknowns $N(-ik), N(ik)$ both integrated over $(0, \infty)$. (4.1.15) gives $N(-ik)$ on $\mathbb{R}^+$ in terms of the known transforms $D(-ik), \widetilde{f}(-k)$ and (4.1.14) gives $N(ik)$ on $\mathbb{R}^+$ in terms of the known transforms $D(ik), \widehat{f}(-k)$. Substituting these into (2.1.52) gives (4.1.4). The singularity at $k = 0$ on the contour is removable once F is expressed using (2.1.58).

3. This follows in exactly the same way as for the modified Helmholtz equation except the signs of all the $1/k$ terms are reversed. This affects the regions of validity of (4.1.8)-(4.1.11). Lemma 3.1.5 part (c) implies the GR (4.1.8) is valid for k in the set

$$\{k \in \mathbb{C} : k = -l, l \in L_{out}\} \setminus \{\pm 1\}, \tag{4.1.16}$$

where the contour L_{out} is given by Figure 2.3(a). $k \mapsto -1/k$ maps this domain to itself, so (4.1.9) is valid here too. Thus (4.1.11) is valid on L_{out} where it is used to give $N(-ik)$ in terms of known functions. The fact that the expression for u (4.1.6) satisfies the radiation condition (3.1.22) can be verified using the method of steepest descent.

$\square$

Remark 4.1.2 (Rigorous considerations – verifying the boundary condition) *This proceeds by evaluating the solution on the boundary. Regarding the forcing term: in §2.1.4 we noted that a spectral representation of the forcing term could only be obtained by splitting the Ω. However, when z is on the boundary, no splitting is necessary since Ω is convex, hence $F(z, \bar{z})$ is given in terms of integrals of the transforms of f appearing in the GR, $\widehat{f}(k)$ (and also $\widetilde{f}(k)$ for the Poisson equation).*

1. *Using* (2.1.34), $F(x, 0)$ *is given by*

$$F(x, 0) = \frac{1}{4\pi} \int_0^{-\infty} \frac{dk}{k} e^{i\beta\left(k - \frac{1}{k}\right)x} \widehat{f}(k). \tag{4.1.17}$$

Using $k \mapsto -1/k$ *this cancels with the integral of* $\widetilde{f}(-k)$ *in* (4.1.3) *to leave*

$$u(x, 0) = \frac{1}{2\pi} \int_0^{\infty} \frac{dk}{k} e^{i\beta\left(k - \frac{1}{k}\right)x} \beta \left(k + \frac{1}{k}\right) D(-ik). \tag{4.1.18}$$

The boundary condition (4.1.2) *follows from*

$$\delta(x-\xi)=\frac{\beta}{2\pi}\int_0^{\infty}\frac{dk}{k}\left(k+\frac{1}{k}\right)e^{i\beta\left(k-\frac{1}{k}\right)(x-\xi)}. \tag{4.1.19}$$

This completeness relation can be obtained from the usual Fourier transform completeness relation

$$\delta(x-\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dl e^{il(x-\xi)} \tag{4.1.20}$$

by letting $l=\beta\left(k-\frac{1}{k}\right)$ *and noting that when* $k\in(0,\infty)$ *then* $l\in(-\infty,\infty)$.

2. *Using* (2.1.58), $F(x,0)$ *is given by*

$$F(x,0)=\frac{1}{4\pi}\int_0^{-\infty}\frac{dk}{k}e^{ikx}\widehat{f}(k)+\frac{1}{4\pi}\int_0^{-\infty}\frac{dk}{k}e^{-ikx}\widetilde{f}(k), \tag{4.1.21}$$

which cancels with the integrals of $\widetilde{f}(-k)$ *in* $\widehat{f}(-k)$(4.1.4) *using* $k\mapsto -k$ *to leave*

$$u(x,0)=\frac{1}{2\pi}\left(\int_0^{\infty}dke^{ikx}D(-ik)+\int_0^{\infty}dke^{-ikx}D(ik)\right). \tag{4.1.22}$$

The boundary condition (4.1.2) *follows from the usual Fourier transform inversion after using* $k\mapsto -k$ *in the second term.*

3. *Using* (2.1.46), $F(x,0)$ *is given by*

$$F(x,0)=\frac{1}{4\pi}\int_{\{k=-l,\, l\in L_{out}\}}\frac{dk}{k}e^{i\beta\left(k+\frac{1}{k}\right)x}\widehat{f}(k). \tag{4.1.23}$$

Using $k\mapsto 1/k$ *this cancels with the integral of* $\widetilde{f}(-k)$ *in* (4.1.6) *to leave*

$$u(x,0)=\frac{1}{2\pi}\int_{L_{out}}\frac{dk}{k}e^{i\beta\left(k+\frac{1}{k}\right)x}\beta\left(k-\frac{1}{k}\right)D(-ik). \tag{4.1.24}$$

The boundary condition (4.1.2) *follows from*

$$\delta(x-\xi)=\frac{\beta}{2\pi}\int_{L_{out}}\frac{dk}{k}\left(k-\frac{1}{k}\right)e^{i\beta\left(k+\frac{1}{k}\right)(x-\xi)}. \tag{4.1.25}$$

This completeness relation can be obtained from the usual Fourier transform completeness relation (4.1.20) *by letting* $l=\beta\left(k+\frac{1}{k}\right)$. *There are several choices of integration contour in* k *(which all lead to equivalent answers). One is* $k\in L_{out}$, *that is if* k *in the set*

$$(0,-1)\cup\{e^{i\theta},\,-\pi\le\theta\le 0\}\cup(1,\infty) \tag{4.1.26}$$

then $l\in(-\infty,\infty)$ *(Note that proposition 2.1.2 contains a similar discussion).*

Remark 4.1.3 (Orientation of half plane) *Here we chose the half plane domain to be $y > 0$, i.e. $z_1 = -\infty$, $z_2 = \infty$ because this is the usual convention. However in Proposition 3.1.11 we decided to orient Ω so that $\alpha_1 = \pi/2$, which would be the choice of half plane $x < 0$, ($z_1 = -\infty$, $z_2 = -\infty$). In this case the GR is valid in $i\mathbb{R}^+$ and the SC in $i\mathbb{R}^-$, which immediately gives $N(\pm ik)$ on the integration contours in terms of known functions, without needing to use any transformation such as $k \mapsto -k$.*

Remark 4.1.4 (The Dirichlet to Neumann map) *For simplicity consider the Laplace equation. The GR* (4.1.12) *and SC* (4.1.15) immediately *give the Dirichlet to Neumann map*

$$N(-ik) = |k|D(-ik), \quad k \in \mathbb{R}, \tag{4.1.27}$$

which can be inverted to give either the Neumann or Dirichlet boundary values in terms of the other. Classically this relation is only *obtained after* solving both *the Neumann and Dirichlet problems: [OHLM03, §5.5.1.2 page 178].*

Remark 4.1.5 (Comparison with classical transforms) *Consider the Poisson equation. The Dirichlet problem can be solved using* either *the Fourier transform in x, giving*

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx-|k|y} D(-ik) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{dk}{k}\, e^{ikx} \int_0^{\infty} d\eta\, e^{-|k||y-\eta|} \int_{-\infty}^{\infty} d\xi\, e^{-ik\xi} f(\xi,\eta) \tag{4.1.28}$$

which converges uniformly at $y = 0$, or *the sine transform in y, giving*

$$u(x,y) = \frac{1}{\pi}\int_0^{\infty} dk \sin ky \left(\int_{-\infty}^{\infty} d\xi\, e^{-k|x-\xi|}\left(u(\xi,0) + \int_0^{\infty} d\eta\, \sin k\eta f(\xi,\eta)\right)\right) \tag{4.1.29}$$

which does not *converge uniformly at $y = 0$. The expression* (4.1.4) *is easily shown to be equal to* (4.1.28). *Indeed, the terms involving $D(-ik)$ in* (4.1.4) *are equal to the term involving $D(-ik)$ in* (4.1.28) *using $k \mapsto -k$. To show the forcing terms of* (4.1.4) *are equal to those of* (4.1.28), *use* (2.1.58) *to obtain a spectral representation of $F(z,\bar{z})$ by splitting the domain for $\eta \lesseqgtr y$ ($\theta_F = 0$). The sine transform solution* (4.1.29) *can be shown to be equivalent to the Fourier transform solution* (4.1.28) *only by* deforming contours. *Similar statements hold for the modified Helmholtz and Helmholtz equations, although for the Helmholtz there is the additional difficulty of ensuring that the branches of the square roots in the exponentials are such that the solution satisfies the radiation condition.*

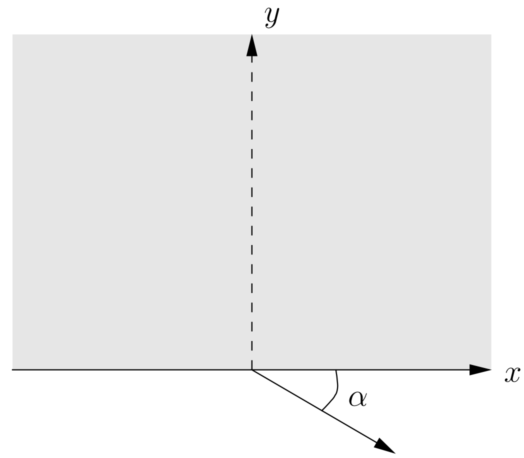


Figure 4.1: The upper half plane with the angle of the oblique boundary condition.

4.1.2 Oblique Robin boundary conditions

One of the advantages of the Fokas method is that the solution Steps are independent of the boundary conditions, which makes this section very similar to the previous section on Dirichlet boundary conditions.

Proposition 4.1.6 *Let the complex-valued function $u(x, y)$ satisfy* (1.1.1) *in the upper half plane*

$$\Omega = \{-\infty < x < \infty,\, y > 0\}, \tag{4.1.30}$$

with the condition that $u \to 0$ at infinity, and the oblique Robin boundary conditions

$$-\sin\alpha\, u_y(x,0) + \cos\alpha\, u_x(x,0) + \gamma u(x,0) = g(x), \quad -\infty < x < \infty, \tag{4.1.31}$$

where g has sufficient decay at infinity (e.g. $g \in L_1$). This problem is well-posed if $0 < \alpha < \pi$ and $\gamma \geq 0$ (see Remark 4.2.10). The first two terms correspond to the derivative of u at angle α to the boundary, see Figure (4.1)*. The Neumann and Robin problems correspond to the folowing particular choices of α and γ:*

$$\textit{Neumann: } \alpha = \frac{\pi}{2},\ \gamma = 0; \qquad \textit{Robin: } \alpha = \frac{\pi}{2},\ \gamma > 0. \tag{4.1.32}$$

Then the solution $u(x, y)$ is given by

1. *if $\lambda = -4\beta^2$ (the modified Helmholtz equation)*

$$u(z,\bar{z}) = \frac{1}{4\pi i}\int_0^\infty \frac{dk}{k} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} \left(\frac{-2\beta\left(k + \frac{1}{k}\right) G(-ik) + iH(k)\widetilde{f}(-k)}{\overline{H(-\bar{k})}} \right) + F(z,\bar{z}), \tag{4.1.33}$$

where

$$G(k) = \int_{-\infty}^{\infty} ds\, e^{\beta\left(k+\frac{1}{k}\right)s}\, g(s), \tag{4.1.34}$$

$$H(k) = \beta\left(ke^{i\alpha} - \frac{1}{ke^{i\alpha}}\right) - i\gamma, \tag{4.1.35}$$

and

$$\widetilde{f}(k) = \widehat{f}(1/k) \tag{4.1.36}$$

where $\widehat{f}(k)$ *is defined by* (3.1.6), *and* $F(z,\bar{z})$ *is given by* (2.1.33).

2. *if* $\lambda = 0$ *(the Poisson equation)*

$$\begin{aligned} u(z,\bar{z}) =& \frac{1}{4\pi i}\int_0^{\infty}\frac{dk}{k}e^{ikz}\left(\frac{-2kG(-ik)+iH(k)\widetilde{f}(-k)}{\overline{H(-\bar{k})}}\right) \\ &- \frac{1}{4\pi i}\int_0^{\infty}\frac{dk}{k}e^{-ik\bar{z}}\left(\frac{-2kG(ik)-i\overline{H(\bar{k})}\widehat{f}(-k)}{H(-k)}\right) + F(z,\bar{z}), \end{aligned} \tag{4.1.37}$$

where

$$G(k) = \int_{-\infty}^{\infty} ds\, e^{ks}\, g(s), \tag{4.1.38}$$

$$H(k) = ke^{i\alpha} - i\gamma, \tag{4.1.39}$$

and $\widehat{f}(k), \widetilde{f}(k)$ *are given by* (3.1.8), *and* $F(z,\bar{z})$ *is given by* (2.1.57). *If* $\gamma = 0$ *then a solution exists if and only if* g *and* f *satisfy the solvability condition*

$$G(0) = -\sin\alpha\,\widehat{f}(0). \tag{4.1.40}$$

3. *if* $\lambda = 4\beta^2$ *(the Helmholtz equation) and* u *satisfies the radiation condition* (3.1.22) *for* $0 < \arg z < \pi$, *then*

$$u(z,\bar{z}) = \frac{1}{4\pi}\int_{L_{out}}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\left(\frac{-2\beta\left(k-\frac{1}{k}\right)G(-ik)+iH(k)\widetilde{f}(-k)}{\overline{H(-\bar{k})}}\right) + F(z,\bar{z}), \tag{4.1.41}$$

where

$$G(k) = \int_{-\infty}^{\infty} ds\, e^{\left(k-\frac{1}{k}\right)s}\, g(s), \tag{4.1.42}$$

$$H(k) = \beta\left(ke^{i\alpha} + \frac{1}{ke^{i\alpha}}\right) - i\gamma, \tag{4.1.43}$$

and

$$\widetilde{f}(k) = \widehat{f}(-1/k), \tag{4.1.44}$$

where $\widehat{f}(k)$ *is defined by* (3.1.10), *and* $F(z,\bar{z})$ *is given by* (2.1.45).

Proof This is very similar to that of Proposition 4.1.1.

1. Rearrange (4.1.31) to give the Neumann boundary value $-u_y(x,0)$ in terms of $u_x(x,0)$, $u(x,0)$, and $g(x)$:

$$-u_y(x,0) = \frac{1}{\sin\alpha}\left(g(x) - \cos\alpha u_x(x,0) - \gamma u(x,0)\right), \quad -\infty < x < \infty, \tag{4.1.45}$$

integrate by parts the term involving $u_x(x,0)$ (using $u \to 0$ at infinity) to yield

$$\widehat{u}_1(k) = \frac{H(k)D(-ik) + iG(-ik)}{\sin\alpha}, \tag{4.1.46}$$

where

$$D(k) = \int_{-\infty}^{\infty} ds\, e^{\beta\left(k+\frac{1}{k}\right)s} u(s,0). \tag{4.1.47}$$

The GR (3.1.5) is then

$$H(k)D(-ik) + iG(-ik) = -i\sin\alpha \widehat{f}(k), \quad k \in \mathbb{R}^-, \tag{4.1.48}$$

where, as for the Dirichlet problem, the region of validity is dictated by lemma 3.1.5 part b). Let $k \mapsto 1/k$ (or equivalently take the Schwartz conjugate) to give the second equation

$$\overline{H(\bar{k})}D(ik) - iG(ik) = i\sin\alpha \widetilde{f}(k), \quad k \in \mathbb{R}^-. \tag{4.1.49}$$

$k \mapsto -k$ yields the analogues of (4.1.10) and (4.1.11) as

$$H(-k)D(ik) + iG(ik) = -i\sin\alpha \widehat{f}(-k), \quad k \in \mathbb{R}^+, \tag{4.1.50}$$

$$\overline{H(-\bar{k})}D(-ik) - iG(-ik) = i\sin\alpha \widetilde{f}(-k), \quad k \in \mathbb{R}^+. \tag{4.1.51}$$

(4.1.51) gives $D(-ik)$ on $\mathbb{R}^+$ in terms of the known transforms $G(-ik), \widetilde{f}(-k)$. Substituting this into (2.1.28) gives (4.1.33). There is the possibility that $\overline{H(-\bar{k})}$ has zeros on the contour, which would mean that (4.1.33) is not well-defined. The two zeros of $\overline{H(-\bar{k})}$ are at

$$k = \left(\frac{i\gamma}{2\beta} \pm \sqrt{1 - \left(\frac{\gamma}{2\beta}\right)^2}\right) e^{i\alpha}. \tag{4.1.52}$$

For $\frac{\gamma}{2\beta} < 1$ these are on the unit circle, but never on $\mathbb{R}^+$ since $-\alpha < \arg k < \pi + \alpha$ and $0 < \alpha < \pi$. For $\frac{\gamma}{2\beta} > 1$ these are on $i\mathbb{R}^+ e^{i\alpha}$ so never on $\mathbb{R}^+$.

2. Proceeding identically as in case 1 we find that (4.1.48) – (4.1.51) are valid for $H(k)$ given by (4.1.39), $G(k)$ given by (4.1.38), and

$$D(k) = \int_{-\infty}^{\infty} ds\, e^{ks} u(s,0). \tag{4.1.53}$$

(4.1.51) gives $D(-ik)$ on $\mathbb{R}^+$ in terms of the known transforms $D(-ik), \widetilde{f}(-k)$ and (4.1.50) gives $D(ik)$ on $\mathbb{R}^+$ in terms of the known transforms $D(ik), \widehat{f}(-k)$. Substituting these into (2.1.52) gives (4.1.37). Now $\overline{H(-\bar{k})}$ has its zero at $k = i\gamma e^{i\alpha}$ and $H(-k)$ has its zero at $k = -i\gamma e^{-i\alpha}$, both of which are never on $\mathbb{R}^+$ except in the case $\gamma = 0$ (oblique Neumann boundary conditions) when both are zero. In this case evaluating the GR (4.1.48) at $k = 0$ gives (4.1.40) and so the parts of the two integrals in (4.1.37) near $k = 0$ are equal to

$$-\frac{1}{4\pi} \int_0^{\varepsilon} \frac{dk}{k} \left(\widetilde{f}(0) + \widehat{f}(0)\right) \tag{4.1.54}$$

(remember $\widetilde{f}(0) = \widehat{f}(0)$). This singularity at $k = 0$ is removable once F is expressed using (2.1.58).

3. This follows in exactly the same manner as the modified Helmholtz equation except the signs of all the $1/k$ terms are reversed. As for the Dirichlet problem the GR (4.1.48) and SC (4.1.49) are valid for k in the set

$$\{k \in \mathbb{C} : k = -l, l \in L_{out}\} \tag{4.1.55}$$

where the contour L_{out} is given by Figure 2.3(a). Now the two zeros of $\overline{H(-\bar{k})}$ are at

$$k = i\left(\frac{\gamma}{2\beta} \pm \sqrt{1 + \left(\frac{\gamma}{2\beta}\right)^2}\right) e^{i\alpha}. \tag{4.1.56}$$

One $(+)$ is on $i\mathbb{R}^+e^{i\alpha}$ and has modulus greater than one, the other $(-)$ is on $i\mathbb{R}^+e^{-i\alpha}$ and has modulus less than one; so both are never on L_{out}. The fact that the expression for u (4.1.41) satisfies the radiation condition (3.1.22) can be verified using the method of steepest descent.

□

Remark 4.1.7 (Rigorous considerations – verifying the boundary condition) *This follows in an identical way to Remark 4.1.2 using*

$$\left(-\sin\alpha\,\frac{\partial}{\partial y}+\cos\alpha\,\frac{\partial}{\partial x}+\gamma\right)e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}=-i\overline{H(-\bar{k})},\tag{4.1.57}$$

$$\left(-\sin\alpha\,\frac{\partial}{\partial y}+\cos\alpha\,\frac{\partial}{\partial x}+\gamma\right)F(x,0)=-i\overline{H(-\bar{k})}\frac{1}{4\pi}\int_0^{-\infty}\frac{dk}{k}e^{i\beta\left(k-\frac{1}{k}\right)x}\hat{f}(k),\tag{4.1.58}$$

and

$$\overline{H(1/\bar{k})}=-H(k),\tag{4.1.59}$$

for the modified Helmholtz equation, and similar expressions for the Poisson and Helmholtz equations.

Remark 4.1.8 (Comparison with the classical solutions) *There does* not *exist an appropriate transform in y for solving* (1.1.1) *in the upper half plane with the boundary conditions* (4.1.31)*, except for the case of Robin boundary conditions, $\alpha=\pi/2$, where the appropriate transform is obtained by spectral analysis of the differential operator*

$$-\frac{d^2}{dy^2},\quad 0<y<\infty,\quad u'(0)+\gamma u(0)=0\tag{4.1.60}$$

and appears in [Sta67, p. 295, Ex. 4.24]. The appropriate transform in x is the Fourier transform, which yields an expression equivalent to (4.1.33),(4.1.37) *and* (4.1.41) *(although for the Helmholtz equation, ensuring that the branches of the square roots are such that the solution satisfies the radiation condition is awkward).*

The solution of this boundary value problem by the method of images is given in [GT01] for the Laplace equation and [Kel81] for other PDEs including the modified Helmholtz and Helmholtz equations. The Green's function is given by

$$\begin{aligned}G(\xi,\eta;x,y)=&E(\xi,\eta;x,y)-E(\xi,\eta;x,-y)\\&+2\sin\alpha\,\frac{\partial}{\partial y}\int_0^{\infty}ds\,e^{-\gamma s}E(\xi,\eta;x+s\cos\alpha,-y-s\sin\alpha).\end{aligned}\tag{4.1.61}$$

(In [Kel81] his a equals our $-\gamma$, and his $b=(-\cos\alpha,\sin\alpha)$.) Extending the discussion of §1.4.3 to the case where the image solution is given as a finite sum plus a semi-infinite integral of images: if the boundary conditions are such that their transforms can be computed explicitly then the solutions (4.1.33), (4.1.37) *and* (4.1.41) *are the best possible representations of the solution. If the boundary conditions are such that their transforms cannot be computed explicitly, for the Poisson equation both* (4.1.61) *and* (4.1.37) *give*

the Green's function as an infinite integral of elementary functions, so neither has a clear advantage. For the modified Helmholtz and Helmholtz equations, (4.1.33) *and* (4.1.41) *give the Green's function as an integral of elementary functions, whereas* (4.1.61) *gives it as an integral of special functions (which themselves must be computed via an integral representation), thus the Fokas method solutions* (4.1.33) *and* (4.1.41) *are superior to the image method solution* (4.1.61).

Remark 4.1.9 (Solvability conditions) *The existence of a solvability condition for the Poisson equation is well known for Neumann boundary conditions:*

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds = -\int_{\Omega} f d\xi d\eta, \tag{4.1.62}$$

which in this case is

$$-\int_{-\infty}^{\infty} dx\, u_y(x,0) = -\int_0^{\infty} dy \int_{-\infty}^{\infty} dx\, f(x,y). \tag{4.1.63}$$

Surprisingly the existence of a solvability condition for oblique Neumann boundary conditions(4.1.31) *with* $\gamma = 0$ *appears not to be well known (for example, [GT01] and [Kel81] do not mention it). For these boundary conditions the appropriate condition is obtained by multiplying* (4.1.63) *by* $-\sin\alpha$ *and adding*

$$\cos\alpha \int_{-\infty}^{\infty} dx\ u_x(x,0)$$

which equals zero by integration. In general, the global relation provides the easiest way for determining whether the given boundary conditions and forcing must satisfy a solvability condition. This is achieved by seeing whether there exist any k *in the region of validity at which the coefficients of the unknown boundary values vanish, leaving a condition on the known boundary values and forcing.*

4.2 The Helmholtz equation in the quarter plane

In this section we use the integral representation of propositions 2.1.13 applied to the interior of the polygon defined by $z_1 = 0, z_2 = \infty, z_3 = i\infty$, see Figure 4.2.

Since $\Omega = \Omega^{(i)}$ is convex, the contours of integration do not depend on the position of z (see remark 2.1.11). The contour $L_{out\,1}$ is just L_{out}, and l_2 is L_{out} rotated by $\pi/2$ anticlockwise. To simplify notation we write L_j instead of $L_{out\,j}$, see Figure 4.3.

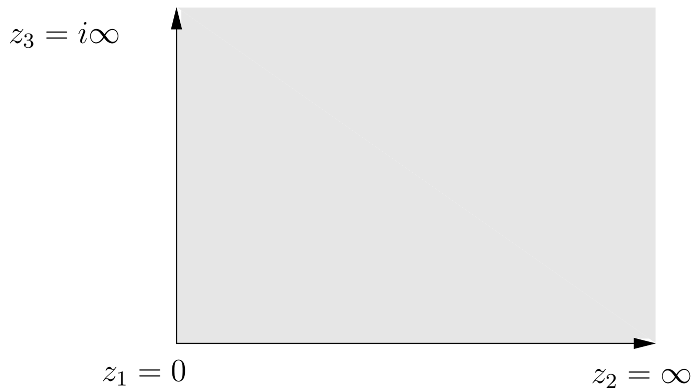
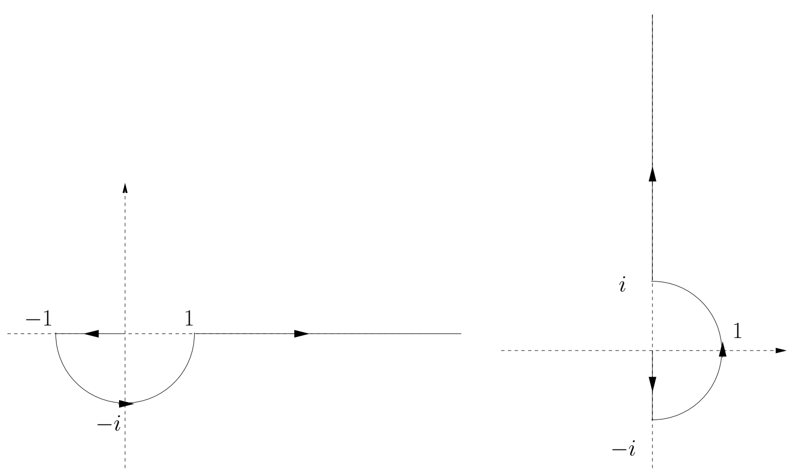


Figure 4.2: The quarter plane.



(a) The contour L_1 (b) The contour L_2

Figure 4.3: The contours in the IR of the Helmholtz equation in the quarter plane.

4.2.1 Dirichlet boundary conditions

Proposition 4.2.1 *Let the complex-valued function $u(x,y)$ satisfy the Helmholtz equation in the quarter plane,*

$$\Omega = \{0 < x < \infty,\, 0 < y < \infty\}, \tag{4.2.1}$$

with the Dirichlet boundary conditions

$$u(x,0) = d_1(x),\, 0 < x < \infty, \quad u(0,y) = d_2(y), \quad 0 < y < \infty; \tag{4.2.2}$$

where the complex-valued functions d_1 and d_2 have sufficient decay at infinity (e.g. $d_j \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$), and with the radiation condition (3.1.22). *Then the solution $u(x,y)$ is given by*

$$\begin{aligned} u(z,\bar{z}) =& \frac{1}{4\pi}\int_{L_1}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\Big[2\beta\left(k-\frac{1}{k}\right)D_1(-ik) + 2i\beta\left(k+\frac{1}{k}\right)D_2(-k) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \widehat{f}(-k) - \widetilde{f}(-k)\Big] \\ &+ \frac{1}{4\pi}\int_{L_2}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\left[-2\beta\left(k-\frac{1}{k}\right)D_1(ik) - 2i\beta\left(k+\frac{1}{k}\right)D_2(k) - \widetilde{f}(k)\right] \\ &+ F(z,\bar{z}) \end{aligned} \tag{4.2.3}$$

where

$$D_1(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})s}d_1(s)ds, \quad D_2(k) = \int_0^\infty e^{\beta(k-\frac{1}{k})s}d_2(s)ds, \tag{4.2.4}$$

$$\widetilde{f}(k) = \widehat{f}(-1/k), \tag{4.2.5}$$

where $\widehat{f}(k)$ is defined by (3.1.10) *and $F(z,\bar{z})$ is given by* (2.1.45).

Proof Parametrise side 1 by $z = s$, $0 < s < \infty$, and side 2 by $z = is$, $0 < s < \infty$, to give

$$\hat{u}_1(k) = iN_1(-ik) + i\beta\left(k-\frac{1}{k}\right)D_1(-ik), \tag{4.2.6}$$

$$\hat{u}_2(k) = iN_2(k) + \beta\left(k+\frac{1}{k}\right)D_2(k), \tag{4.2.7}$$

where

$$N_2(k) = -\int_0^\infty e^{\beta\left(k-\frac{1}{k}\right)s}q_x(0,s)ds,$$

$$N_1(k) = -\int_0^\infty e^{\beta\left(k-\frac{1}{k}\right)s}q_y(s,0)ds,$$

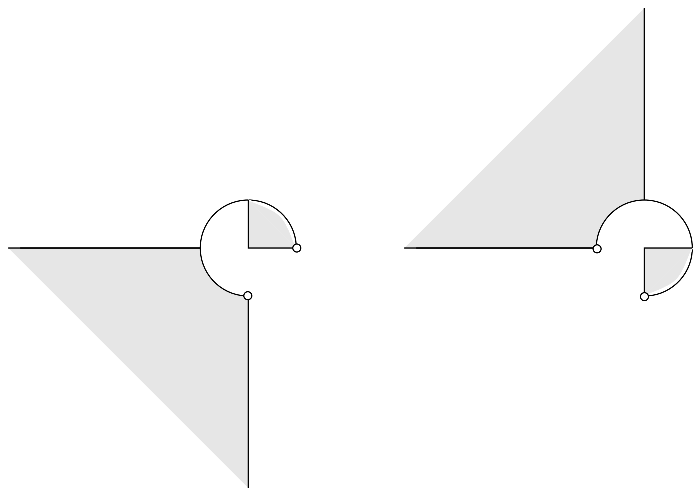


Figure 4.4: The domains D and $\widetilde{D}$.

The global relation (3.1.9) is

$$N_1(-ik) + \beta\left(k - \frac{1}{k}\right) D_1(-ik) + N_2(k) - i\beta\left(k + \frac{1}{k}\right) D_2(k) = -\widehat{f}(k), \quad k \in D, \tag{4.2.8}$$

where the domain D is given by

$$D = \left\{|k| > 1,\, \pi \le \arg k \le \frac{3\pi}{2}\right\} \cup \left\{|k| < 1,\, 0 \le \arg k \le \frac{\pi}{2}\right\} \cup \left\{|k| = 1,\, 0 < \arg k < \frac{3\pi}{2}\right\} \tag{4.2.9}$$

using Lemma 3.1.5 part c) with $\theta_1 = 0$, $\theta_2 = \pi/2$, see Figure 4.4 Let $k \mapsto -1/k$ to give the second equation

$$N_1(ik) + \beta\left(k - \frac{1}{k}\right) D_1(ik) + N_2(k) + i\beta\left(k + \frac{1}{k}\right) D_2(k) = -\widetilde{f}(k), \quad k \in \widetilde{D}, \tag{4.2.10}$$

where $\widetilde{D}$ is obtained from D via the transformation $k \mapsto -1/k$:

$$\widetilde{D} = \left\{|k| < 1,\, -\frac{\pi}{2} \le \arg k \le 0\right\} \cup \left\{|k| > 1,\, \frac{\pi}{2} \le \arg k \le \pi\right\} \cup \left\{|k| = 1,\, -\frac{\pi}{2} < \arg k < \pi\right\},$$

see Figure 4.4. Note that the second equation (4.2.10) *cannot* be obtained from (4.2.8) by Schwartz-conjugation (as in the case of the Poisson and modified Helmholtz equations) since the act of complex conjugation does *not* preserve the radiation condition (3.1.22).

In the IR we have two unknown functions $N_1(-ik)$ and $N_2(k)$ on L_1 and L_2 respectively. We have four equations: the GR, SC and $k \mapsto -k$ in both. The SC and $k \mapsto -k$ in the

GR are valid on L_2, and $k \mapsto -k$ in the SC and $k \mapsto -k$ in the GR are valid on L_1. Following steps 1 and 2 of the solution procedure outlined in §1.3, our plan is to express both $N_1(-ik)$ and $N_2(k)$ in terms of one unknown (here we chose $N_1(ik)$) and known functions, and for the contribution from that one unknown to vanish by analyticity.

Both (4.2.8) and (4.2.10) are valid in the intersection of D and $\widetilde{D}$ which is $\{k = -l, l \in L_1\} \setminus \{\pm 1\}$. Subtract (4.2.10) from (4.2.8) to eliminate $N_2(k)$ and obtain

$$N_1(-ik) - N_1(ik) + \beta\left(k - \frac{1}{k}\right)(D_1(-ik) - D_1(ik)) - 2i\beta\left(k + \frac{1}{k}\right)D_2(k) = -\widehat{f}(k) + \widetilde{f}(k), \quad \{k = -l, l \in L_1\}.$$

Let $k \mapsto -k$ and rearrange to find

$$N_1(-ik) = N_1(ik) - \beta\left(k - \frac{1}{k}\right)(D_1(-ik) - D_1(ik)) + 2i\beta\left(k + \frac{1}{k}\right)D_2(k) + \widehat{f}(-k) - \widetilde{f}(-k), \tag{4.2.11}$$

for $k \in L_1$. Substitute this into (4.2.6) to give

$$\hat{u}_1(k) = iN_1(ik) + 2i\beta\left(k - \frac{1}{k}\right)D_1(-ik) - i\beta\left(k - \frac{1}{k}\right)D_1(ik) - 2\beta\left(k + \frac{1}{k}\right)D_2(-k) + i(\widehat{f}(-k) - \widetilde{f}(-k)), \quad k \in L_1.$$

The SC (4.2.10) gives $N_2(k)$ in terms of known functions and $N_1(ik)$, and is valid on L_2, hence

$$\hat{u}_2(k) = -iN_1(ik) + 2\beta\left(k + \frac{1}{k}\right)D_2(k) - i\beta\left(k - \frac{1}{k}\right)D_1(ik) - i\widetilde{f}(-k), \quad k \in L_2. \tag{4.2.12}$$

Substitute these expressions for $\hat{u}_1(k)$ and $\hat{u}_2(k)$ into (2.1.13) to yield the solution in terms of integrals of known functions plus

$$\frac{1}{4\pi}\int_{L_1 - L_2} \frac{dk}{k} e^{i\beta\left(kz + \frac{\bar{z}}{k}\right)} N_1(ik). \tag{4.2.13}$$

This term equals zero by Cauchy's theorem. Indeed $N_1(ik)$ is analytic and bounded at infinity in D_1 given by

$$D_1 = \Big\{|k| \le 1,\ \Im k \le 0\Big\} \cup \Big\{|k| \ge 1,\ \Im k \ge 0\Big\} \tag{4.2.14}$$

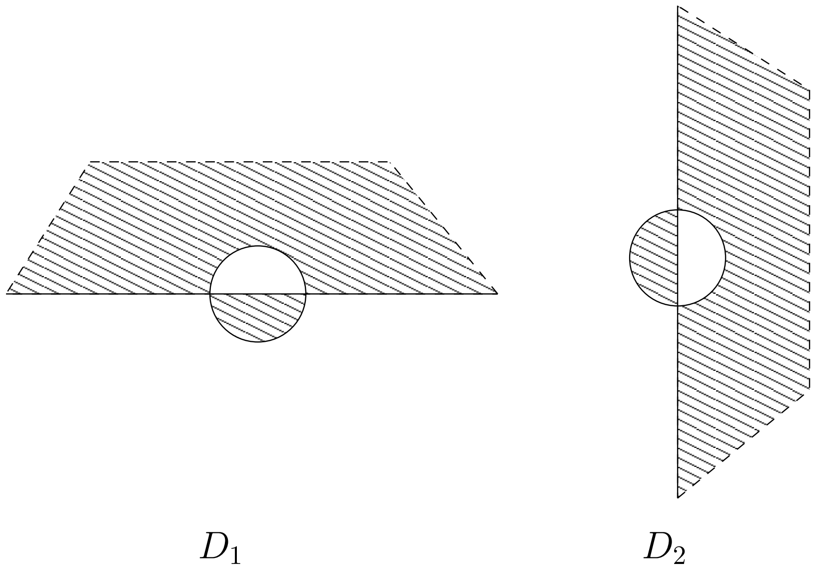


Figure 4.5: The domains D_1 and D_2.

see Figure 4.5, Now, $e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}$ is analytic for all $k \in \mathbb{C}$ and bounded at infinity in $0 \leq \arg k \leq \frac{\pi}{2}$ and at zero in $\pi \leq \arg k \leq \frac{3\pi}{2}$. What remains is equal to (4.2.3) after using the analyticity considerations above to deform the contour for the $D_1(ik)$ term on L_2 to L_1. The fact that the expression for u (4.2.3) satisfies the radiation condition (3.1.22) can be verified by the method of steepest descent. □

Remark 4.2.2 (Rigorous considerations – verifying the boundary condition) *When $y = 0$*

$$u(x,0) = \frac{1}{2\pi}\int_{L_1} \frac{dk}{k} e^{i\beta\left(k+\frac{1}{k}\right)x}\beta\left(k - \frac{1}{k}\right) D(-ik). \tag{4.2.15}$$

which equals $d_1(x)$ by (4.1.25). This follows from the following considerations: When $k \mapsto 1/k$ then

- $L_2 \mapsto -L_2$ *(ie. L_2 oriented the other way),*
- $D_2(k) \mapsto D_2(-k)$*,*
- $D_1(ik) \mapsto D_1(ik)$*,*
- $e^{i\beta\left(k+\frac{1}{k}\right)x} \mapsto e^{i\beta\left(k+\frac{1}{k}\right)x}$*,*

and

- *$D_1(ik)$ and $e^{i\beta\left(k+\frac{1}{k}\right)x}$ are analytic in D_1.*
- *$D_2(-k)$ is analytic in D_2.*
- *$\widehat{f}(k)$ is analytic in $D_1 \cap D_2$.*

where

$$D_2 = \Big\{k \in \mathbb{C}, \{|k| \leq 1\} \cap \{\Re k \geq 0\}, \quad \{|k| \geq 1\} \cap \{\Re k \leq 0\}\Big\}. \tag{4.2.16}$$

see Figure 4.5. Using these facts, when $y = 0$:

- *the $D_2(k)$ term on L_2 combined with the $D_2(-k)$ term on L_1 are equal to zero using $k \mapsto 1/k$ and analyticity.*
- *the $D_1(ik)$ term on L_2 equals zero ($k \mapsto 1/k$ shows it equals minus itself).*
- *the $\widehat{f}(-k)$ term on L_1 combined with the $\widetilde{f}(k)$ term on L_2 are equal to zero using $k \mapsto 1/k$ and analyticity.*
- *using (2.1.46), $F(x,0)$ is given by*

$$F(x,0) = \frac{1}{4\pi}\int_{\{k=-l, l\in L_1\}} \frac{dk}{k} e^{i\beta\left(k+\frac{1}{k}\right)x}\widehat{f}(k), \tag{4.2.17}$$

and using $k \mapsto 1/k$ this cancels with the $\widetilde{f}(-k)$ term on L_1.

All that remains is (4.2.15).

When $x = 0$

$$u(0,y) = \frac{i\beta}{2\pi}\int_{L_2} \frac{dk}{k}\left(k + \frac{1}{k}\right) e^{-\beta\left(k-\frac{1}{k}\right)y} D_2(k), \tag{4.2.18}$$

which equals $d_2(y)$ by

$$\delta(y-\eta) = \frac{i\beta}{2\pi}\int_{L_2} \frac{dk}{k}\left(k + \frac{1}{k}\right) e^{-\beta\left(k-\frac{1}{k}\right)(y-\eta)}., \tag{4.2.19}$$

which can be obtained from (4.1.25) by letting $k \mapsto ik$. This follows in a very similar way to $y = 0$ with the following additional considerations: when $k \mapsto -1/k$

- *$L_1 \mapsto -L_1$ (ie. L_1 oriented the other way),*
- *$D_2(k) \mapsto D_2(k)$,*

- $D_1(-ik) \mapsto D_1(ik)$,
- $e^{-\beta\left(k-\frac{1}{k}\right)y} \mapsto e^{-\beta\left(k-\frac{1}{k}\right)y}$,

and

- $e^{-\beta\left(k-\frac{1}{k}\right)y}$ *is analytic in* D_2.

Using these facts we deduce that, when $y = 0$,

- *the* $D_1(-ik)$ *term on* L_1 *combined with the* $D_1(ik)$ *term on* L_1 *are equal to zero using* $k \mapsto -1/k$ *and analyticity,*
- *the* $D_2(-k)$ *term on* L_1 *equals zero (*$k \mapsto -1/k$ *shows it equals minus itself),*
- *the* $\widehat{f}(-k)$ *term on* L_1 *and the* $\widetilde{f}(-k)$ *term on* L_1 *cancel using* $k \mapsto 1/k$.
- *using* (2.1.46) $F(0, y)$ *is given by*

$$F(x,0) = \frac{1}{4\pi} \int_{\{k=-l, l\in L_2\}} \frac{dk}{k} e^{-\beta\left(k-\frac{1}{k}\right)y} \widehat{f}(k), \tag{4.2.20}$$

  and using $k \mapsto -1/k$ *this cancels with the* $\widetilde{f}(k)$ *term on* L_2.

All that remains is (4.2.18).

Remark 4.2.3 (Comparison with classical) *The Dirichlet problem can be solved using the Sine transform in* x *or in* y*, However the resulting expressions are* not *uniformly convergent on the boundary, either at* $x = 0$ *if the* x *transform is used, or at* $y = 0$ *if the* y *transform is used. These two expressions can be obtained from the solution obtained by the Fokas method* (4.2.3) *by using appropriate contour deformations.*

The Dirichlet problem can be solved using the method of images,

$$G = E(\xi, \eta; x, y) - E(\xi, \eta; x, -y) - E(\xi, \eta; -x, y) + E(\xi, \eta; -x, -y), \tag{4.2.21}$$

and the solution is uniformly convergent at the boundary. By the discussion of §1.4.3, if the boundary conditions are such that their transforms can be computed explicitly, then the Fokas method solution (4.2.3) *is superior to the image method solution* (4.2.21). *In all other cases, neither* (4.2.21) *nor* (4.2.3) *has a clear advantage over the other.*

4.2.2 Oblique Robin boundary conditions

As for the half plane, the method of solution for oblique Robin boundary conditions is very similar to that for Dirichlet boundary conditions. There are two main differences:

1. The transforms of the boundary values in the GR are now multiplied by rational functions of k involving α_j, γ_j ($H_j(k)$). This has two effects:
   (a) The $H_j(k)$ must satisfy a certain condition for the transform of the unknown boundary values to vanish by analyticity, this imposes some restrictions on α_j, γ_j (given by (4.2.25) and (4.2.26) below, see also Remark 4.2.6).
   (b) The contours of integration L_j must be deformed to avoid poles of the integrand at zeros of the $H_j(k)$.
2. The solution contains the value of u at the corner of the domain. This arises from the fact that the boundary conditions involve $u_x(x,0), u_y(0,y)$ (derivatives of the Dirichlet data) which are integrated by parts, introducing $u(0,0)$. For certain α_j these contributions cancel. Exactly the same situation arises if the problem can be solved in the physical space (by images), see remark 4.2.8.

Proposition 4.2.4 *Let the complex-valued function $u(x,y)$ satisfy the Helmholtz equation in the quarter plane,*

$$\Omega = \{0 < x < \infty,\, 0 < y < \infty\}, \tag{4.2.22}$$

with the radiation condition (3.1.22) *and the oblique Robin boundary conditions*

$$-\sin\alpha_1\, u_y(x,0) + \cos\alpha_1\, u_x(x,0) + \gamma_1 u(x,0) = g_1(x), \quad 0 < x < \infty, \tag{4.2.23a}$$

$$-\sin\alpha_2\, u_x(0,y) + \cos\alpha_2\, u_y(0,y) + \gamma_2 u(0,y) = g_2(y), \quad 0 < y < \infty, \tag{4.2.23b}$$

where g_j have sufficient decay at infinity (e.g. $g_j \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$). This problem is well-posed if $0 < \alpha_j < \pi$ and $\gamma_j \in \mathbb{R}^+$ (see Remark 4.2.10). The first two terms in each boundary condition correspond to the derivative of u at angle α_j to the boundary, see Figure (4.6) *The Neumann and Robin problems correspond to the following particular choices of α_j and γ_j:*

$$\textit{Neumann: } \alpha_j = \frac{\pi}{2},\ \gamma_j = 0; \qquad \textit{Robin: } \alpha_j = \frac{\pi}{2},\ \gamma_j > 0. \tag{4.2.24}$$

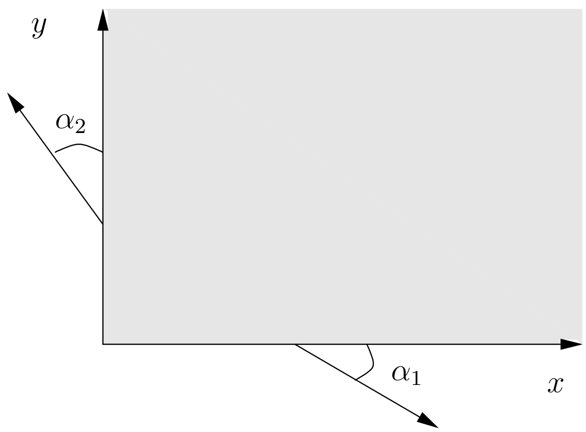


Figure 4.6: The quarter plane with the angle of the oblique Robin boundary conditions.

Suppose the α_j and γ_j satisfy the following two conditions:

$$\alpha_1 + \alpha_2 = \frac{n\pi}{2},\ n = 1, 2, 3, \tag{4.2.25}$$

and

$$\gamma_2^2 \sin 2\alpha_1 - \gamma_1^2 \sin 2\alpha_2 = 0, \tag{4.2.26}$$

then the solution $u(x, y)$ is given by

$$\begin{aligned} u(z,\bar{z}) =& \frac{1}{4\pi i}\int_{\mathcal{L}_1} \frac{dk}{k} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} \Bigg[-\frac{2\beta \sin\alpha_1 \left(k - \frac{1}{k}\right)\left(G_1(-ik) + \cot\alpha_1 d_1\right) + H_1(k)\widetilde{f}(-k)}{\overline{H_1(-\bar{k})}} \\ & - \frac{H_1(k)\left(2\beta\sin\alpha_2\left(k + \frac{1}{k}\right)\left(G_2(-k) + \cot\alpha_2 d_2\right) + \widetilde{f}(-k)\right)}{H_2(-k)\overline{H_1(-\bar{k})}} \Bigg] \\ & \frac{1}{4\pi i}\int_{\mathcal{L}_2} \frac{dk}{k} e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)} \Bigg[-\frac{2\beta \sin\alpha_2 \left(k + \frac{1}{k}\right)\left(G_2(k) + \cot\alpha_2 d_2\right) - H_2(k)\widetilde{f}(k)}{\overline{H_2(\bar{k})}} \\ & - \frac{H_2(k) 2\beta\sin\alpha_1\left(k - \frac{1}{k}\right)\left(G_1(ik) + \cot\alpha_1 d_1\right)}{H_1(-k)\overline{H_2(\bar{k})}} \Bigg] + F(z,\bar{z}), \end{aligned} \tag{4.2.27}$$

where

$$G_j(k) = \frac{1}{\sin\alpha_j}\int_0^\infty ds\, e^{\beta\left(k - \frac{1}{k}\right)s}\, g_j(s), \tag{4.2.28}$$

$$H_1(k) = \beta\left(ke^{i\alpha_1} + \frac{1}{ke^{i\alpha_1}}\right) - i\gamma_1, \tag{4.2.29}$$

$$H_2(k) = \beta\left(ke^{-i\alpha_2} - \frac{1}{ke^{-i\alpha_2}}\right) - \gamma_2, \tag{4.2.30}$$

$$\widetilde{f}(k) = \widehat{f}(1/k) \tag{4.2.31}$$

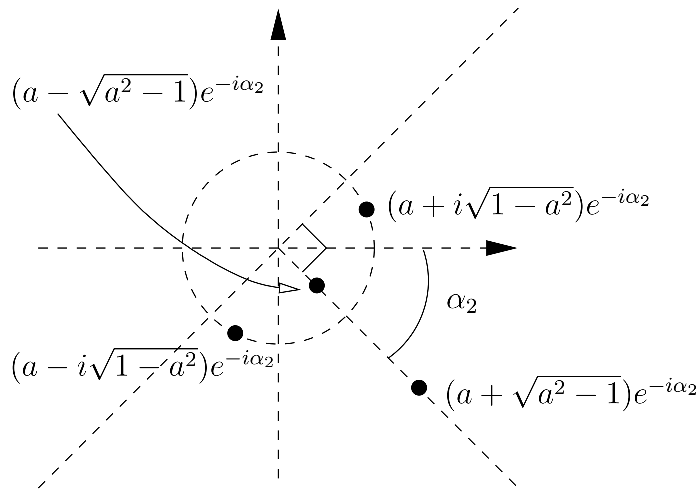


Figure 4.7: The zeros of $\overline{H_2(\bar{k})}$ in the k plane for $|a| \lesseqgtr 1$ where $a = \frac{\gamma_2}{2\beta}$.

where $\widehat{f}(k)$ is defined by (3.1.10)*and* $F(z, \bar{z})$ *is given by* (2.1.45)*, and where the contours $\mathcal{L}_1$ and $\mathcal{L}_2$ are deformations of L_1 and L_2, which depend on α_2, and are shown in Figures 4.2.4 and 4.2.4 for $0 < \alpha_2 < \pi/2$ and $\pi/2 \le \alpha_2 < \pi$ respectively, where the black dots are zeros of $\overline{H_2(\bar{k})}$ shown in Figure 4.7.*

Proof Parametrise side 1 by $z = s$, $0 < s < \infty$ and rearrange (4.2.23a) to give the Neumann boundary value $-u_y(x, 0)$ in terms of $u_x(x, 0)$, $u(x, 0)$, and $g_1(x)$:

$$-u_y(x,0) = \frac{1}{\sin\alpha_1}\left(g_1(x) - \cos\alpha_1 u_x(x,0) - \gamma_1 u(x,0)\right), \quad 0 < x < \infty, \tag{4.2.32}$$

integrate by parts the term involving $u_x(x, 0)$ (using $u \to 0$ at infinity) to yield

$$\widehat{u}_1(k) = H_1(k)D_1(-ik) + iG_1(-ik) + i\cot\alpha_1 d_1, \tag{4.2.33}$$

where

$$D_1(k) = \frac{1}{\sin\alpha_1}\int_0^\infty ds\, e^{\beta\left(k-\frac{1}{k}\right)s} u(s,0) \tag{4.2.34}$$

and

$$d_1 = u(0^+, 0). \tag{4.2.35}$$

Parametrise side 2 by $z = is$, $0 < s < \infty$, and rearrange (4.2.23b) to give the Neumann boundary value $-u_x(0, y)$ in terms of $u_y(0, y)$, $u(0, y)$, and $g_2(y)$:

$$-u_x(0,y) = \frac{1}{\sin\alpha_2}\left(g_2(y) - \cos\alpha_2 u_y(0,y) - \gamma_2 u(0,y)\right), \quad 0 < y < \infty, \tag{4.2.36}$$

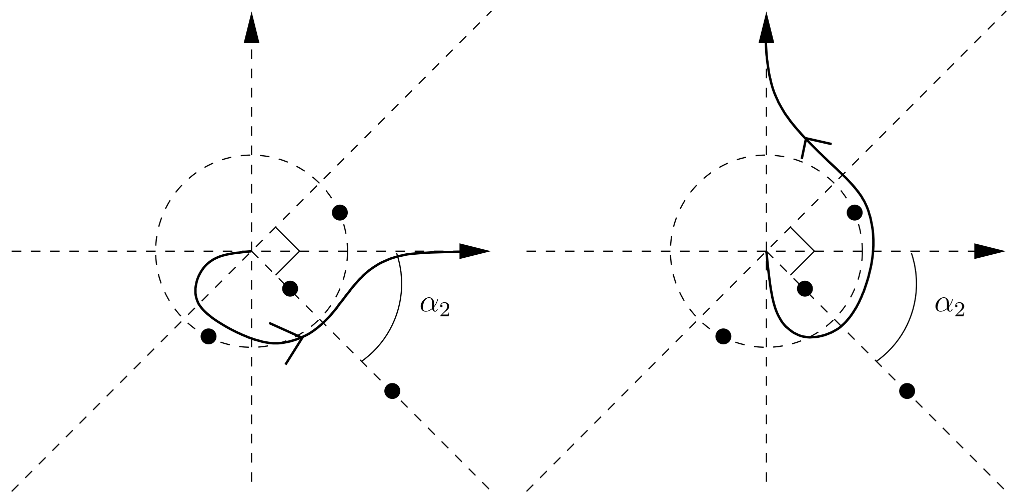


Figure 4.8: The contours $\mathcal{L}_1$ and $\mathcal{L}_2$ for $0 < \alpha_2 < \pi/2$ (black dots are zeros of $\overline{H_2(\bar{k})}$ shown in Figure 4.7)

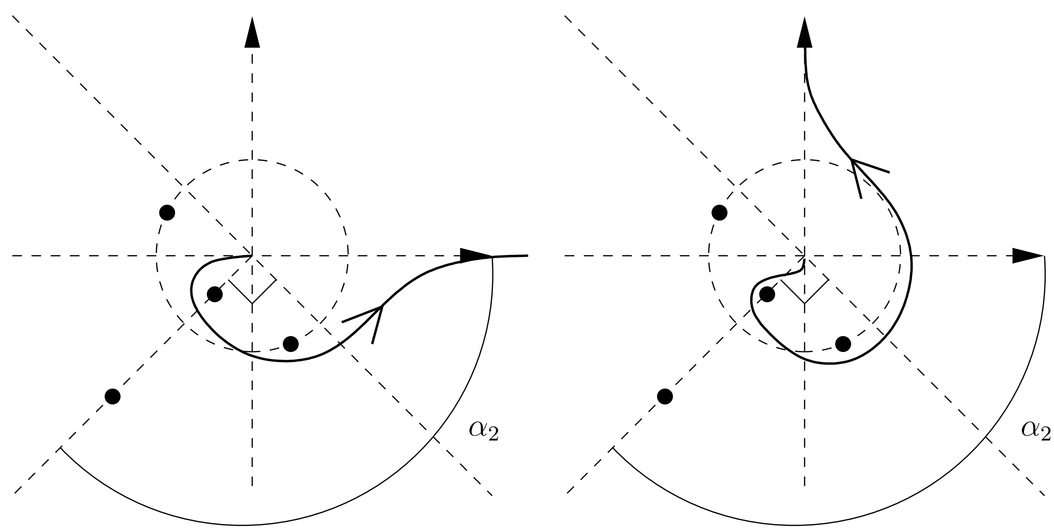


Figure 4.9: The contours $\mathcal{L}_1$ and $\mathcal{L}_2$ for $\pi/2 \leq \alpha_2 < \pi$ (black dots are zeros of $\overline{H_2(\bar{k})}$ shown in Figure 4.7)

integrate by parts the term involving $u_y(0,y)$ (using $u \to 0$ at infinity) to yield

$$\hat{u}_2(k) = iH_2(k)D_2(k) + iG_2(k) + i\cot\alpha_2 d_2, \tag{4.2.37}$$

where

$$D_2(k) = \frac{1}{\sin\alpha_2}\int_0^\infty ds\, e^{\beta\left(k-\frac{1}{k}\right)s}u(0,s) \tag{4.2.38}$$

and

$$d_2 = u(0,0^+). \tag{4.2.39}$$

The global relation (3.1.9) is

$$H_1(k)D_1(-ik)+iG_1(-ik)+i\cot\alpha_1 d_1+iH_2(k)D_2(k)+iG_2(k)+i\cot\alpha_2 d_2 = -\widehat{f}(k), \quad k \in D, \tag{4.2.40}$$

where the domain D is given by (4.2.9) as for the Dirichlet problem. Let $k \mapsto -1/k$ (or equivalently take the Schwartz conjugate) to give the second equation

$$\overline{H_1(\bar{k})}D_1(ik) - iG_1(ik) - i\cot\alpha_1 d_1 - i\overline{H_2(\bar{k})}D_2(k) - iG_2(k) - i\cot\alpha_2 d_2 = -\widetilde{f}(k), \quad k \in \widetilde{D}. \tag{4.2.41}$$

Eliminate $D_2(k)$ from (4.2.40) and (4.2.41), let $k \mapsto -k$ and rearrange to give $D_1(-ik)$ in terms of $D_1(ik)$ and known functions on L_1:

$$\begin{aligned} D_1(-ik) = \frac{\overline{H_2(-\bar{k})}}{\overline{H_1(-\bar{k})}}\Bigg[& -\frac{H_1(-k)}{H_2(-k)}D_1(ik) - i\frac{G_1(ik)+G_2(-k)}{H_2(-k)} + i\frac{G_1(-ik)+G_2(-k)}{\overline{H_2(-\bar{k})}} \\ & + i(\cot\alpha_1 d_1 + \cot\alpha_2 d_2)\left(\frac{1}{\overline{H_2(-\bar{k})}} - \frac{1}{H_2(-k)}\right) - \frac{\widehat{f}(k)}{H_2(-k)} - \frac{\widetilde{f}(-k)}{\overline{H_2(-\bar{k})}}\Bigg], \ k \in L_1. \end{aligned} \tag{4.2.42}$$

Rearrange (4.2.41) to give $D_2(k)$ in terms of $D_1(ik)$ and known functions on L_2@

$$iD_2(k) = \frac{1}{\overline{H_2(\bar{k})}}\left[\overline{H_1(\bar{k})}D_1(ik) - iG_1(ik) - i\cot\alpha_1 d_1 - iG_2(k) - i\cot\alpha_2 d_2 + \widetilde{f}(k)\right]. \tag{4.2.43}$$

Substitute (4.2.42) into (4.2.33) and (4.2.43) into (4.2.37) to yield the solution in terms of integrals of known functions plus

$$\frac{1}{4\pi}\left(-\int_{L_1}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}D_1(ik)\frac{H_1(k)H_1(-k)\overline{H_2(-\bar{k})}}{\overline{H_1(-\bar{k})}H_2(-k)} + \int_{L_2}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}D_1(ik)\frac{H_2(k)\overline{H_1(\bar{k})}}{\overline{H_2(\bar{k})}}\right). \tag{4.2.44}$$

Now, $D_1(ik)$ is analytic in D_1, $e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}$ is analytic in $D_1 \cap D_2$, so (4.2.44) equals zero by Cauchy's theorem if and only if

1. the factors multiplying $D_1(ik)$ and the exponential in both integrals are equal, that is

$$H_2(k)\, H_2(-k)\, \overline{H_1(\bar{k})}\, \overline{H_1(-\bar{k})} = H_1(k)\, H_1(-k)\, \overline{H_2(\bar{k})}\, \overline{H_2(-\bar{k})},\ \forall k \in \mathbb{C}, \tag{4.2.45}$$

2. the integrand has no non-zero poles in the intersection of D_1 and D_2 (poles at zero pose no problem due to the decay of $e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}$).

Regarding 1: the condition (4.2.45) is a polynomial with powers of k and $1/k$ up to k^4 and $1/k^4$. Satisfying the condition imposes some constraints on α_j, γ_j. The $\mathcal{O}(k)$, $\mathcal{O}(1/k)$, $\mathcal{O}(k^3)$, $\mathcal{O}(1/k^3)$ and $\mathcal{O}(1)$ terms are all identities with no additional constraints on α_j, γ_j. The $\mathcal{O}(k^4)$ and $\mathcal{O}(1/k^4)$ terms both impose the condition

$$e^{4i(\alpha_1+\alpha_2)} = 1, \tag{4.2.46}$$

which is equivalent to (4.2.25) ($n = 0$ is ruled out since $\alpha_j > 0$); and the $\mathcal{O}(k^2), \mathcal{O}(1/k^2)$ terms impose the condition (4.2.26).

Regarding 2: under the condition (4.2.45), the non-zero poles of (4.2.44) lie at the zeros of $\overline{H_2(\bar{k})}$:

$$k = e^{-i\alpha_2}\left(\frac{\gamma_2}{2\beta} \pm \sqrt{\left(\frac{\gamma_2}{2\beta}\right)^2 - 1}\right), \tag{4.2.47}$$

which are on $e^{-i\alpha_2}\Re^+$ if $\frac{\gamma_2}{2\beta} \geq 1$ and on the unit circle with $-\pi/2 - \alpha_2 < \arg k < \pi/2 - \alpha_2$ if $\frac{\gamma_2}{2\beta} \leq 1$, see Figure 4.7. Given the IR (2.1.40), we have the freedom to deform the contours anywhere in $\mathbb{C}$ provided that the directions of approach to zero and infinity are unchanged. With the deformations $\mathcal{L}_1$ and $\mathcal{L}_2$ of L_1 and L_2 shown in Figures 4.2.4 and 4.2.4 for $0 < \alpha_2 < \pi/2$ and $\pi/2 \leq \alpha_2 < \pi$ respectively, the integrand in (4.2.44) is analytic in the region bounded by $\mathcal{L}_1 - \mathcal{L}_2$ (the deformation of $D_1 \cap D_2$) and so the integral is zero by Cauchy's theorem.

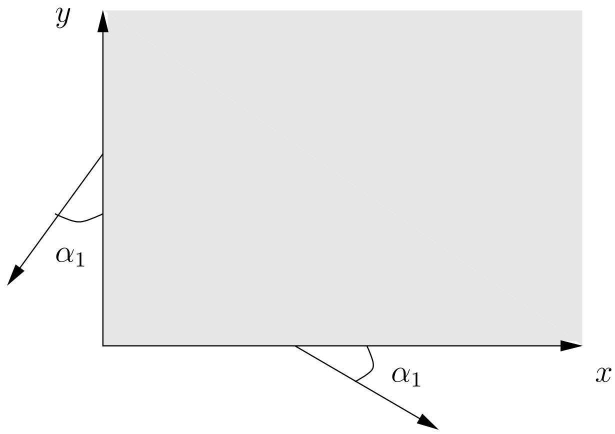


Figure 4.10: Particular oblique Robin boundary conditions satisfying the constraints (4.2.25),(4.2.26).

Finally it is straightforward to check that the zeros of $H_1(-k), \overline{H_1(-\bar{k})}$ and $H_2(-k)$ do not lie on $\mathcal{L}_1$ or $\mathcal{L}_2$, and so the solution (4.2.27) is well-defined. $\square$

Remark 4.2.5 (The value of u at the corner) *If the g_j are such that $d_1 = d_2$, i.e. $u(0^+,0) = u(0,0^+)$ then the corner contributions vanish if $\cot\alpha_1 = -\cot\alpha_2$, that is*

$$\alpha_1 + \alpha_2 = m\pi,\ m \in \mathbb{Z}. \tag{4.2.48}$$

This is not easy to see from (4.2.27). The easiest way to see it is that if $\cot\alpha_1 = -\cot\alpha_2$ then the d_j terms cancel in the GR (4.2.40) and hence in every equation derived from it. This leaves the only contribution from the d_j to the solution as

$$\frac{d_1\cot\alpha_1}{4\pi}\int_{L_1-L_2}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)} \tag{4.2.49}$$

(from (4.2.33),(4.2.37)), and this integral is zero by analyticity.

Remark 4.2.6 (The constraints on α_j, γ_j) *A particular solution of (4.2.25),(4.2.26) is*

$$\alpha_2 = \pi - \alpha_1, \quad \gamma_1 = \gamma_2, \tag{4.2.50}$$

see Figure 4.10. Under these conditions the corner terms cancel, see remark 4.2.5. These boundary conditions are considered in [Gau88], see remark 4.2.9.

Remark 4.2.7 (Rigorous considerations – verifying the boundary conditions) *This follows in a similar way to remark 4.2.2 using*

$$\left(-\sin\alpha_1\frac{\partial}{\partial y}+\cos\alpha_1\frac{\partial}{\partial x}+\gamma_1\right)e^{i\beta\left(kz+\frac{\bar z}{k}\right)}=-i\overline{H_1(-\bar k)},\tag{4.2.51}$$

$$\left(-\sin\alpha_2\frac{\partial}{\partial x}+\cos\alpha_2\frac{\partial}{\partial y}+\gamma_2\right)e^{i\beta\left(kz+\frac{\bar z}{k}\right)}=-\overline{H_2(\bar k)}.\tag{4.2.52}$$

Remark 4.2.8 (Green's integral representation in the physical space) *This is given by*

$$u(x,y)=\int_0^\infty d\xi\Big(uE_\eta-Eu_\eta\Big)(\xi,0)-\int_0^\infty d\eta\Big(Eu_\xi-uE_\xi\Big)(0,\eta)+\int_0^\infty d\xi\int_0^\infty d\eta\, f\, E$$

In order to bring it into the form for which an appropriate Green's function eliminates the unknown boundary values, use (4.2.32) *and* (4.2.36) *for the Neumann boundary values, and integrate by parts to eliminate the* $u_x(x,0)$ *and* $u_y(0,y)$ *terms to yield*

$$\begin{aligned}u(x,y)=&\int_0^\infty d\xi\left(\frac{E\,g_1}{\sin\alpha_1}-\frac{u}{\sin\alpha_1}\left(-\sin\alpha_1E_\eta-\cos\alpha_1E_\xi+\gamma_1E\right)\right)(\xi,0)\\&-\int_0^\infty d\eta\left(-\frac{E\,g_2}{\sin\alpha_2}+\frac{u}{\sin\alpha_2}\left(-\cos\alpha_2E_\eta-\sin\alpha_2E_\xi+\gamma_2E\right)\right)(0,\eta)\\&+E(0,0;x,y)\left(u(0^+,0)\frac{\cos\alpha_1}{\sin\alpha_1}-u(0,0^+)\frac{\cos\alpha_2}{\sin\alpha_2}\right)+\int_0^\infty d\xi\int_0^\infty d\eta\, f\, E\,.\end{aligned}\tag{4.2.53}$$

This shows that the appropriate Green's function for the oblique Robin boundary conditions (4.2.23) *must satisfy the same boundary conditions but with* α_j *replaced by* $\pi-\alpha_j$, *see Figure 4.11. In addition, even if the Green's function can be found, the resulting IR* (4.2.53) *still contains the value of* u *at the corner, unless the* α_j *satisfy* (4.2.48).

Remark 4.2.9 (Comparison with the classical solutions) *If one of the* α_j *equals* $\pi/2$, *with loss of generality* α_2, *then the problem can be solved using the transform in the* x *co-ordinate discussed in remark 4.1.8. However the solution is* not *uniformly convergent at* $x=0$. *Otherwise this boundary value problem* cannot *be solved using a classical transform.*

The application of the method of images to a wedge of angle π/N *with Robin and oblique Robin boundary conditions is considered in [Gau88]. For the Robin problem (both* α_j *equal to* $\pi/2$*) the Green's function is given as a source point, plus infinite lines of images, plus*

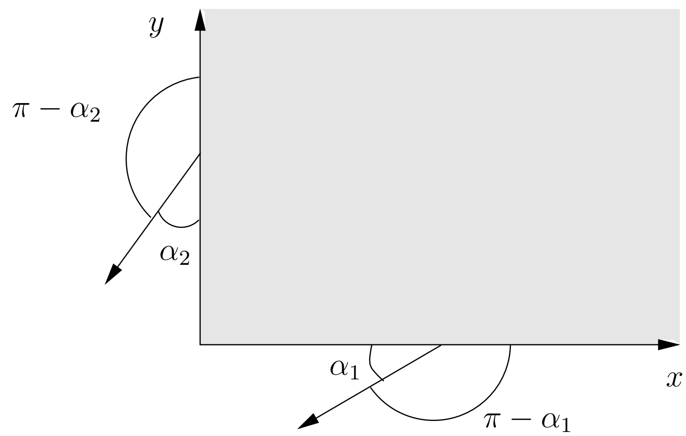


Figure 4.11: The quarter plane with oblique Robin boundary conditions for the Green's function in order that the solution satisfy the boundary conditions (4.2.23) shown in Figure 4.6.

infinite regions of images. Furthermore [Gau88] shows that the method of images can solve the oblique Robin problem with $\alpha_2 = \pi - \alpha_1 \neq \pi/2$ only if N is odd and under some additional restrictions on α_1. Thus, the boundary value problem considered in this section (with $N = 2$) cannot *be solved using images.*

Remark 4.2.10 (Impedance boundary conditions) *In §4.1.2 and §4.2.2 we considered oblique Robin boundary conditions with $\gamma \in \mathbb{R}^+$. We did this so that we could compare our expressions for the solutions with those obtained by the method of images in [Kel81] and [Gau88] – see Remarks 4.1.8 and 4.2.9. However, in the case of the Helmholtz equation impedance boundary condtions, i.e. $\gamma \in i\mathbb{R}^+$, are physically more interesting. The relevant BVP with these boundary conditions can be solved by the new method in a similar way.*

4.3 The Helmholtz equation in the exterior of the circle

As explained in §1.2.2 this particular boundary value problem played a significant role in the development of the classic theory. We now revisit it with the Fokas method.

For simplicity consider the Dirichlet problem. Other boundary conditions can be considered similarly.

Proposition 4.3.1 *Let Ω be the domain exterior to a disc of radius a centered at the origin, i.e. be given by*

$$\Omega = \{a < r < \infty, \quad 0 < \theta < 2\pi\}. \tag{4.3.1}$$

Let $u(a,\theta)$ satisfy the PDE (1.1.1), *the radiation condition* (2.2.3), *and the Dirichlet boundary conditions*

$$u(a,\theta) = d(\theta), \quad 0 < \theta < 2\pi, \tag{4.3.2}$$

where $d(\theta), f(r,\theta)$ are given. Furthermore let u satisfy the following boundary conditions on $\theta = 0$ (which ensure the solution is periodic).

$$u(r,2\pi) = u(r,0), \quad u_\theta(r,2\pi) = u_\theta(r,0). \tag{4.3.3}$$

Then u is given by

$$\begin{aligned} u(r,\theta) =& \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)} e^{ik\theta} D(-ik) + \frac{1}{2\pi}\int_{ABC} dk \frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)} \frac{e^{ik\theta}D(-ik) + e^{-ik\theta}D(ik)}{1-e^{-2\pi ik}} \\ & - \lim_{\varepsilon\to 0}\frac{i}{4}\int_{ABD} dk\, I(\varepsilon)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)} J_k(\beta a)\frac{e^{ik\theta}F(k,-ik) + e^{-ik\theta}F(k,ik)}{1-e^{-2\pi ik}} \\ & - \frac{i}{4}\int_0^{\infty} dk\, J_k(\beta a)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(e^{ik\theta}F(k,-ik) + e^{-ik\theta}F(k,ik)\right) \\ & + \iint_\Omega d\rho\, d\phi\, \rho\, f(\rho,\phi) E_s(\rho,\phi;r,\theta) \end{aligned} \tag{4.3.4}$$

where the known functions $\{D(\pm ik), F(k,\pm ik), I(\varepsilon)\}$ and the contours $\{ABC, ABD\}$ are defined as follows:

$$D(\pm ik) = \int_0^{2\pi} d\phi\, e^{\pm ik\phi}\, d(\phi), \quad k \in \mathbb{C}, \tag{4.3.5}$$

$$F(k,\pm ik) = \int_0^{\infty} d\rho \int_0^{2\pi} d\phi\, \rho\, H_k^{(1)}(\beta\rho) e^{\pm ik\phi} f(\rho,\phi), \quad k \in \mathbb{C}, \tag{4.3.6}$$

$$I(\varepsilon) = \begin{cases} e^{\varepsilon k^2} & \text{if } |\Im k| > 1, \\ 1 & \text{otherwise,} \end{cases} \tag{4.3.7}$$

the contours ABC and ABD are shown in Figure 4.12, where $\Im B < \Im k_1$, where k_1 is the zero of $H_k^{(1)}(\beta a)$ in the first quadrant of the complex k plane with the smallest imaginary part. The function E_s appearing in the last term of (4.3.2) *is given by either* (2.2.6) *or* (2.2.5).

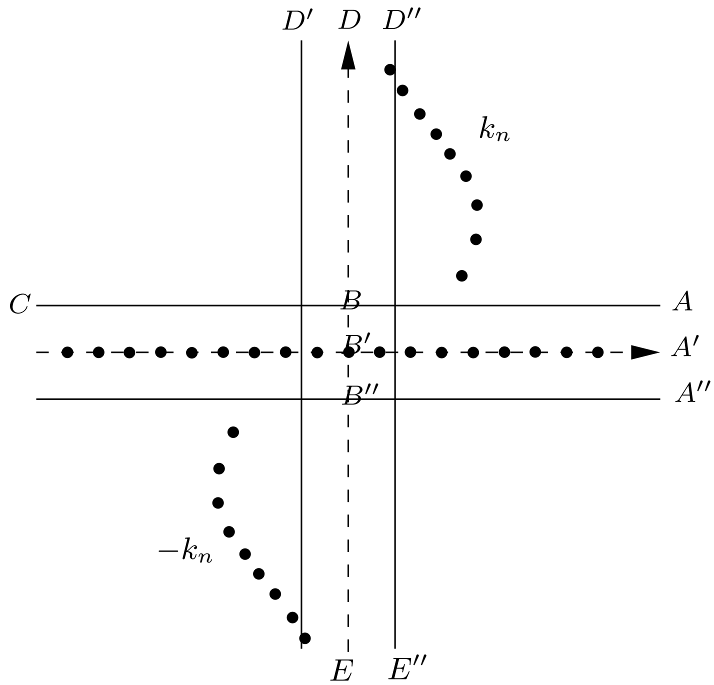


Figure 4.12: The poles of the integrands of (4.3.4) and the contours of integration in the complex k plane.

Proof The integral representation of u in the domain Ω is given by

$$
\begin{aligned}
u(r,\theta) = \lim_{\varepsilon\to 0}\frac{i}{4}\Bigg(&\int_0^{i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r)\left[-\left(e^{ik\theta}+e^{ik(2\pi-\theta)}\right) ikD_0(k) - \left(e^{ik\theta}-e^{ik(2\pi-\theta)}\right) N_0(k)\right]\\
&+\int_0^{-i\infty} e^{\varepsilon k^2} J_k(\beta r)\left[\left(e^{-ik\theta}+e^{-ik(2\pi-\theta)}\right) ikD_0(k) - \left(e^{-ik\theta}-e^{-ik(2\pi-\theta)}\right) N_0(k)\right]\Bigg)\\
&-\frac{ia}{4}\Bigg(\int_0^{\infty} dk e^{ik\theta} H_k^{(1)}(\beta r)\left(J_k(\beta a)N(-ik)-\beta J_k'(\beta a)D(-ik)\right)\\
&+\int_0^{\infty} dk e^{-ik\theta} H_k^{(1)}(\beta r)\left(J_k(\beta a)N(ik)-\beta J_k'(\beta a)D(ik)\right)\Bigg)\\
&+\iint_\Omega d\rho\, d\phi\, \rho\, f(\rho,\phi)E_s(\rho,\phi;r,\theta)
\end{aligned}
\tag{4.3.8}
$$

where

$$
D_0(k) = \int_a^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho)\, u(\rho,0), \quad N_0(k) = \int_a^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho)\, u_\theta(\rho,0), \tag{4.3.9}
$$

$$
N(\pm ik) = \int_0^{2\pi} d\phi\, e^{\pm ik\phi}\, u_r(a,\phi). \tag{4.3.10}
$$

This follows from (2.2.35) with $\alpha = 2\pi$ and u periodic (4.3.3).

The GR is given by (3.2.7), again with $\alpha = 2\pi$ and u periodic (4.3.3),

$$-aH_k^{(1)}(\beta a)N(\pm ik)+a\beta H_k^{(1)'}(\beta a)D(\pm ik)+(1-e^{\pm 2\pi ik})\left(\pm ikD_0(k) - N_0(k)\right) = -F(k,\pm ik) \tag{4.3.11}$$

We use the three Steps (1)-(3) outlined in Chapter 1 §3 to eliminate the unknown transforms D_0, N_0, and N from the representation (4.3.8).

Before we proceed we note the following three facts: First, $H_{-k}^{(1)}(x) = e^{ik\pi}H_k^{(1)}(x)$, which implies that $D_0(-k) = e^{ik\pi}D_0(k)$ (and simliarly for N_0).

Second, the following identity holds for any function $B(k)$ (provided the integrals exist):

$$\begin{aligned}&\int_0^{i\infty} dk\, J_k(\beta r)H_k^{(1)}(\beta a)B(k) + \int_0^{-i\infty} dk\, J_k(\beta r)H_k^{(1)}(\beta a)B(-k)\\ &= \int_0^{i\infty} dk\, J_k(\beta a)H_k^{(1)}(\beta r)B(k) + \int_0^{-i\infty} dk\, J_k(\beta a)H_k^{(1)}(\beta r)B(-k).\end{aligned} \tag{4.3.12}$$

This identity can be derived by expanding $H_k^{(1)}$ as a linear combination of J_k and J_{-k} (using its definition), and then by letting $k \mapsto -k$ in the term involving J_{-k} (this identity shows reciprocity in r and ρ in the expression (2.2.5), and a similar identity ((2.2.17)) is used when solving problems using the Kontorovich-Lebedev transform [Jon80, §5], [Jon86, §9.19, page 587]).

Third, the zeros of $H_k^{(1)}(\beta a)$ are in the 1st and 3rd quadrants of the complex k plane [KRG63]. If the zeros in the first quadrant are denoted by k_n, then the zeros in the third quadrant are given by $-k_n$ (using the symmetry property of Hankel functions mentioned earlier). For our purposes the only fact that we will require about these zeros is that $\arg k_n \to \pi/2$ as $n \to \infty$; more detailed information about their behaviour (including their asymptotics as $\beta a \to \infty$) is given in [KRG63].

Step 1 The two global relations (4.3.11) involve four unknown functions: $N(\pm ik), N_0(k)$, and $D_0(k)$ (treating $N(\pm ik)$ as two unknowns). These two equations can therefore express any one unknown in terms of two others. Here we shall use (4.3.11) to express $N(\pm ik)$ in terms of $N_0(k)$ and $D_0(k)$, however if one takes another choice (such as expressing $N_0(k)$ and $D_0(k)$ in terms of $N(\pm ik)$) and goes through the steps this will also yield the solution (4.3.4).

Solving (4.3.11) for $N(\pm ik)$

$$aN(\pm ik) = \frac{1}{H_k^{(1)}(\beta a)} \left(a\beta H_k^{(1)'}(\beta a) D(\pm ik) + (1 - e^{\pm 2\pi ik}) \left(\pm ik D_0(k) - N_0(k) \right) + F(k, \pm ik) \right),$$

and substituting these expressions into (4.3.8) the unknown parts of the integrals over $(0,\infty)$ in (4.3.8) are

$$\begin{aligned} -\frac{i}{4} \Bigg(&\int_0^\infty dk e^{ik\theta} \frac{H_k^{(1)}(\beta r) J_k(\beta a)}{H_k^{(1)}(\beta a)} (1 - e^{-2\pi ik}) \left(-ik D_0(k) - N_0(k) \right) \\ &+ \int_0^\infty dk e^{-ik\theta} \frac{H_k^{(1)}(\beta r) J_k(\beta a)}{H_k^{(1)}(\beta a)} (1 - e^{2\pi ik}) \left(ik D_0(k) - N_0(k) \right) \Bigg). \end{aligned} \tag{4.3.13}$$

(For brevity of presentation, we will focus only on the unknown terms in (4.3.8) and not display the known terms involving $D(\pm ik)$ and $F(k, \pm ik)$.)

Since u is a solution of (1.1.1) with the outgoing radiation condition (2.2.3) we expect the r dependence of the solution to be of the form $H_k^{(1)}(\beta r)$. The integrals over $(0,\infty)$ in (4.3.8) are of this form, but those over $(0, i\infty)$ are not. To rectify this situation we multiply and divide by $H_k^{(1)}(\beta a)$ in the first two terms of (4.3.8) and use the identity (4.3.12). Indeed, introducing $H_k^{(1)}(\beta a)$ in this way, and noting that there are no zeros of $H_k^{(1)}(\beta a)$ on the contour, we find

$$\lim_{\varepsilon \to 0} \frac{i}{4} \left(\int_0^{i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) H_k^{(1)}(\beta a) A(k) + \int_0^{-i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r) H_k^{(1)}(\beta a) A(-k) \right)$$

where

$$A(k) = -\left(e^{ik\theta} + e^{ik(2\pi - \theta)} \right) ik \frac{D_0(k)}{H_k^{(1)}(\beta a)} - \left(e^{ik\theta} - e^{ik(2\pi - \theta)} \right) \frac{N_0(k)}{H_k^{(1)}(\beta a)},$$

and then the identity (4.3.12) with $B(k) = e^{\varepsilon k^2} A(k)$ allows the arguments of H_k and J_k to be interchanged. When the resulting terms are combined with (4.3.13), the unknowns in the IR (4.3.8) are given by

$$\lim_{\varepsilon\to 0}\frac{i}{4}\left(\int_{A'B'D} dk\, I(\varepsilon)\, H_k^{(1)}(\beta r)J_k(\beta a)\left[e^{ik\theta}\frac{-ikD_0(k)-N_0(k)}{H_k^{(1)}(\beta a)}+e^{ik(2\pi-\theta)}\frac{-ikD_0(k)+N_0(k)}{H_k^{(1)}(\beta a)}\right]\right.$$
$$\left.+\int_{A'B'E} dkI(\varepsilon)H_k^{(1)}(\beta r)J_k(\beta a)\left[e^{-ik\theta}\frac{ikD_0(k)-N_0(k)}{H_k^{(1)}(\beta a)}+e^{-ik(2\pi-\theta)}\frac{ikD_0(k)+N_0(k)}{H_k^{(1)}(\beta a)}\right]\right). \tag{4.3.14}$$

Our aim is to show that this expression is equal to integrals only involving the known transforms $D(\pm ik)$ and $F(k,\pm ik)$.

Step 2 We now look at the analyticity and decay properties of the integrands of (4.3.14). The branch cuts of $H_k^{(1)}$ and J_k as functions of k are taken on the negative imaginary axis, and thus $D_0(k)$ and $N_0(k)$ are analytic in the cut plane. Recall that, as $|k|\to\infty$ for $\Re k>0$, $H_k^{(1)}(\beta r)J_k(\beta a)$ decays exponentially and $D_0(k)/H_k^{(1)}(\beta a)$ is bounded (except at zeros of $H_k^{(1)}(\beta a)$) (these facts can be established by using (2.2.12) and other standard results about the asymptotics of Bessel functions for large orders, see e.g. [AS65, §9.3]),

Other than the branch cuts on the negative real axis, the only singularities of the integrands of (4.3.14) are poles at the zeros of $H_k^{(1)}(\beta a)$, which are in the first and third quadrants of the complex k plane. Since there are no zeros in the fourth quadrant, the integral on $A'B'E$ in (4.3.14) is zero. Indeed, deform the contour of this integral from $A'B'E$ to $A''B''E'$ where $-\pi/2<\arg E'<-\pi/4$ (so $e^{\varepsilon k^2}$ still decays). This integral now converges absolutely even when $\varepsilon=0$ and by the dominated convergence theorem ε can be set to zero. When the contour is closed at infinity in the fourth quadrant, this term equals zero by Cauchy's theorem (the contribution from the integral at infinity is zero since the integrand decays exponentially).

Step 3 Deform the contour of the first integral in (4.3.14) from $A'B'D$ to ABD. (Note that we cannot deform off $i\mathbb{R}^+$ and get rid of the regularizing factor $I(\varepsilon)$ as we did for the integral over $A'B'E$ above since the zeros of $H_k^{(1)}(\beta a)$ are arbitrarily close to $i\mathbb{R}^+$.) The

GR (4.3.11) implies that

$$-ikD_0(k) - N_0(k) = \frac{a\left(H_k^{(1)}(\beta a)N(-ik) - \beta H_k^{(1)'}(\beta a)D(-ik) - F(k,-ik)\right)}{1-e^{-2\pi ik}},$$

$$-ikD_0(k) + N_0(k) = \frac{-a\left(H_k^{(1)}(\beta a)N(ik) - \beta H_k^{(1)'}(\beta a)D(ik) - F(k,ik)\right)}{1-e^{2\pi ik}}.$$

and these can be used in the integral over ABD since $1-e^{\pm 2\pi ik} \neq 0$ on this contour. When these expressions are substituted into the integral over ABD, the terms involving $N(\pm ik)$ vanish by analyticity. Indeed, when the contour ABD is closed at infinity in the first quadrant there are no poles of the integrand inside the contour: the $H_k^{(1)}(\beta a)$ term in the denominator is cancelled by the same term appearing in the numerator, and the contour does not enclose any of the zeros of $1-e^{\pm 2\pi ik}$ which are on the real axis. The contribution from the integral at infinity is zero since the asymptotics

$$\frac{e^{ik\theta}N(-ik)}{1-e^{-2\pi ik}} \sim e^{ik\theta}\int_0^{2\pi} e^{ik(2\pi-\phi)}u_\theta(a,\phi)d\phi \sim e^{ik\theta}\frac{\left(u_\theta(a,2\pi) - e^{ik2\pi}u_\theta(a,0)\right)}{-ik}, \quad (4.3.15a)$$

$$\frac{e^{-ik\theta}N(ik)}{1-e^{-2\pi ik}} \sim e^{ik(2\pi-\theta)}\int_0^{2\pi} e^{ik\phi}u_\theta(a,\phi)d\phi \sim e^{ik(2\pi-\theta)}\frac{\left(e^{ik2\pi}u_\theta(a,2\pi) - u_\theta(a,0)\right)}{-ik} \quad (4.3.15b)$$

show the integrand decays exponentially as $|k| \to \infty$, $\Im k > 0$ for $0 < \theta < 2\pi$.

In summary, we have eliminated all the unknown transforms from the IR. The remained terms equal (4.3.4). To show this, we use the Wronskian

$$J_k(\beta a)H_k^{(1)'}(\beta a) = J_k'(\beta a)H_k^{(1)}(\beta a) + \frac{2i}{\pi a\beta} \quad (4.3.16)$$

to simplify both integrals on $(0,\infty)$ as well as the integral over ABD. The latter integral equals

$$\lim_{\varepsilon\to 0}\frac{i}{4}\int_{ABD} dk I(\varepsilon)\frac{H_k^{(1)}(\beta r)J_k(\beta a)H_k^{(1)'}(\beta a)}{H_k^{(1)}(\beta a)(1-e^{-2\pi ik})}\left(e^{ik\theta}D(-ik) + e^{-ik\theta}D(ik)\right). \quad (4.3.17)$$

When (4.3.16) is substituted into (4.3.17), the resulting term involving $H_k^{(1)}(\beta a)$ in the numerator is zero; this can be shown by deforming the contour to ABD'' and then setting $\varepsilon = 0$, the $H_k^{(1)}(\beta a)$ terms in the numerator and the denominator cancel and the resulting integrand is analytic and exponentially decaying at infinity in the first quadrant. The contour of the remaining term can be deformed to ABD', ε can be set to zero, and then the contour of the resulting integral can be deformed to ABC to yield the second term of (4.3.4). $\square$

Remark 4.3.2 (Rigorous considerations - verifying the boundary condition) *To show the boundary condition* (4.3.2) *note that when* $r = a, \quad 0 < \theta < 2\pi$*, the first term of* (4.3.4) *equals* $d(\theta)$ *by the Fourier transform inversion theorem. The second term equals zero by analyticity (closing the contour in the upper half plane). To deal with the terms involving the forcing, use the angular expansion of* E_s, (2.2.6)*, to write* $E_s(\rho, \phi; a, \theta)$ *as*

$$E_s(\rho, \phi; a, \theta) = \frac{i}{4}\left(\int_0^\infty dk\, H_k^{(1)}(\beta\rho) J_k(\beta a) e^{ik(\theta-\phi)} + \int_0^\infty dk\, H_k^{(1)}(\beta\rho) J_k(\beta a) e^{-ik(\theta-\phi)}\right) \tag{4.3.18}$$

so

$$\iint_\Omega f(\rho, \theta') E_s(\rho, \theta'; a, \theta) \rho d\rho d\theta' = \frac{i}{4}\int_0^\infty dk\, J_k(\beta a)\left(e^{ik\theta} F(k, -ik) + e^{-ik\theta} F(k, ik)\right) e^{ik(\theta-\phi)} \tag{4.3.19}$$

which cancels with the penultimate term of (4.3.4) *when* $r = a$*, leaving the only contribution from the forcing as*

$$\lim_{\varepsilon\to 0} -\frac{i}{4}\int_{ABD} dk\, I(\varepsilon)\, J_k(\beta a) \frac{e^{ik\theta} F(k, -ik) + e^{-ik\theta} F(k, ik)}{1 - e^{-2\pi i k}}, \tag{4.3.20}$$

which is zero by analyticity. Indeed, deform ABD *to* ABD''*, set* $\varepsilon = 0$ *and close the contour in the first quadrant. The integrand is analytic here and the asymptotics* (4.3.15) *and* (2.2.12) *imply that*

$$J_k(\beta a) \frac{e^{ik\theta} F(k, -ik) + e^{-ik\theta} F(k, ik)}{1 - e^{-2\pi i k}} \to 0 \text{ as } |k| \to \infty, \quad 0 < \arg k < \pi/2, \quad 0 < \theta < 2\pi. \tag{4.3.21}$$

Finally, to show (4.3.3)*, take the differences*

$$u(r, 2\pi) - u(r, 0), \quad u_\theta(r, 2\pi) - u_\theta(r, 0).$$

The terms involving $D(\pm ik)$ *cancel in a straightforward way by deforming* ABC *to* $\mathbb{R}$*, and* *using* $k \mapsto -k$*. For the forcing terms use* (2.2.5) *to give*

$$\begin{aligned}\iint_\Omega d\rho\, d\phi\, \rho\, f\, E_s = \frac{i}{4}\Bigg(&\int_0^\infty dk\, e^{ik\theta} H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,-ik)\\ &+\int_0^\infty dk\, e^{-ik\theta} H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,ik)\\ &+\int_0^\infty dk\, e^{ik\theta} J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,-ik)\\ &+\int_0^\infty dk\, e^{-ik\theta} J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,ik)\Bigg)\end{aligned} \tag{4.3.22}$$

where

$$\hat{f}(\rho,\pm ik) = \int_0^{2\pi} e^{\pm ik\phi} f(\rho,\phi)d\phi.$$

(This expression involves several different transforms of f, but they combine to give $F(k,\pm ik)$ later). Then deform all the integrals over $(0,\infty)$ *to either* $(0,i\infty)$ *or* $(0,-i\infty)$ *depending on whether the integrands are analytic and decay at infinity in either the upper or lower half planes (the factor $I(\varepsilon)$ must be added to ensure convergence of the resulting integrals on $i\mathbb{R}$). Then the differences are equal to zero using* (4.3.12) *and* $k \mapsto -k$*.*

Remark 4.3.3 (Proving completeness of classical transforms) *Evaluating the expression for the solution of a given BVP obtained by the new method on part of the boundary of the domain yields a completeness relation. If this completeness relation has* not *been used to obtain the solution, then this process gives a proof that the associated transform is complete. For example, in our case evaluating the solution* (4.3.4) *on $r = a$ yields the completeness relation in the angular variables* (2.2.8) *(see the previous remark), but of course this completeness relation was used to obtain the expression for E_s* (2.2.6) *which was then used to find the solution* (4.3.4)*.*

4.3.1 Recovery of the classical representations of the solution

In this section we show how the two classical representations of the solution to the BVP of Proposition 4.3.1 (via the radial and angular eigenfunction expansions) can be obtained from the expression for the solution given by the new method (4.3.4).

4.3.1.1 Derivation of the radial series expansion

The classical radial series expansion is given by

$$u(r,\theta) = -\pi i \sum_{n=1}^{\infty} \frac{k_n H_{k_n}^{(1)}(\beta r) J_{k_n}(\beta a)}{\dot{H}_{k_n}^{(1)}(\beta a)} \hat{u}(k_n,\theta). \tag{4.3.23}$$

where

$$\hat{u}(k_n,\theta) = \int_0^{2\pi} d\phi\, w(\phi,\theta)\Big(-a\beta H_{k_n}^{(1)'}(\beta a) u(a,\phi) - \int_a^{\infty} d\rho\, \rho\, f(\rho,\phi) H_{k_n}^{(1)}(\beta\rho)\Big) \tag{4.3.24}$$

and

$$w(\phi,\theta) = \frac{e^{ik|\theta-\phi|} + e^{2\pi i k} e^{-ik|\theta-\phi|}}{2ik(1-e^{2\pi i k})}. \tag{4.3.25}$$

Indeed, the spectral analysis of the radial ODE (2.2.9) on (a,∞) with the boundary condition $u(a,\theta) = 0$, and with the additional condition that the eigenfunctions satisfy the *outgoing* radiation condition (2.2.3), yields

$$\rho\, \delta(r-\rho) = -\pi i \sum_{n=1}^{\infty} \frac{k_n H_{k_n}^{(1)}(\beta\rho) H_{k_n}^{(1)}(\beta r) J_{k_n}(\beta a)}{\dot{H}_{k_n}^{(1)}(\beta a)}, \tag{4.3.26}$$

where

$$\dot{H}_{k_n}^{(1)}(\beta a) = \frac{d}{dk} H_k^{(1)}(\beta a)\Big|_{k=k_n}$$

and k_n are the zeros of $H_k^{(1)}(\beta a)$ in the first quadrant.

Integration by parts implies that the ODE satisfied by the transform

$$\hat{u}(k_n,\theta) = \int_a^{\infty} \frac{d\rho}{\rho} u(\rho) H_{k_n}^{(1)}(\beta\rho),$$

is given by

$$\frac{d^2\hat{u}}{d\theta^2} + k_n^2 \hat{u} = -a\beta H_{k_n}^{(1)'}(\beta a) u(a,\theta) - \widehat{\rho^2 f}(k_n,\theta),$$

Solving this ODE by employing an appropriate Green's function yields (4.3.23).

Note that (4.3.24) can be rewritten as

$$\hat{u}(k_n,\theta) = \frac{1}{2ik_n}\left(I(ik_n,\theta)e^{ik_n\phi} - I(-ik_n,\theta)e^{-ik_n\phi}\right)\left(a\beta H_{k_n}^{(1)'}(\beta a)u(a,\phi) + \widehat{\rho^2 f}(k_n,\phi)\right)$$

where the integral operator $I(ik,\theta)$ is defined by

$$I(ik,\theta)=\frac{e^{-ik\theta}}{1-e^{-2i\pi k}}\left(\int_0^{\theta}d\varphi+e^{-2i\pi k}\int_{\theta}^{2\pi}d\varphi\right).$$

The Wronskian (4.3.16) evaluated at $k=k_n$ implies that

$$J_{k_n}(\beta a)H^{(1)'}_{k_n}(\beta a)=\frac{2i}{\pi\beta a}$$

and furthermore

$$\begin{aligned}I(ik,\theta)e^{ik\phi}&-I(-ik,\theta)e^{-ik\phi}\\&=\frac{e^{-ik\theta}\int_0^{2\pi}d\phi e^{ik\phi}+e^{ik\theta}\int_0^{2\pi}d\phi e^{-ik\phi}}{1-e^{-2\pi ik}}-e^{-ik\theta}\int_{\theta}^{2\pi}d\phi e^{ik\phi}-e^{ik\theta}\int_0^{\theta}d\phi e^{-ik\phi}.\end{aligned}$$

Hence the solution (4.3.23) can be written as

$$\begin{aligned}u(r,\theta)=&-i\sum_{n=1}^{\infty}\frac{H^{(1)}_{k_n}(\beta r)}{\dot H^{(1)}_{k_n}(\beta a)}\left(\frac{e^{ik_n\theta}D(-ik_n)+e^{-ik_n\theta}D(ik_n)}{1-e^{-2\pi ik_n}}-e^{-ik_n\theta}D_R(ik_n)-e^{ik_n\theta}D_L(-ik_n)\right)\\&-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{H^{(1)}_{k_n}(\beta r)J_{k_n}(\beta a)}{\dot H^{(1)}_{k_n}(\beta a)}\left(\frac{e^{ik_n\theta}F(k_n,-ik_n)+e^{-ik_n\theta}F(k_n,ik_n)}{1-e^{-2\pi ik_n}}\right.\\&\qquad\left.-e^{-ik_n\theta}F_R(k_n,ik_n)-e^{ik_n\theta}F_L(k_n-ik_n)\right),\end{aligned}\tag{4.3.27}$$

where

$$D_L(\pm ik)=\int_0^{\theta}d\phi\, e^{\pm ik\phi}d(\phi),\quad D_R(\pm ik)=\int_{\theta}^{2\pi}d\phi\, e^{\pm ik\phi}d(\phi),$$

and

$$\begin{aligned}F_L(k,\pm ik)&=\int_0^{\infty}d\rho\int_0^{\theta}d\phi\,\rho\, H^{(1)}_k(\beta\rho)e^{\pm ik\phi}f(\rho,\phi),\\F_R(k,\pm ik)&=\int_0^{\infty}d\rho\int_{\theta}^{2\pi}d\phi\,\rho\, H^{(1)}_k(\beta\rho)e^{\pm ik\phi}f(\rho,\phi).\end{aligned}$$

The above series solution is *not* uniformly convergent at $r=a$ since each term contains $H^{(1)}_{k_n}(\beta a)$ which equals zero.

Convergence of the radial series. Cohen considered the case of $d=0$ and showed that this series does not converge outside the geometric shadow of f [Coh64a]. That is, if f has support in $\alpha_1<\theta<\alpha_2$ then (4.3.27) is only valid for $0\le\theta<\alpha_1$, $\alpha_2\le\theta\le 2\pi$, see Figure (4.13). The same proof can be used to show the analogous result for $d(\theta)$: if d has support in $\gamma_1<\theta<\gamma_2$ then (4.3.27) is only valid for $0\le\theta<\gamma_1$, $\gamma_2\le\theta\le 2\pi$.

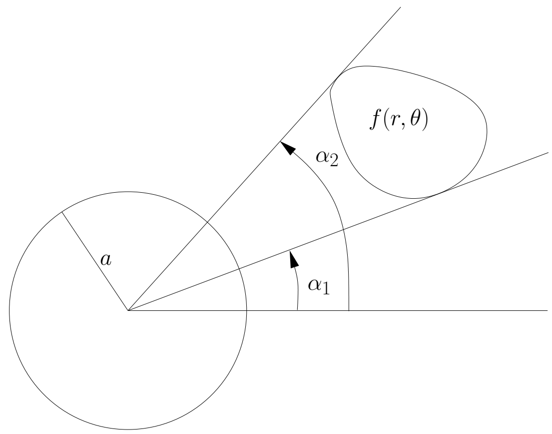


Figure 4.13: The circle and support of the forcing f.

In what follows we show that (4.3.27) can be obtained from (4.3.4). This can be achieved by closing the contours in the upper half complex k-plane and evaluating the integrals via the residues at the zeroes of $H_k^{(1)}(\beta a)$ in the first quadrant. Recall that $H_k^{(1)}(\beta r)/H_k^{(1)}(\beta a)$ decays exponentially as $|k| \to \infty$ for $\arg k \neq \pi/2$, thus the above construction requires that the terms involving $e^{ik\theta}, e^{-ik\theta}$ and the transforms of d or f must have appropriate decay on $i\mathbb{R}^+$. The asymptotics (4.3.15) imply that the second term of (4.3.4) can be evaluated via residues to yield the first term on the first line of (4.3.27). Regarding the first term of (4.3.4), we first split $D(-ik) = D_L(-ik) + D_R(-ik)$ and then let $k \mapsto -k$ in the second term to obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(e^{ik\theta} D_L(-ik) + e^{-ik\theta} D_R(ik)\right) \frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}. \tag{4.3.28}$$

This term will yield the last two terms on the first line of (4.3.27) if both $e^{ik\theta} D_L(-ik)$ and $e^{-ik\theta} D_R(-ik)$ decay faster than $\mathcal{O}(1/k)$ as $|k| \to \infty$ for $\arg k = \pi/2$. (The residues give the sum, and this condition is required for the integral at infinity to be zero.) By integration by parts, both $e^{ik\theta} D_L(-ik)$ and $e^{-ik\theta} D_R(-ik)$ are $\mathcal{O}(1/k)$ as $\Im k \to \infty$ where the coefficient of this term is proportional to $d(\theta)$. Thus (4.3.28) can be evaluated purely via residues to give the relevant sum in (4.3.27) *only if θ is not in the support of d* (and thus $d(\theta) = 0$), which confirms Cohen's result. However, we emphasise that if the regularising factor $e^{\varepsilon k^2}$ is inserted in the integral (4.3.28), then the resulting integral *can* be evaluated purely as residues to give a series involving terms similar to the terms in (4.3.27) involving D_L and D_R, where now the term $e^{\varepsilon k_n^2}$ is included in the sum and the limit $\varepsilon \to 0$ is outside.

Regarding the terms involving the forcing in (4.3.4), we first split the integral over Ω depending on whether $\theta \lessgtr \phi$ and then use the radial representation (2.2.5) to obtain

$$\iint_\Omega d\rho\, d\phi\, \rho\, f\, E_s = \lim_{\varepsilon\to 0} \frac{i}{4}\left(\int_0^{i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r)\left[e^{ik\theta}F_L(k,-ik)+e^{-ik\theta}F_R(k,ik)\right]\right.$$
$$\left.+\int_0^{-i\infty} dk\, e^{\varepsilon k^2} J_k(\beta r)\left[e^{-ik\theta}F_L(k,ik)+e^{ik\theta}F_R(k,-ik)\right]\right).$$

Introducing $H_k^{(1)}(\beta a)/H_k^{(1)}(\beta a)$ in the integrands and using (4.3.12) we obtain

$$\iint_\Omega d\rho\, d\phi\, \rho\, f\, E_s = \lim_{\varepsilon\to 0} \frac{i}{4}\left(\int_0^{i\infty} dk\, e^{\varepsilon k^2} \frac{J_k(\beta a) H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left[e^{ik\theta}F_L(k,-ik)+e^{-ik\theta}F_R(k,ik)\right]\right.$$
$$\left.+\int_0^{-i\infty} dk\, e^{\varepsilon k^2} \frac{J_k(\beta a) H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left[e^{-ik\theta}F_L(k,ik)+e^{ik\theta}F_R(k,-ik)\right]\right).$$

Combining this expression with the third and fourth terms of (4.3.4), using $F(k,\pm k) = F_L(k,\pm k) + F_R(k,\pm k)$, we find that the terms involving the forcing equal

$$\lim_{\varepsilon\to 0} \frac{i}{4}\left(-\int_{ABD} dk I(\varepsilon)\, J_k(\beta a)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left[\frac{e^{ik\theta}F(k,-ik)+e^{-ik\theta}F(k,ik)}{1-e^{-2\pi ik}}\right]\right.$$
$$+\int_{A'B'D} dk I(\varepsilon)\, J_k(\beta a)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left[e^{ik\theta}F_L(k,-ik)+e^{-ik\theta}F_R(k,ik)\right]$$
$$\left.+\int_{A'B'E} dk I(\varepsilon)\, J_k(\beta a)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left[e^{-ik\theta}F_L(k,ik)+e^{ik\theta}F_R(k,-ik)\right]\right). \tag{4.3.29}$$

The third term in (4.3.29) vanishes by analyticity (by deforming $A'B'E$ to $A'B'E'$ and setting $\varepsilon = 0$). The first term can be evaluated via residues and gives the second line of (4.3.27): the asymptotics (4.3.21) imply that the contour can be closed at infinity in the first quadrant. The second term in (4.3.29) can be evaluated in terms of an infinite number of residues: we deform $A'B'D$ to $A'B'D''$, set $\varepsilon = 0$, close the contour in the first quadrant, and evaluate the resulting integral as residues. Thus the contribution from the second term in (4.3.29) equals

$$\lim_{\varepsilon\to 0} \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{H_{k_n}^{(1)}(\beta r) J_{k_n}(\beta a)}{\dot{H}_{k_n}^{(1)}(\beta a)} e^{\varepsilon k_n^2}\left(e^{-ik_n\theta}F_R(k_n, ik_n)+e^{ik_n\theta}F_L(k_n - ik_n)\right). \tag{4.3.30}$$

Integration by parts of F_L and F_R shows that *if θ is not in the support of f* then this sum converges absolutely even with $\varepsilon = 0$ producing the result (4.3.27). Otherwise (4.3.27) does not converge, again in agreement with Cohen.

Remark 4.3.4 (A complete radial eigenfunction expansion) *The discussion above shows that the radial expansion of the solution* (4.3.27) is *valid provided that the regularising factor $e^{\varepsilon k_n^2}$ is inserted in the sums. This suggests that the following completeness relation is valid:*

$$\rho\,\delta(r-\rho) = -\pi i \lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} e^{\varepsilon k_n^2}\frac{k_n H_{k_n}^{(1)}(k\rho)H_{k_n}^{(1)}(kr)J_{k_n}(ka)}{\dot{H}_{k_n}^{(1)}(ka)}. \tag{4.3.31}$$

However there is no boundary in r on which to evaluate the solution (4.3.4) *and therefore we cannot use Remark 4.3.3 to prove the validity of* (4.3.31)*. Nevertheless, the new method can indeed be used to prove that* (4.3.31) *holds by solving a BVP for the Helmholtz equation in the domain D defined by* (2.2.27)*: if arbitrary Dirichlet boundary conditions are prescribed on $\theta = 0$ and, for simplicity, zero Dirichlet boundary conditions are prescribed elsewhere on ∂D, then evaluating the solution to this BVP obtained by the new method on $\theta = 0$ yields* (4.3.31) *(solving this BVP in D is very similar to solving the BVP in Ω of Proposition 4.3.1). Thus, there* does *exist a complete eigenfunction expansion for the radial ODE* (2.2.9)*, albeit with a regularising factor.*

4.3.1.2 Derivation of the angular series expansion

Spectral analysis of the angular ODE (2.2.7) on $0 < \theta < 2\pi$ under periodicity (4.3.3) yields the completeness relation

$$\delta(\theta-\phi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)},$$

i.e. the standard Fourier series. The classical angular solution obtained using this transform is

$$
\begin{aligned}
u(r,\theta) =& \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{H_n^{(1)}(\beta r)}{H_n^{(1)}(\beta a)}e^{in\theta}D(-in) \\
&+\frac{i}{4}\sum_{n=-\infty}^{\infty}e^{in\theta}\left\{H_n^{(1)}(\beta r)\int_a^r d\rho\,\rho\left(J_n(\beta\rho)-\frac{J_n(\beta a)}{H_n^{(1)}(\beta a)}H_n^{(1)}(\beta\rho)\right)\int_0^{2\pi}d\phi\, e^{-in\phi}f(\rho,\phi)\right. \\
&\left.+\left(J_n(\beta r)-\frac{J_n(\beta a)}{H_n^{(1)}(\beta a)}H_n^{(1)}(\beta r)\right)\int_a^r d\rho\,\rho\, H_n^{(1)}(\beta\rho)\int_0^{2\pi}d\phi\, e^{-in\phi}f(\rho,\phi)\right\},
\end{aligned}
\tag{4.3.32}
$$

[Coh64a],[KL59]. This expression can also be obtained from (4.3.4) in the following way: We first consider the term involving the integral over ABC. We deform the contour down to the real axis, indented above the poles at $k=n$, $n\in\mathbb{Z}$; we denote this contour by $(0,\infty^+)$ and we denote the corresponding contour indented below the poles as $(0,\infty^-)$. We split the integral into $\int_{-\infty}^{i\varepsilon}+\int_{i\varepsilon}^{\infty}$ and let $k\mapsto -k$ in the integral on $(-\infty,i\varepsilon)$ to obtain

$$
\begin{aligned}
&-\frac{1}{2\pi}\int_{i\varepsilon}^{\infty^+}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(e^{ik\theta}D(-ik)+\frac{e^{ik(\theta-2\pi)}D(-ik)+e^{-ik\theta}D(ik)}{1-e^{-2\pi ik}}\right) \\
&-\frac{1}{2\pi}\int_{-i\varepsilon}^{\infty^-}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(e^{-ik\theta}D(ik)-\frac{e^{ik(\theta-2\pi)}D(-ik)+e^{-ik\theta}D(ik)}{1-e^{-2\pi ik}}\right).
\end{aligned}
$$

Combining these integrals we find

$$
\begin{aligned}
&-\frac{1}{2\pi}\int_{\varepsilon}^{\infty}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(e^{ik\theta}D(-ik)+e^{-ik\theta}D(ik)\right)-\frac{1}{2\pi}\int_{i\varepsilon}^{\varepsilon}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}e^{ik\theta}D(-ik) \\
&-\frac{1}{2\pi}\int_{-i\varepsilon}^{\varepsilon}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}e^{-ik\theta}D(ik)+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{H_n^{(1)}(\beta r)}{H_n^{(1)}(\beta a)}\left(e^{in(\theta-2\pi)}D(-in)+e^{-in\theta}D(in)\right) \\
&-\frac{1}{2\pi}\int_{\{k=\varepsilon e^{i\phi},\,-\pi\le\phi\le\pi\}}dk\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(\frac{e^{ik(\theta-2\pi)}D(-ik)+e^{-ik\theta}D(ik)}{1-e^{-2\pi ik}}\right),
\end{aligned}
$$

where the sum arises by evaluating an integral as residues at its poles at $\mathbb{Z}^+$. Taking the limit as $\varepsilon\to 0$ the first term cancels with the first term of (4.3.4) (using $k\mapsto -k$), the second and third terms tend to zero, and the final term tends to

$$
\frac{1}{2\pi}\frac{H_n^{(1)}(\beta r)}{H_n^{(1)}(\beta a)}D(0),
$$

giving rise to the first term of (4.3.32).

For the terms involving the forcing, the contours in the second and third terms of (4.3.4) cannot be deformed. Instead, we first split the area integral depending on whether $r \lessgtr \rho$ and then use the angular representation (2.2.6) to obtain (4.3.22). and we then introduce poles using

$$
\begin{aligned}
1 &= \frac{1}{1-e^{-2i\pi k}} + \frac{1}{1-e^{2i\pi k}} \\
&\sim -e^{2i\pi k} \qquad\quad + 1, \quad |k| \to \infty,\ \Im k > 0 \\
&\sim 1 \qquad\qquad\quad - e^{-2i\pi k}, \quad |k| \to \infty,\ \Im k < 0.
\end{aligned}
\tag{4.3.33}
$$

For simplicity, in what follows we will ignore the pole at $k = 0$. The same argument can be modified to include this pole by introducing a parameter ε and letting $\varepsilon \to 0$, as we did for the terms involving $D(\pm ik)$. In (4.3.22) we deform all contours from $(0, \infty)$ to $(0, \infty^+)$ (slightly above the real axis), use (4.3.33) and combine terms to write the integrand as two fractions, one decaying at infinity in the first quadrant and the other decaying in the fourth quadrant. By deforming the contours from $(0, \infty^+)$ to $(0, i\infty)$ and $(0, -i\infty)$ respectively, we obtain an infinite number of residues from the poles at $\mathbb{Z}^+$ of the second term. Thus $\iint_\Omega d\rho\, d\phi\, \rho\, f\, E_s$ equals

$$
\begin{aligned}
&\frac{i}{4}\lim_{\varepsilon\to 0}\Bigg[\int_0^{i\infty} dk\, I(\varepsilon)\Bigg(e^{ik\theta}\frac{H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,-ik) + J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,-ik)}{1-e^{-2i\pi k}} \\
&\qquad\qquad + e^{-ik\theta}\frac{H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,ik) + J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,ik)}{1-e^{-2i\pi k}}\Bigg) \\
&\quad + \int_0^{-i\infty} dk\, I(\varepsilon)\Bigg(e^{ik\theta}\frac{H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,-ik) + J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,-ik)}{1-e^{2i\pi k}} \\
&\qquad\qquad + e^{-ik\theta}\frac{H_k^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_k(\beta\rho)\hat{f}(\rho,ik) + +J_k(\beta r)\int_r^\infty d\rho\,\rho\, H_k^{(1)}(\beta\rho)\hat{f}(\rho,ik)}{1-e^{2i\pi k}}\Bigg)\Bigg] \\
&\quad + \frac{i}{4}\sum_{n=1}^{\infty}\Bigg(e^{in\theta}\left(H_n^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_n(\beta\rho)\hat{f}(\rho,-in) + J_n(\beta r)\int_r^\infty d\rho\,\rho\, H_n^{(1)}\hat{f}(\rho,-in)\right) \\
&\qquad\qquad + e^{-in\theta}\left(H_n^{(1)}(\beta r)\int_a^r d\rho\,\rho\, J_n(\beta\rho)\hat{f}(\rho,in) + J_n(\beta r)\int_r^\infty d\rho\,\rho\, H_n^{(1)}\hat{f}(\rho,in)\right)\Bigg).
\end{aligned}
\tag{4.3.34}
$$

Letting $k \mapsto -k$ in the integral over $(0, -i\infty)$ and combining this term with the integral over $(0, i\infty)$ (using the definition of H_k in terms of J_k and J_{-k} to combine the transforms of f) we find

$$\frac{i}{4}\lim_{\varepsilon\to 0}\int_0^{i\infty} dk\, I(\varepsilon)\frac{1-e^{2\pi ik}}{2}\,\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(\frac{e^{ik\theta}F(k,-ik)+e^{-ik\theta}F(k,ik)}{1-e^{-2\pi ik}}\right),$$

which looks similar to the third term of (4.3.4). Introducing $H_k^{(1)}(\beta a)/H_k^{(1)}(\beta a)$ into the integrand and using the identity

$$\frac{1-e^{2\pi ik}}{2}\,H_k^{(1)}(\beta r)H_k^{(1)}(\beta a) = H_k^{(1)}(\beta r)J_k(\beta a) - H_{-k}^{(1)}(\beta r)J_{-k}(\beta a) \tag{4.3.35}$$

(which follows from the definition of H_k in terms of J_k and J_{-k}) we see that the first term cancels with the third term of (4.3.4). Using $k \mapsto -k$ in the second term and deforming from $(0, -i\infty)$ to $(0, \infty^-)$ (there are no poles of $H_k^{(1)}(\beta a)$ in the third quadrant), we see that the contribution from the forcing to the solution is given by the sum in (4.3.34) plus the term

$$\begin{aligned}\frac{i}{4}\Bigg(&\int_0^{\infty^+} dk\,\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}J_k(\beta a)\left(\frac{e^{ik\theta}F(k,-ik)+e^{-ik\theta}F(k,ik)}{1-e^{-2\pi ik}}\right)\\ &-\int_0^{\infty} dk\, J_k(\beta a)\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}\left(e^{ik\theta}F(k,-ik)+e^{-ik\theta}F(k,ik)\right)\\ &\int_0^{\infty^-} dk\,\frac{H_k^{(1)}(\beta r)}{H_k^{(1)}(\beta a)}J_k(\beta a)\left(\frac{e^{ik\theta}F(k,-ik)+e^{-ik\theta}F(k,ik)}{1-e^{2\pi ik}}\right)\Bigg).\end{aligned} \tag{4.3.36}$$

Using (4.3.33) in the third term of (4.3.36), the two integrals over $(0, \infty)$ cancel leaving one integral which can be evaluated via the residues at the poles on $\mathbb{Z}^+$:

$$-\frac{i}{4}\sum_{n=1}^{\infty}\frac{H_n^{(1)}(\beta r)}{H_n^{(1)}(\beta a)}J_n(\beta a)\left(e^{in\theta}F(n,-in)+e^{-in\theta}F(n,in)\right). \tag{4.3.37}$$

The above sums, together with the $n = 0$ term obtained by introducing ε near $k = 0$ and letting $\varepsilon \to 0$, equal the forcing contribution in the classical angular solution (4.3.32).

Chapter 5

Solution of boundary value problems in a non-separable domain

Summary:

- This Chapter presents the solution of the Dirichlet problem for the Poisson and modified Helmholtz equations in the interior of a right isosceles triangle.

- In this domain the Fokas method, yielding the solution as an integral, has a huge advantage over the classical solution of a bi-infinite sum of eigenfunctions.

- The solution procedure involves Steps 1-3 of §1.3.

- We also include an example of how to obtain the Dirichlet to Neumann map directly (without going via the solution) for the Dirichlet problem.

In this chapter we implement the Fokas method to the Poisson and modified Helmholtz equations in the interior of a right isosceles triangle. The Helmholtz equation can be considered similarly, see remark 5.3.3.

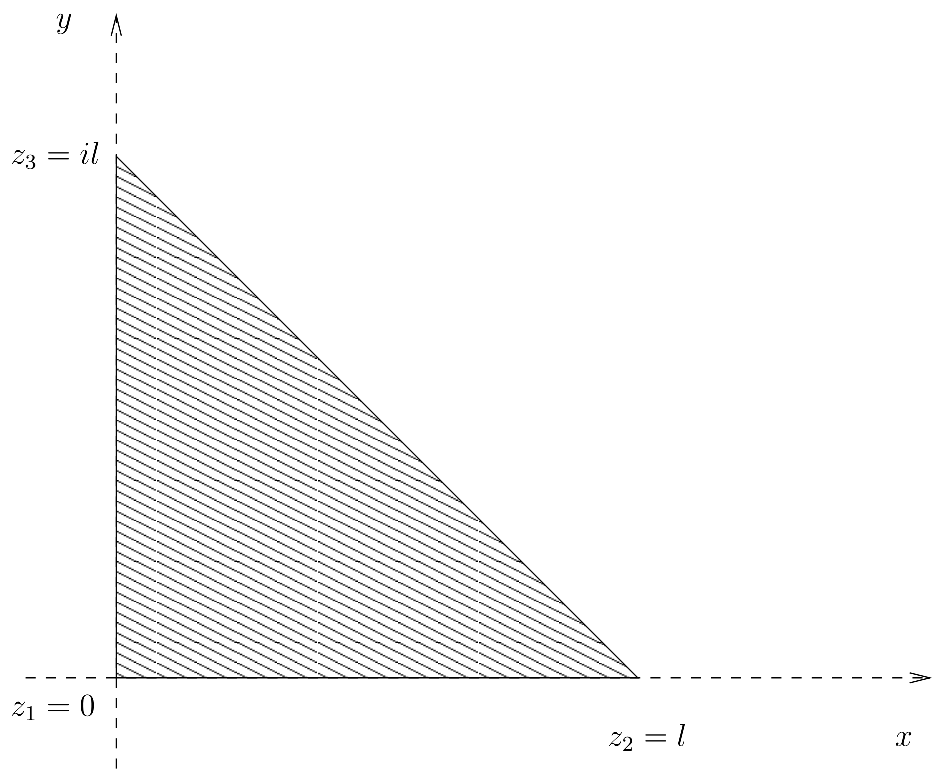


Figure 5.1: The right isosceles triangle.

5.1 The global relation

We use the integral representations of chapter 2, (2.1.28), (2.1.52) and global relations of chapter 3, (3.1.5), (3.1.7). We begin by parametrising the transforms of the boundary values following §3.1.3. Parametrising side (1) by $z = s,\ 0 \le s \le l$, side (2) by $z = l + e^{3\pi i/4}s,\ 0 \le s \le \sqrt{2}l$, and side (3) by $z = is,\ 0 \le s \le l$, the spectral functions $\hat{u}_j(k)$ become

$$\hat{u}_1(k) = iN_1(-ik) - H(-ik)D_1(-ik), \tag{5.1.1}$$

$$\hat{u}_2(k) = E(-ik)\left(iN_2(e^{i\pi/4}k) - H(ke^{i\pi/4})D_2(e^{i\pi/4}k)\right), \tag{5.1.2}$$

$$\hat{u}_3(k) = iN_3(k) + H(k)D_3(k), \tag{5.1.3}$$

where N_j are transforms of the Neumann boundary values and D_j are transforms of the Dirichlet boundary values given by (3.1.38) for the Poisson equation and (3.1.42) for the modified Helmholtz equation, where

$$H(k) = \beta l \left(k - \frac{1}{k} \right), \quad E(k) = e^{\beta\left(k+\frac{1}{k}\right)l}, \tag{5.1.4}$$

for the modified Helmholtz equation and

$$H(k) = k, \quad E(k) = e^{kl}, \tag{5.1.5}$$

for the Poisson equation. We note that on side 1, $u_n(s) = -u_y(s, 0)$, $u(s) = u(s, 0)$, on side 3, $u_n(s) = -u_x(0, s)$, $u(s) = u(0, s)$ and on side 2, $u_n(s) = (u_x + u_y)(s, \sqrt{2}l - s)/\sqrt{2}$, $u(s) = u(s, \sqrt{2}l - s)$.

From example 3.1.10, the GR and SC are

$$N_1(-ik) + E(-ik)N_2(e^{i\pi/4}k) + N_3(k) = G(k), \tag{5.1.6a}$$

$$N_1(ik) + E(ik)N_2(e^{-i\pi/4}k) + N_3(k) = \overline{G(\bar{k})}, \tag{5.1.6b}$$

where $G(k)$ is the known function given by

$$G(k) = -i \left[H(-ik)D_1(-ik) + H(e^{i\pi/4})E(-ik)D_2(e^{i\pi/4}k) - H(k)D_3(k) + i\widehat{f}(k) \right] \tag{5.1.7}$$

and the additional equations we require are

$$N_1(-k) + E(-k)N_2(e^{-i\pi/4}k) + N_3(-ik) = G(-ik), \tag{5.1.8a}$$

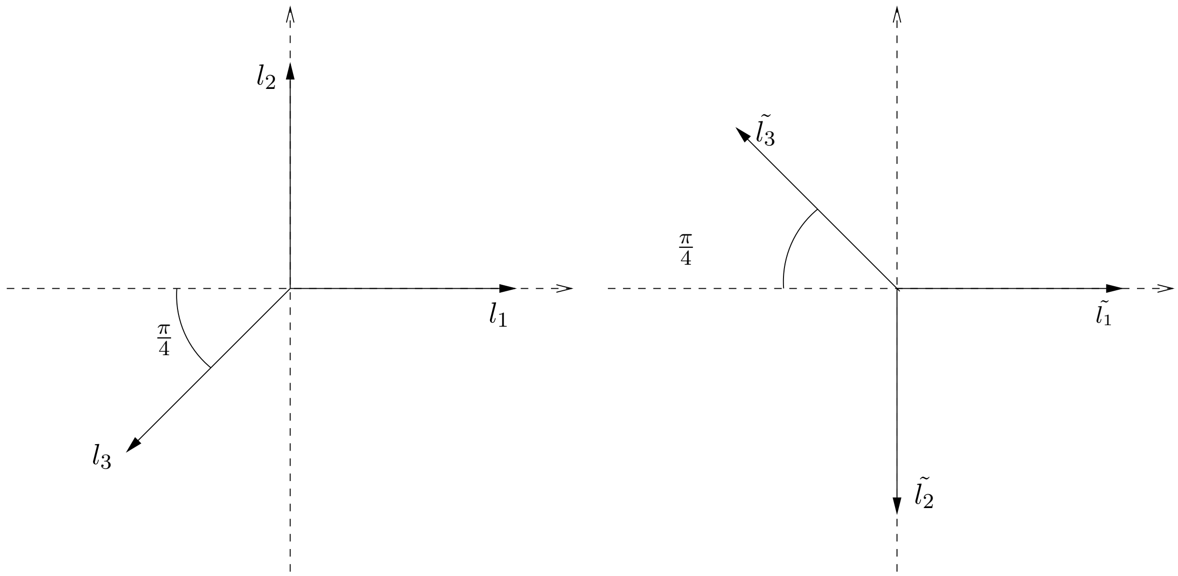
$$N_1(-k) + E(-k)N_2(e^{i\pi/4}k) + N_3(ik) = \overline{G(-i\bar{k})}, \tag{5.1.8b}$$

$$N_1(ik) + E(ik)N_2(e^{-i\pi/4}k) + N_3(-k) = G(-k), \tag{5.1.9a}$$

$$N_1(-ik) + E(-ik)N_2(-e^{-i\pi/4}k) + N_3(-k) = \overline{G(-\bar{k})}. \tag{5.1.9b}$$

$$N_1(k) + E(k)N_2(-e^{-i\pi/4}k) + N_3(ik) = G(ik), \tag{5.1.10a}$$

$$N_1(k) + E(k)N_2(-e^{i\pi/4}k) + N_3(-ik) = \overline{G(i\bar{k})}. \tag{5.1.10b}$$



(a) The contours l_j for the Poisson equation and the modified Helmholtz equation.

(b) The contours $\tilde{l_j}$ for the Poisson equation.

Figure 5.2: The contours for the right isosceles triangle

That is, **8 equations** (5.1.6), (5.1.8), (5.1.9), (5.1.10) for **12 unknowns**: $N_j(\pm ik), N_j(\pm k), j = 1, 3$ and $N_2(\pm e^{i\pi/4}k)$, $N_2(\pm e^{-i\pi/4}k)$.

The contours l_j and $\tilde{l_j}$ in the integral representation for the right isosceles triangle are illustrated in Figure 5.2.

5.2 The symmetric Dirichlet problem

In preparation for the Dirichlet problem, first consider particular Dirichlet boundary conditions in addition to forcing $f(x, y) = f(y, x)$, so that the solution is *symmetric* in x and y, that is

$$u(x, y) = u(y, x), \quad 0 \leq x, y \leq l. \tag{5.2.1}$$

We do this because under these conditions, the system of equations derived from the GR (5.1.6), (5.1.8), (5.1.9), (5.1.10) simplifies.

First we find the solution u, then we find the Dirichlet to Neumann map .

5.2.1 The solution u

Proposition 5.2.1 (The symmetric Dirichlet problem for the modified Helmholtz equation) *Let the complex-valued function $u(x, y)$ satisfy (1.1.1) with $\lambda = -4\beta^2$ in the right isosceles triangle, see figure 5.1, with the Dirichlet boundary conditions*

$$u(s, 0) = u(0, s) = d(s), \quad 0 \le s \le l, \tag{5.2.2}$$

$$u(s, \sqrt{2}l - s) = d_2(s), \quad 0 \le s \le \sqrt{2}l, \tag{5.2.3}$$

where $d_2(s) = d_2(\sqrt{2}l - s)$, and forcing

$$f(x, y) = f(y, x). \tag{5.2.4}$$

Then

$$u = \frac{1}{4\pi}\left\{-\int_{ABC'}\frac{dk}{k}\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)}e(k, z, \bar{z}) - \int_{DEF'}\frac{dk}{k}\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)}e(k, z, \bar{z})E(-k)E(-ik)\right\} + Q_1(z), \tag{5.2.5}$$

where $\Delta_1(k)$ and $\Delta_2(k)$ are defined by

$$\Delta_1(k) = E(-ik) + E(-k) \tag{5.2.6}$$

and

$$\Delta_2(k) = E(ik) + E(-k), \tag{5.2.7}$$

$Q_1(z)$ is given by

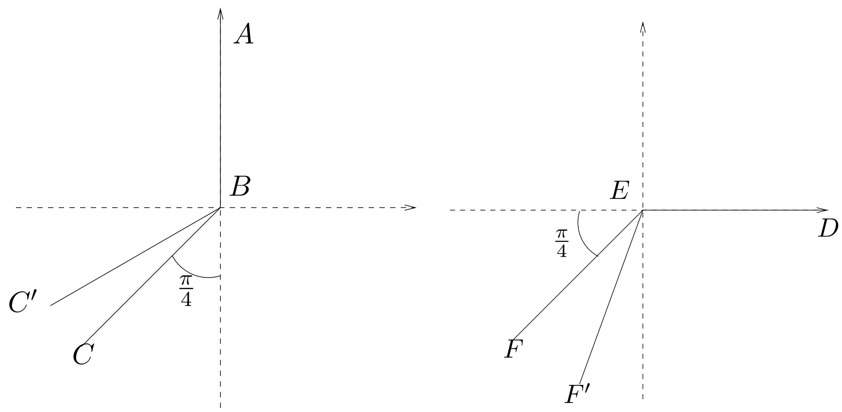
$$Q_1(z) = \frac{1}{4\pi}\left\{\int_{l_1}\frac{dk}{k}e(k, z, \bar{z})\frac{\widetilde{\widetilde{G(k)}}}{E(-ik) + E(-k)} + \int_{l_2}\frac{dk}{k}e(k, z, \bar{z})E(-ik)\frac{\widetilde{G(k)}}{E(-ik) + E(-k)} + \int_{l_3}\frac{dk}{k}e(k, z, \bar{z})\overline{G(\bar{k})}\right\} + Q_0(z), \tag{5.2.8}$$

where $Q_0(z)$ is given by

$$Q_0(z) = -\frac{1}{4\pi i}\left\{\int_{l_1}\frac{dk}{k}e(k, z, \bar{z})H(-ik)D_1(-ik) + \int_{l_2}\frac{dk}{k}e(k, z, \bar{z})E(-ik)H(ke^{i\pi/4})D_2(e^{i\pi/4}k) - \int_{l_3}\frac{dk}{k}e(k, z, \bar{z})H(k)D_3(k)\right\} + F(z, \bar{z}), \tag{5.2.9}$$

and $\widetilde{G(k)}$ and $\widetilde{\widetilde{G(k)}}$ are given in terms of the known function $G(k)$ (5.1.7) by

$$\widetilde{G(k)} = G(k) - \overline{G(\bar{k})} - G(-ik) + \overline{G(-i\bar{k})}, \tag{5.2.10}$$



(a) The contours ABC and ABC' (b) The contours DEF and DEF'

Figure 5.3: The contours ABC and DEF of proposition 5.2.1.

and

$$\widetilde{\widetilde{G(k)}} = E(-ik)(G(-ik) - \overline{G(-i\bar{k})}) + E(-k)(G(k) - \overline{G(\bar{k})}). \tag{5.2.11}$$

In addition

$$e(k, z, \bar{z}) = e^{i\beta\left(kz - \frac{\bar{z}}{k}\right)}, \tag{5.2.12}$$

and the contours ABC' and DEF' are given by figure 5.2.1.

Proof The symmetry properties (5.2.2), (5.2.3), (5.2.4) imply that u satisfies (5.2.1), and the transform functions satisfy

$$N_1(k) = N_3(k) \equiv N(k), \tag{5.2.13}$$

$$E(-ik)N_2(e^{i\pi/4}k) = E(k)N_2(-e^{i\pi/4}k). \tag{5.2.14}$$

$$D_1(k) = D_3(k) \equiv D(k), \tag{5.2.15}$$

$$E(-ik)D_2(e^{i\pi/4}k) = E(k)D_2(-e^{i\pi/4}k), \tag{5.2.16}$$

and

$$\widehat{f}(k) = \overline{\widehat{f}(i\bar{k})}, \tag{5.2.17}$$

so that

$$G(k) = \overline{G(i\bar{k})}. \tag{5.2.18}$$

Then the system of equations (5.1.6), (5.1.8),(5.1.9),(5.1.10) becomes

$$N(-ik) + E(-ik)N_2(e^{i\pi/4}k) + N(k) = G(k), \tag{5.2.19}$$

$$N(ik) + E(ik)N_2(e^{-i\pi/4}k) + N(k) = \overline{G(\bar{k})}, \tag{5.2.20}$$

$$N(-k) + E(-k)N_2(e^{-i\pi/4}k) + N(-ik) = G(-ik), \tag{5.2.21}$$

$$N(-k) + E(-k)N_2(e^{i\pi/4}k) + N(ik) = \overline{G(-i\bar{k})}. \tag{5.2.22}$$

Hence we have four equations for six unknown functions $N(\pm k), N(\pm ik), N_3(e^{\pm i\pi/4}k)$.

We begin the solution procedure by performing **Step 1** of §1.3: use the GR, and equations derived from it, to express the transforms of the unknown boundary values appearing in the IR, $N(-ik), N(k)$ and $N_2(e^{i\pi/4}k)$, in terms of the smallest possible subset of the functions appearing in the GR.

Having four equations and six unknowns implies that each of the unknowns $N(-ik), N(k)$ and $N_2(e^{i\pi/4}k)$ can be expressed in terms of two other unknown functions. We have three choices:

1. $N(ik)$ and $N_2(e^{-i\pi/4}k)$,
2. $N(-k)$ and $N_2(e^{-i\pi/4}k)$,
3. $N(-k)$ and $N(ik)$.

However, the symmetry of the system means that we can get one equation only involving two unknowns, $N_2(e^{i\pi/4}k)$ and $N_2(e^{-i\pi/4}k)$ ((5.2.24) below), and it seems sensible to use this through either using choices 1 or 2. Here we proceed using choice 1, and note that choice 2 proceeds in a similar way. It appears that under choice 3, the unknown boundary values *cannot* be eliminated from the IR, hence the explicit requirement in Step 1 of §1.3 that we express the functions appearing in the IR in terms of the *smallest possible* subset of the functions appearing in the GR.

Simple algebra gives

$$N(k) = -N(ik) - E(ik)N_2(e^{-i\pi/4}k) + \overline{G(\bar{k})}, \tag{5.2.23}$$

$$N_2(e^{i\pi/4}k) = \frac{E(ik)+E(-k)}{E(-ik)+E(-k)}N_2(e^{-i\pi/4}k) + \frac{\widetilde{G(k)}}{E(-ik)+E(-k)}, \tag{5.2.24}$$

$$N(-ik) = N(ik) + E(-k)\frac{E(ik)-E(-ik)}{E(-ik)+E(-k)}N_2(e^{-i\pi/4}k) + \frac{\widetilde{\widetilde{G(k)}}}{E(-ik)+E(-k)}. \tag{5.2.25}$$

Using the definitions of $\hat{u}_j(k)$, (5.1.1),(5.1.2),(5.1.3), the integral representation for u becomes

$$u = \frac{1}{4\pi}\left\{\int_{l_1}\frac{dk}{k}e(k,z,\bar{z})N(-ik) + \int_{l_2}\frac{dk}{k}e(k,z,\bar{z})E(-ik)N_2(e^{i\pi/4}k) + \int_{l_3}\frac{dk}{k}e(k,z,\bar{z})N(k)\right\} + Q_0(z), \tag{5.2.26}$$

where the known function $Q_0(z)$ is given by (5.2.9).

Using (5.2.23),(5.2.24),(5.2.25) we find

$$u = \frac{1}{4\pi}\left\{I_1(z)+I_2(z)+I_3(z)+I_4(z)\right\} + Q_1(z), \tag{5.2.27}$$

where

$$\begin{aligned}
I_1(z) &= \int_{l_1-l_3}\frac{dk}{k}e(k,z,\bar{z})N(ik),\\
I_2(z) &= \int_{l_1-l_3}\frac{dk}{k}e(k,z,\bar{z})\frac{E(-k)E(ik)}{E(-ik)+E(-k)}N_2(e^{-i\pi/4}k),\\
I_3(z) &= \int_{l_2-l_3}\frac{dk}{k}e(k,z,\bar{z})\frac{N_2(e^{-i\pi/4}k)}{E(-ik)+E(-k)},\\
I_4(z) &= \int_{l_2-l_1}\frac{dk}{k}e(k,z,\bar{z})\frac{E(-k)E(-ik)}{E(-ik)+E(-k)}N_2(e^{-i\pi/4}k).
\end{aligned} \tag{5.2.28}$$

Step 2 of §1.3 (look at the analyticity of the integrands): $N(ik)$ is analytic and bounded at $k=\infty$ and $k=0$ for Im $k \geq 0$, and $e(k,z,\bar{z})$ is analytic and bounded for $0 \leq \arg k \leq \pi/2$ (since z is in Ω, $0 < \arg z < \pi/2$). So, closing the contour of I_1 at ∞ in the first quadrant $I_1(z) = 0$ by Cauchy's theorem – the contribution from the integral at ∞ is zero by Jordan's lemma, see e.g. [AF03, Lemma 4.2.2, page 222] (in fact the full strength of Jordan's lemma is only needed if $z \in \partial\Omega$). We now use Cauchy's theorem again to evaluate I_2, I_3 and I_4.

Using the definition of $\Delta_1(k)$, (5.2.6), straightforward calculations and integration by parts for $N_2(e^{-i\pi/4}k)$ yield

$$\begin{gathered}\frac{1}{\Delta_1(k)} \sim E(ik), \quad N_2(e^{-i\pi/4}k) = O\left(\frac{1}{k}e^{kl-ikl}\right), \quad k \to \infty, \quad \arg k \in (-\pi/4, 3\pi/4), \\ \frac{1}{\Delta_1(k)} \sim E(k), \quad N_2(e^{-i\pi/4}k) = O\left(\frac{1}{k}\right), \quad k \to \infty, \quad \arg k \in [3\pi/4, 7\pi/4].\end{gathered} \tag{5.2.29}$$

These facts imply as $k \to \infty$:

- the integrand of I_2 for $\arg k \in [0, \pi/2]$ is asymptotic to $\frac{1}{k}e^{ik(z+l)}$,
- the integrand of I_3 for $\arg k \in [0, 5\pi/4]$ is asymptotic to $\frac{1}{k}e^{ik(z-il)}$,
- the integrand of I_4 for $\arg k \in [-3\pi/4, 0]$ is asymptotic to $\frac{1}{k}e^{ik(z-l)}$.

By considering the right isosceles triangle (see Figure 1):

$$\begin{aligned}&\arg(z+l) \in (0, \pi/4), \\ &\arg(z-il) \in (-\pi/2, -\pi/4), \\ &\arg(z-l) \in (3\pi/4, \pi),\end{aligned}$$

and so, closing the contour of the integral I_2 at ∞ in the first quadrant, its integrand is bounded in the interior of the closed contour. In a similar way, closing the contours of I_3 and I_4 at ∞ for $\arg k \in (\pi/2, 5\pi/4)$ and $\arg k \in (-3\pi/4, 0)$ respectively, the integrands are bounded in the interior of the closed contours. (All the contributions from the integrals at ∞ are zero by Jordan's lemma.) The only singularities of the integrands are at the zeros k_n of $\Delta_1(k)$ given by

$$k_n = e^{\frac{i3\pi}{4}}\left(\frac{(2n+1)\pi}{l\beta 2\sqrt{2}} \pm \sqrt{\frac{(2n+1)^2\pi^2}{8l^2\beta^2} + 1}\right), \quad n \in \mathbb{Z}, \tag{5.2.30}$$

that is, on $e^{\frac{i3\pi}{4}}\mathbb{R}$. Since none of the k_n is in the first quadrant, $I_2 = 0$.

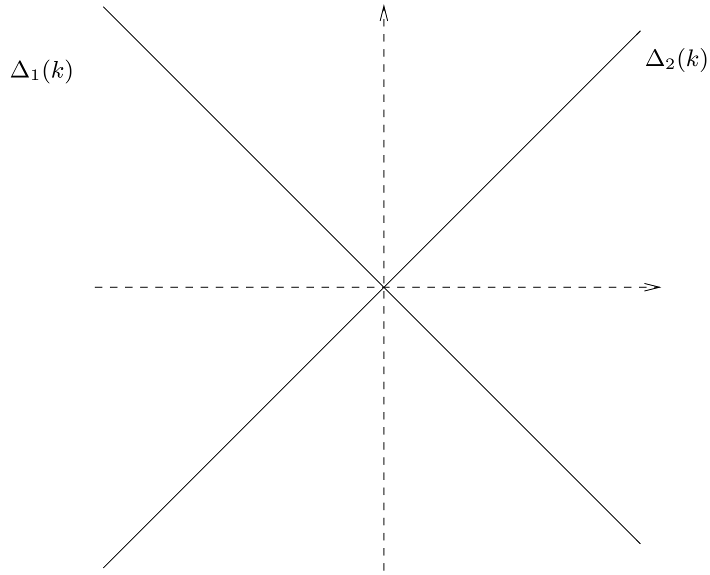


Figure 5.4: The lines on which the zeros of $\Delta_1(k)$ and $\Delta_2(k)$ lie

Step 3 of §1.3 (deform contours and use the GR again): focus on I_3. Equation (5.2.24) can be rewritten as

$$\frac{N_2(e^{i\pi/4}k)}{\Delta_2(k)} = \frac{N_2(e^{-i\pi/4}k)}{\Delta_1(k)} + \frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)}, \tag{5.2.31}$$

where $\Delta_2(k)$ is given by (5.2.7). The zeros of $\Delta_1(k)$ are on $\exp(\frac{3\pi i}{4})\mathbb{R}$ and those of $\Delta_2(k)$ are on $\exp(\frac{\pi i}{4})\mathbb{R}$, see Figure 5.4.

We deform the contour of I_3 as shown in Figure 5.3(a) from ABC to ABC' and use (5.2.31) to give

$$I_3 = \int_{ABC'} \frac{dk}{k} \frac{N_2(e^{i\pi/4}k)}{\Delta_2(k)} e(k,z,\bar{z}) - \int_{ABC'} \frac{dk}{k} \frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} e(k,z,\bar{z}). \tag{5.2.32}$$

The first integral in (5.2.32) is zero. Indeed, similar calculations to those in (5.2.29) show that the integral can be evaluated by closing the contour at infinity, and since the zeros of $\Delta_2(k)$ are outside the contour (this was achieved by deforming from ABC to ABC'), the integral is zero by Cauchy's theorem. Hence, I_3 is given by

$$I_3 = -\int_{ABC'} \frac{dk}{k} \frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} e(k,z,\bar{z}). \tag{5.2.33}$$

In exactly the same way,

$$I_4 = -\int_{DEF'} \frac{dk}{k} \frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} e(k, z, \bar{z}) E(-k) E(-ik), \tag{5.2.34}$$

see Figure 5.3(b). Hence we arrive at the (5.2.5). □

Remark 5.2.2 (Series solution) *Evaluating the integrals in* (5.2.5) *over the contours ABC' and DEF' as residues in the 2nd and 4th quadrants respectively yields the solution as integrals plus an infinite series which is given in [FK03, Example 2.2].*

Using the short–cut of remark 2.1.16, it is straightforward to convert this result into the corresponding one for the Poisson equation.

Proposition 5.2.3 (The symmetric Dirichlet problem for the Poisson equation) *Let u satisfy the conditions of proposition 5.2.1 except with $\lambda = 0$. Then u is given by* (5.2.5) *with the following changes:*

- $e(k, z, \bar{z}) = e^{ikz}$,
- *$H(k)$ and $E(k)$ are given by* (5.1.5) *instead of* (5.1.4),
- *to every term, except for $F(z, \bar{z})$, one must add its complex conjugate,*
- *$D_j(k)$ are given by* (3.1.38) *instead of* (3.1.42).

Proof This is identical to that of proposition 5.2.1. The only difference is that the zeros of $\Delta_1(k)$ are now at

$$k_n = e^{\frac{i3\pi}{4}} \frac{\pi}{l\sqrt{2}}(2n+1), \quad n \in \mathbb{Z},$$

but these are still on $e^{\frac{i3\pi}{4}}\mathbb{R}$, so the proof is unchanged. □

5.2.2 The Dirichlet to Neumann map

We now solve the problem of finding the unknown boundary values directly without finding the solution in Ω. For simplicity, consider the Poisson equation.

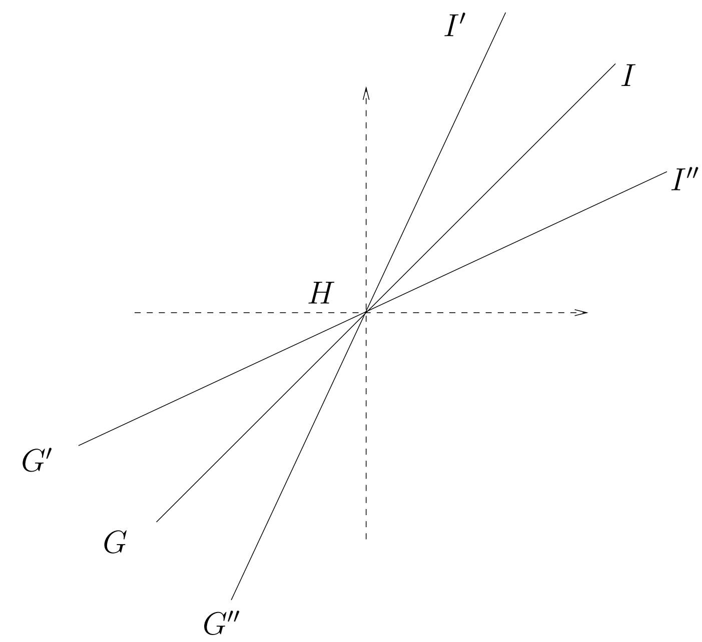


Figure 5.5: The contours GHI,$G'HI'$ and $G''HI''$

Proposition 5.2.4 (The Dirichlet to Neumann map for the symmetric Dirichlet problem of the Poisson equation) *Let u satisfy the conditions of Proposition 5.2.1 except with $\lambda = 0$. Then the unknown Neumann boundary values on sides (1)/(3) $u_n(s)$ and side (2) $u_n^{(2)}(s)$ are given by*

$$u_N(s) = -\frac{1}{2\pi}\int_{JKL} dk\, e^{iks-kl+ikl}\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} + \frac{1}{2\pi}\int_{MKN} dk\, e^{iks-kl-ikl}\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)}$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{iks}\frac{\widetilde{\widetilde{G(k)}}}{\Delta_1(k)}, \tag{5.2.35}$$

$$u_N^{(2)}(s) = \frac{e^{-i\pi/4}}{2\pi}\Bigg[-\int_{G'HI'} dk\, e^{-e^{i\pi/4}ks}E(ik)\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)}$$
$$-\int_{G'HI'} dk\, e^{-e^{i\pi/4}ks}E(-k)\frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} + \int_{-\infty e^{i\pi/4}}^{\infty e^{i\pi/4}} dk\, e^{-e^{i\pi/4}ks}\frac{\widetilde{G(k)}}{\Delta_1(k)}\Bigg] \tag{5.2.36}$$

where the contours are shown in Figures 5.6 and 5.5 and $E(k)$ and $H(k)$ are given by (5.1.5).

Proof

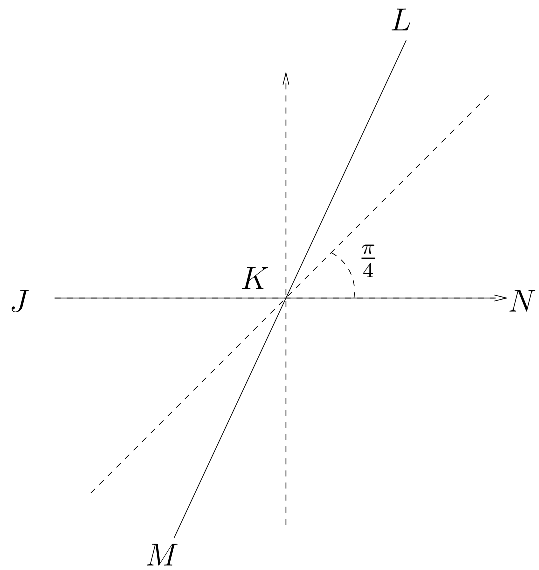


Figure 5.6: The contours JKL and MKN

Side (2) Letting $k \mapsto ik$ in the definition of $N_2(k)$, the equation (3.1.38), we obtain

$$N_2(ik) = \int_0^{\sqrt{2}l} e^{iks} u_N^{(2)}(s) ds.$$

The Fourier inversion formula gives

$$u_N^{(2)}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-iks} N_2(ik), \quad s \in [0, \sqrt{2}l]. \tag{5.2.37}$$

Multiplying (5.2.24) by $\Delta_1(k)$ we have

$$\Delta_1(k) N_2(e^{i\pi/4}k) = (E(ik) + E(-k)) N_2(e^{-i\pi/4}k) + \widetilde{G(k)}, \tag{5.2.38}$$

where $\widetilde{G(k)}$ is given by (5.2.10). This motivates us to let $k \mapsto e^{-i\pi/4}k$ in (5.2.37) (so that $ik \mapsto e^{i\pi/4}k$) and use (5.2.38) to obtain

$$u_N^{(2)}(s) = \frac{e^{-i\pi/4}}{2\pi} \int_{-\infty e^{i\pi/4}}^{\infty e^{i\pi/4}} dk\, \frac{e^{-e^{i\pi/4}ks}}{\Delta_1(k)} \left[(E(ik) + E(-k)) N_2(e^{-i\pi/4}k) + \widetilde{G(k)} \right]. \tag{5.2.39}$$

The contour $(-\infty e^{i\pi/4}, \infty e^{i\pi/4})$ splits the complex k-plane into two halves

$$\mathcal{D}^+ = \left\{ k \in \mathbb{C} : \frac{\pi}{4} < \arg k < \frac{5\pi}{4} \right\}, \tag{5.2.40}$$

$$\mathcal{D}^- = \left\{ k \in \mathbb{C} : -\frac{3\pi}{4} < \arg k < \frac{\pi}{4} \right\}, \tag{5.2.41}$$

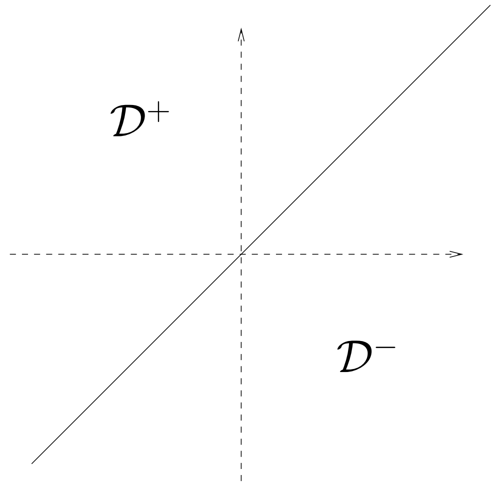


Figure 5.7: The regions $\mathcal{D}^+$ and $\mathcal{D}^+$ in the complex k plane

see Figure 5.7.

Using (5.2.29) we find that

$$\frac{1}{\Delta_1(k)} e^{-e^{i\pi/4}ks} E(ik) N_2(e^{-i\pi/4}k) \text{ is bounded for } k \in \mathcal{D}^+, \tag{5.2.42}$$

$$\frac{1}{\Delta_1(k)} e^{-e^{i\pi/4}ks} E(-k) N_2(e^{-i\pi/4}k) \text{ is bounded for } k \in \mathcal{D}^-. \tag{5.2.43}$$

(This follows from considering the asymptotic behaviour of the integrands in each of the four sectors $\mathcal{D}^\pm \cup (-\frac{\pi}{4}, \frac{2\pi}{4})$, $\mathcal{D}^\pm \cup (\frac{2\pi}{4}, \frac{7\pi}{4})$.)

First we consider the integral of the term (5.2.42). We can deform the contour slightly into $\mathcal{D}^+$ to $G'HI'$, as shown in Figure 5.5. Then using (5.2.38) the integral becomes

$$\int_{G'HI'} dk \, \frac{N_2(e^{i\pi/4}k)}{\Delta_2(k)} e^{-e^{i\pi/4}ks} E(ik) - \int_{G'HI'} dk \, \frac{\widetilde{G(k)}}{\Delta_1(k)\Delta_2(k)} e^{-e^{i\pi/4}ks} E(ik).$$

The first integral is zero: its integrand is bounded at infinity, and the poles lie outside the contour. The term (5.2.43) follows in the same way (except we now deform in $\mathcal{D}^-$ to $G''HI''$), and finally we obtain (5.2.36).

Side (1)/(3) As with side (2) we seek to change variables in the definition of $N(k)$ and use the Fourier inversion formula. Since we have an equation for $N(-ik)$, (5.2.25), we let

$k \mapsto -ik$ in the definition of $N(k)$ in (3.1.38) to get

$$N(-ik) = \int_0^l e^{-iks} q_N(s) ds,$$

and inverting this gives

$$q_N(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iks} N(-ik) dk, \quad s \in [0, l], \tag{5.2.44}$$

where $q_N(s)$ is the Neumann data on sides (1) and (2). Using (5.2.25) we obtain

$$q_N(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{iks} \left[N(ik) + E(-k)(E(ik) - E(-ik)) \frac{N_2(e^{-i\pi/4}k)}{\Delta_1(k)} + \frac{\widetilde{\widetilde{G(k)}}}{\Delta_1(k)} \right], \tag{5.2.45}$$

where $\widetilde{\widetilde{G(k)}}$ is given by (5.2.11).

Now, both e^{iks} and $N(ik)$ are analytic and bounded for $\mathrm{Im} k > 0$, and so, by closing the contour in the upper half k-plane, Cauchy's theorem implies $\int_{-\infty}^{\infty} e^{iks} N(ik) dk = 0$. Using (5.2.29) (and noting that $0 \leq s \leq l$) we find that

$\dfrac{1}{\Delta_1(k)} e^{iks} E(-k) E(ik) N_2(e^{-i\pi/4}k)$ is asymptotic to $e^{ik(s+l)}$ and so is bounded for $\Im k > 0$,

$\dfrac{1}{\Delta_1(k)} e^{iks} E(-k) E(-ik) N_2(e^{-i\pi/4}k)$ is asymptotic to $e^{ik(s-l)}$ and so is bounded for $\Im k < 0$.

Exactly the same argument as above (being careful to deform the contours past the zeros of $\Delta_1(k)$) gives the required Neumann boundary value (5.2.35). $\square$

5.3 General Dirichlet problem

We now consider general Dirichlet boundary conditions (that is, with no symmetry). The fact that we did the symmetric case first proves very useful, since the general case is very similar to the symmetric case, the crucial difference being:

In Step 3 of the symmetric case we used (5.2.31) to change the unknown integrands from having poles on $e^{i3\pi/4}\mathbb{R}$ to $e^{i\pi/4}\mathbb{R}$. The numerators of (5.2.31) were a single function (either $N_2(e^{i\pi/4}k)$ or $N_2(e^{-i\pi/4}k)$). Here we will do the same "shifting poles" trick, but *for it to work, the numerators must be* combinations *of functions (see (5.3.17) and (5.3.18) below.*

Proposition 5.3.1 (The general Dirichlet problem for the modified Helmholtz equation) *Let the complex-valued function $u(x,y)$ satisfy* (1.1.1) *with $\lambda = -4\beta^2$ in the right isosceles triangle, see Figure 5.1, with the Dirichlet boundary conditions*

$$u(s,0) = d_1(s), \quad 0 \le s \le l, \tag{5.3.1}$$

$$u(0,s) = d_3(s), \quad 0 \le s \le l, \tag{5.3.2}$$

$$u(s,\sqrt{2}l - s) = d_2(s), \quad 0 \le s \le \sqrt{2}l, \tag{5.3.3}$$

Then

$$\begin{aligned} u =& \frac{1}{4\pi i}\left\{\int_{ABC'} \frac{dk}{k}\, e(k,z,\bar{z})E(k)\frac{F_2(ik) - F_2(-ik)}{\Delta(-ik)}\right. \\ &- \left.\int_{DEF'} \frac{dk}{k}\, e(k,z,\bar{z})\frac{E(k)E^2(-ik)F_2(ik) + E(-k)F_2(-ik)}{\Delta(-ik)}\right\} + Q_1(z), \end{aligned} \tag{5.3.4}$$

where $Q_1(z)$ is given by (5.2.8) except replacing $\widetilde{\widetilde{G(k)}}$ by $F_1(k)$ and $\widetilde{G(k)}$ by $F_2(k)$, and the functions F_1 and F_2 are given in terms of the transforms of the known boundary conditions by

$$\begin{aligned} F_1(k) =& (\overline{G(i\bar{k})} - G(ik)) + (\overline{G(-i\bar{k})} - G(-ik)) \\ &+ e^{-kl+ikl}(\overline{G(\bar{k})} - G(k)) + e^{kl-ikl}(\overline{G(-\bar{k})} - G(-k)), \end{aligned} \tag{5.3.5}$$

$$\begin{aligned} F_2(k) =& -E(ik)(\overline{G(i\bar{k})} - G(ik)) - E(ik)(\overline{G(-i\bar{k})} - G(-ik)) \\ &- E(k)(\overline{G(\bar{k})} - G(k)) - E(k)(\overline{G(-\bar{k})} - G(-k)), \end{aligned} \tag{5.3.6}$$

and $Q_1(z)$ is given by (5.2.8) except replacing $\widetilde{\widetilde{G(k)}}$ by $F_1(k)$ and $\widetilde{G(k)}$ by $F_2(k)$.

Proof As in the symmetric case we follow Steps 1-3 of §1.3.

Step 1: We now have the eight equations (5.1.6), (5.1.8), (5.1.9), (5.1.10) with twelve unknowns, so we can express the functions appearing in the IR, $N_1(-ik)$, $N_2(e^{i\pi/4}k)$ and $N_3(k)$ in terms of four other unknowns. Actually, we can do it with three unknowns: due to the symmetry of the system, we can get two equations involving $N_2(e^{i\pi/4}k)$, $N_2(e^{-i\pi/4}k)$, $N_2(-e^{i\pi/4}k)$, $N_2(-e^{-i\pi/4}k)$, and therefore one equation for any three of the four. $N_2(e^{i\pi/4}k)$ appears in the IR, so we have three choices for the other two functions: either $\{N_2(e^{-i\pi/4}k), N_2(-e^{-i\pi/4}k)\}$, $\{N_2(e^{-i\pi/4}k), N_2(-e^{i\pi/4}k)\}$, $\{N_2(-e^{-i\pi/4}k), N_2(-e^{i\pi/4}k)\}$.

For our third choice of unknown function, since $N_3(k)$ appears in the IR, if we choose $N_1(ik)$ then we can immediately use the SC (5.1.6b), which is desirable since it minimises algebraic manipulation. Or, since $N_1(-ik)$ appears in the IR, if we choose $N_3(-k)$ then we can use (5.1.9b).

Putting all this together, there are six different choices that give us the lowest number of unknown functions appearing in the IR: either $\{N_2(e^{-i\pi/4}k), N_2(-e^{-i\pi/4}k)\}$, $\{N_2(e^{-i\pi/4}k), N_2(-e^{i\pi/4}k)\}$, or $\{N_2(-e^{-i\pi/4}k), N_2(-e^{i\pi/4}k)\}$, and either $N_1(ik)$ or $N_3(-k)$. We choose $\{N_2(e^{-i\pi/4}k), N_2(-e^{-i\pi/4}k)\}$ and $N_1(ik)$.

In the same way that we manipulated (5.2.19),(5.2.20),(5.2.21) and (5.2.22) to produce (5.2.23),(5.2.24),(5.2.25), we use the eight equations: (5.1.6), (5.1.8), (5.1.9), (5.1.10) to give:

$$\begin{aligned} N_2(k) &= -N_1(ik) - E(ik)N_2(e^{-i\pi/4}k) + \overline{G(\bar{k})}, && (5.3.7)\\ N_1(-ik) &= N_1(ik) + \frac{1}{\Delta(k)}\Big\{ \left(E(-ik) - E(ik)\right) e^{-kl+ikl} N_2(e^{-i\pi/4}k) \\ &\quad + \left(E(ik) - E(-ik)\right) e^{kl-ikl} N_2(-e^{-i\pi/4}k) + F_1(k)\Big\}, && (5.3.8)\\ N_2(e^{i\pi/4}k) &= \frac{1}{\Delta(k)}\Big\{ \left(E(k) - E(-k)\right) E(ik) N_2(e^{-i\pi/4}k) \\ &\quad + \left(E(-ik) - E(ik)\right) E(k) N_2(-e^{-i\pi/4}k) + F_2(k)\Big\}, && (5.3.9) \end{aligned}$$

where

$$\Delta(k) = E(k)E(-ik) - E(-k)E(ik). \qquad (5.3.10)$$

We follow exactly the same steps as in proposition 5.2.1. Using (5.3.7),(5.3.8),(5.3.9) in the definitions of $\rho_j(k)$, (5.1.1),(5.1.2),(5.1.3), and substituting these into the integral representation we find (compare to (5.2.27))

$$q = \frac{1}{4\pi i}\left\{I_1(z) + I_2(z) + I_3(z) + I_4(z) + I_5(z) + I_6(z)\right\} + Q_1(z), \qquad (5.3.11)$$

where

$$I_1(z) = \int_{l_1-l_2} \frac{dk}{k} e(k,z,\bar{z}) N_1(ik), \quad I_2(z) = \int_{l_2-l_1} \frac{dk}{k} e(k,z,\bar{z}) E(-k) E^2(ik) \frac{N_2(e^{-i\pi/4}k)}{\Delta(k)},$$

$$I_3(z) = \int_{l_1-l_3} \frac{dk}{k} e(k,z,\bar{z}) E(-k) \frac{N_2(e^{-i\pi/4}k)}{\Delta(k)}, \quad I_4(z) = \int_{l_3-l_2} \frac{dk}{k} e(k,z,\bar{z}) E(k) \frac{N_2(e^{-i\pi/4}k)}{\Delta(k)},$$

$$I_5(z) = \int_{l_1-l_3} \frac{dk}{k} e(k,z,\bar{z}) E(k) \frac{N_2(-e^{-i\pi/4}k)}{\Delta(k)},$$

and

$$I_6(z) = \int_{l_3-l_1} \frac{dk}{k} e(k,z,\bar{z}) E(k) E^2(-ik) \frac{N_2(-e^{-i\pi/4}k)}{\Delta(k)}.$$

Using the asymptotic calculations for $N_2(e^{-i\pi/4}k)$ in (5.2.29), similar ones for $N_2(-e^{-i\pi/4}k)$ obtained by letting $k \mapsto -k$ in (5.2.29), and

$$\arg(z+2l+il) \in (\tan^{-1}(1/3), \pi/4),$$

$$\arg(z+l-2il) \in (-3\pi/2, -\pi/2),$$

$$\arg(z-2l-il) \in (\pi, 5\pi/4),$$

$$\arg(z+il) \in (\pi/4, \pi/2),$$

$$\tfrac{1}{\Delta(k)} \sim e^{-kl+ikl}, \quad k \to \infty, \quad \arg k \in (-\pi/4, 3\pi/4), \tag{5.3.12}$$

$$\tfrac{1}{\Delta(k)} \sim -e^{kl-ikl}, \quad k \to \infty, \quad \arg k \in (3\pi/4, 7\pi/4), \tag{5.3.13}$$

we obtain the following: as $k \to \infty$

- the integrand of I_1 decays for arg $k \in (0, \pi/2)$,
- the integrand of I_2 decays for arg $k \in (-\tan^{-1}(1/3), 3\pi/4)$,
- the integrand of I_3 decays for arg $k \in (-3\pi/4, \pi/2)$,
- the integrand of I_4 decays for arg $k \in (\pi/2, 3\pi/2)$,
- the integrand of I_5 decays for arg $k \in (0, 5\pi/4)$,
- the integrand of I_6 decays for arg $k \in (-\pi, 0)$.

The zeros of $\Delta(k)$ are at $k = k_n$ where

$$k_n = e^{\frac{i3\pi}{4}}\left(\frac{n\pi}{2\sqrt{2}\beta l} \pm \sqrt{\frac{n^2\pi^2}{8\beta^2 l^2}+1}\right), \quad n \in \mathbb{Z}, \tag{5.3.14}$$

that is, on the line $\exp(\frac{3\pi i}{4})\mathbb{R}$, and so, like the symmetric case, $I_1(z)$ and $I_2(z)$ are 0 by Cauchy's theorem.

We now deform the contour of I_5 from $l_1 - l_3$ to $l_2 - l_3$ (allowed by the fifth bullet point above), and combine it with I_4 to give J_3; and combine I_3 and I_6 to give J_4 where

$$J_3(z) = \int_{l_3-l_2} \frac{dk}{k}\, e(k,z,\bar{z})E(k)\frac{N_2(e^{-i\pi/4}k) - N_2(-e^{-i\pi/4}k)}{\Delta(k)} = \int_{ABC},$$

$$J_4(z) = \int_{l_2-l_1} \frac{dk}{k}\, e(k,z,\bar{z})\frac{E(k)E^2(-ik)N_2(-e^{-i\pi/4}k) - E(-k)N_2(e^{-i\pi/4}k)}{\Delta(k)} = \int_{DEF},$$

and the contours ABC and DEF are shown in Figures 5.3(a) and 5.3(b) respectively. We call these $J_3(z)$ and $J_4(z)$ to emphasise that, in what follows, these are now the analogues of $I_3(z)$ and $I_4(z)$ in the symmetric case. Indeed, using the equation derived from (5.2.14) by $k \mapsto -ik$, J_3 becomes I_3 and J_4 becomes I_4 since $\Delta(k) = \Delta_1(k)\,(E(k) - E(ik))$. (We could have combined I_3 with I_4 and I_5 with I_6, but then it is not possible to close the contours at infinity).

Following the crucial point in bold at the beginning of this section, we now want to express the combinations of functions in the integrands of $J_3(z)$ and $J_4(z)$ in terms of something with poles on $ke^{i\pi/4}$. Letting $k \mapsto -ik$ in (5.3.9) we obtain

$$\begin{aligned} N_2(e^{-i\pi/4}k) =& \frac{1}{\Delta(-ik)}\Big\{(E(-ik) - E(ik))E(k)N_2(-e^{i\pi/4}k) \\ &+ (E(-k) - E(k))E(-ik)N_2(e^{i\pi/4}k) + F_2(-ik)\Big\}, \end{aligned} \tag{5.3.15}$$

and $k \mapsto -k$ gives a similar equation for $N_2(-e^{-i\pi/4}k)$ in terms of $N_2(e^{i\pi/4}k)$ and $N_2(-e^{-i\pi/4}k)$ over $\Delta(ik)$. Now

$$\Delta(ik) = -\Delta(-ik) \tag{5.3.16}$$

and the zeros of $\Delta(-ik)$ are on $\mathbb{R}e^{i\pi/4}$. Using (5.3.15) and its analogue for $N_2(-e^{-i\pi/4}k)$ we obtain

$$\frac{N_2(-e^{-i\pi/4}k) - N_2(e^{-i\pi/4}k)}{\Delta(k)} = \frac{N_2(e^{i\pi/4}k) - N_2(-e^{i\pi/4}k)}{\Delta(-ik)} - \frac{F_2(ik) + F_2(-ik)}{\Delta(k)\Delta(-ik)}, \tag{5.3.17}$$

and

$$\frac{E(k)E^2(-ik)N_2(-e^{-i\pi/4}k) - E(-k)N_2(e^{-i\pi/4}k)}{\Delta(k)} =$$
$$\frac{E^2(-ik)E(-k)N_2(e^{i\pi/4}k) - E(k)N_2(-e^{i\pi/4}k)}{\Delta(-ik)} - \frac{E(k)E^2(-ik)F_2(ik) + E(-k)F_2(-ik)}{\Delta(k)\Delta(-ik)}, \tag{5.3.18}$$

First we concentrate on $J_3(z)$. We deform the contour from ABC to ABC', see Figure 5.3(a), and then use (5.3.17).

$$\begin{aligned}
J_3(z) &= \int_{ABC'} \frac{dk}{k} e(k,z,\bar{z})E(k)\frac{N_2(e^{-i\pi/4}k) - N_2(-e^{-i\pi/4}k)}{\Delta(k)} \\
&= -\int_{ABC'} \frac{dk}{k} e(k,z,\bar{z})E(k)\frac{N_2(e^{i\pi/4}k) - N_2(-e^{i\pi/4}k)}{\Delta(-ik)} \\
&\qquad\qquad + \int_{ABC'} \frac{dk}{k} e(k,z,\bar{z})E(k)\frac{F_2(ik) + F_2(-ik)}{\Delta(k)\Delta(-ik)} \\
&= 0 + \int_{ABC'} \frac{dk}{k} e(k,z,\bar{z})E(k)\frac{F_2(ik) + F_2(-ik)}{\Delta(k)\Delta(-ik)}.
\end{aligned} \tag{5.3.19}$$

Similarly using (5.3.18),

$$J_4(z) = -\int_{DEF'} \frac{dk}{k}\, e(k,z,\bar{z})\frac{E(k)E^2(-ik)F_2(ik) + E(-k)F_2(-ik)}{\Delta(-ik)}. \tag{5.3.20}$$

Thus we obtain (5.3.4). □

Remark 5.3.2 (Solution as an infinite series) *Just as in the symmetric case, evaluating the integrals in* (5.3.4) *over the contours ABC' and DEF' as residues in the 2nd and 4th quadrants respectively yields the solution as integrals plus an infinite series.*

Remark 5.3.3 (The Helmholtz equation) *The solution of the Helmholtz equation, $\lambda = 4\beta^2$, is similar to that for the Poisson and modified Helmholtz equations, but is slightly more complicated due to*

- *the presence of circular arcs in the contours of the IR,*
- *the occurence of eigenvalues.*

The solution is given as a finite number of integrals plus *a finite sum, where the number of terms in the sum grows linearly with β. (This is still a substantial improvement over the classical solution by a bi-infinite sum).*

Remark 5.3.4 (More complicated boundary conditions) *In this chapter we have only considered the Dirichlet problem, however this method can be used to solve more complicated boundary conditions including some Robin and oblique Robin conditions which* cannot *be solved classically. (A particular case of symmetric Robin boundary conditions is solved in [bAF01] and the solution given as integrals plus an infinite sum.) The limiting factor is that the analogue of* (5.3.16) *must hold, which imposes some constraints on the parameters in the boundary conditions.*

Remark 5.3.5 (Solution via reflections) *Actually, it appears that, in principle, it is possible to obtain the expression* (5.3.4) *for the solution of the Dirichlet problem in the right isosceles triangle* without *using the Fokas method. The can be achieved through five steps:*

1. *Use the reflection method of [Prá98] to formulate the BVP for the Green's function in the right isosceles triangle as a BVP in the square.*
2. *Solve the BVP in the square using the classical transform method (i.e. use the Fourier sine series in one variable).*
3. *Use the Watson transformation* (1.2.8) *to convert the expression for the right isosceles Green's function from a series into an integral.*
4. *Split the integrand up, and deform the contours of the resulting integrals so that the resulting expression is uniformly convergent at the boundary of the right isosceles triangle.*
5. *Substitute this expression for the Green's function into Green's IR and interchange the orders of integration.*

Although performing this sequence of steps is possible in principle, we note that Step 4 would be very difficult to perform unless one knew the expression one was aiming for (i.e. (5.3.4)*). (Also note that this sequence of steps would not work for BVPs in the right isosceles triangle with Robin boundary conditions, whereas the Fokas method can solve certain BVPs of this type.)*

Chapter 6

A new numerical method

6.1 Introduction

Summary

- In this Chapter the Dirichlet to Neumann map is solved numerically for the Laplace and modified Helmholtz equations in general convex polygons (with n sides).

- The main idea is that the global relation is valid *for all* $k \in \mathbb{C}$. Expanding the n unknown functions (the unknown boundary value on each side) in some series such as Fourier or Chebyshev up to N terms and then evaluating the global relation at a properly chosen set of nN points (collocation points in the spectral space) yields nN equations for the nN unknowns.

- Numerical experiments suggest that the method inherits the order of convergence of the basis used to expand the unknown functions; namely exponential for a polynomial basis, such as Chebyshev, and algebraic for a Fourier basis.

- However the condition number of the associated linear system is much higher for a polynomial basis than for a Fourier one.

6.1.1 The PDE and the global relation

In this chapter we make a slight notational change, so that the PDE is

$$\triangle q(\boldsymbol{x}) - 4\lambda q(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega \tag{6.1.1}$$

where $\lambda \in \mathbb{R}^+$. This is the Laplace equation for $\lambda = 0$, and the modified Helmholtz equation for $\lambda > 0$.

The change of variables $z = x + iy, \bar{z} = x - iy$ transforms (6.1.1) to the equation

$$q_{z\bar{z}}(z, \bar{z}) - \lambda q(z, \bar{z}) = 0, \quad z \in \Omega. \tag{6.1.2}$$

The two global relations (or one global relation, one Schwartz conjugate (SC)) for (6.1.2) are

$$\int_{\partial\Omega} e^{-ikz - \frac{\lambda}{ik}\bar{z}} \left[(q_z + ikq)dz - \left(q_{\bar{z}} + \frac{\lambda}{ik} q \right) d\bar{z} \right] = 0, \quad k \in \mathbb{C}, \tag{6.1.3a}$$

$$\int_{\partial\Omega} e^{ik\bar{z} + \frac{\lambda}{ik} z} \left[(q_{\bar{z}} - ikq)d\bar{z} - \left(q_z - \frac{\lambda}{ik} q \right) dz \right] = 0 \ , \quad k \in \mathbb{C}. \tag{6.1.3b}$$

Indeed, the four adjoint solutions are

$$v = e^{-ikz - \frac{\lambda}{ik}\bar{z}}, \quad v = e^{ik\bar{z} + \frac{\lambda}{ik} z}. \tag{6.1.4}$$

and two more by letting $k \mapsto -k$ (lemma 3.1.3), and with these (3.1.3) becomes (6.1.3a) and (6.1.3b).

For the Laplace equation, the global relation (6.1.3a) becomes

$$\int_{\partial\Omega} e^{-ikz}[(q_z + ikq)dz - q_{\bar{z}} d\bar{z}] = 0. \tag{6.1.5}$$

Noting that

$$e^{-ikz}[(q_z + ikq)dz - q_{\bar{z}} d\bar{z}] = -d[e^{-ikz} q] + 2e^{-ikz} q_z dz, \tag{6.1.6}$$

it follows that (6.1.5) is equivalent to

$$\int_{\partial\Omega} e^{-ikz} q_z dz = 0, \tag{6.1.7}$$

which is the global relation used in [FFX04] and [SFFS08].

6.1.2 The Numerical Method for a Convex Polygon

This chapter contains a method for solving numerically the global relations (6.1.3a) and (6.1.3b) for the Dirichlet problem for the Laplace and modified Helmholtz equations in the interior of a convex polygon, see Figure 6.1. This method was introduced in [SFFS08] in connection with the simple global relation (6.1.7) (see also [FFX04]). Because this approach involves enforcing the global relations to hold at a set of discrete points in the *spectral* plane, it has been called a spectral collocation method.

The key to this method is that the global relations (6.1.3a) and (6.1.3b) are valid *for all* $k \in \mathbb{C}$. Suppose we expand the n unknown functions (the unknown boundary value on each side) in some series such as Fourier or Chebyshev up to N terms, and then we evaluate either of the global relations at nN points; this would yield nN equations for the nN unknowns, which in principle could be solved. We now face two questions:

1. How to choose the basis.

2. How to choose the points k.

The presence of exponentials in the global relations means that if we choose a Fourier basis it is possible to choose points which single out each Fourier coefficient and produce a linear system for the unknown coefficients for which the block diagonal submatrices are the identity matrix. This means the system has a low condition number, and numerical experiments suggest that the condition number actually grows linearly with the number of basis elements.

For sufficiently smooth functions the employment of a Chebyshev basis gives exponential convergence, and numerical experiments suggest that this is inherited by the method. However, because of the non-orthogonality of the Chebyshev polynomials with the exponentials in the global relations it is not obvious what the optimal choice of points is. In the example computed here we have chosen the same points as for the Fourier basis and in this case it appears that the condition number of the associated matrix grows exponentially with the number of basis elements.

For smooth functions a Fourier basis yields only algebraic convergence; however, if the functions do not have sufficient smoothness then a Fourier basis yields better convergence properties than a polynomial basis.

The method presented in this chapter requires that the given boundary conditions are sufficiently compatible at the corners of the polygon so the solution has a continuous first derivative (however in §6.4 we discuss how this limitation can be bypassed). This requirement makes it possible to compute the values of the unknown Neumann data at the corners of the polygon; then, by subtracting these known values, one obtains unknown functions which vanish at the corners. (Note that this requirement is not that the normal derivative of q is continuous at corners, which would be false for smooth q because the direction of the normal derivative changes discontinuously at the corner. Rather, it is the derivatives q_z and $q_{\bar{z}}$ which are assumed to be continuous.)

A convenient Fourier representation for a function which is zero at the endpoints is a modified sine-Fourier series expansion:

$$f(s) = \sum_{m=1}^{\infty} \left[s_m \sin ms + c_m \cos\left(m - \frac{1}{2}\right) s \right], \quad -\pi < s < \pi, \tag{6.1.8}$$

$$s_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \sin(ms) ds, \;\; c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \cos\left(m - \frac{1}{2}\right) s\, ds, \;\; m = 1, 2, \cdots, \;\; p = 1, \cdots, n. \tag{6.1.9}$$

The advantage of the above expansion is that s_m^p and c_m^p are of order m^{-3} as $m \to \infty$, provided that $f(s)$ has sufficient smoothness. The representation (6.1.8) can be obtained from the usual sine-Fourier series in the interval $(0, \pi)$ by using a change of variables to map this interval to $(-\pi, \pi)$. The analogue of the representation (6.1.8) corresponding to the cosine-Fourier series was introduced in [IN08]. Using the techniques of [Olv09] it is possible to prove that if $f \in C^3(-\pi, \pi)$ and

$$f^N(s) = \sum_{m=1}^{N} \left[s_m \sin ms + c_m \cos\left(m - \frac{1}{2}\right) s \right], \quad p = 1, \cdots, n, \tag{6.1.10}$$

then

$$\|f - f^N\|_\infty = O\left(\frac{1}{N^2}\right). \tag{6.1.11}$$

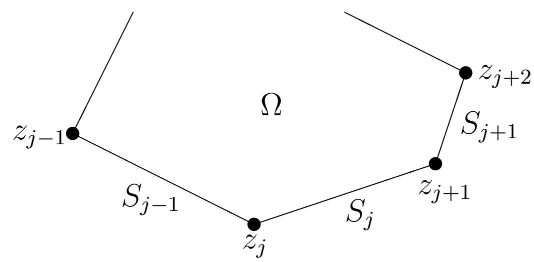


Figure 6.1: The convex polygon Ω with corners z_j and sides S_j

Remark 6.1.1 (The null-field method) *The method described in this chapter is conceptually similar to the so-called "null-field method", see e.g. [CK83, §3.9 p.104], [Mar06, Chapter 7]. In fact, the method of this chapter is based on the numerical solution of the global relation in* polygons, *whereas the null-field method is based on the numerical solution of the global relation in* polar co-ordinates (3.2.1) *with $k \in \mathbb{Z}$ so that the angular variable θ is periodic.*

6.1.3 Outline of Chapter

Numerical schemes for computing the Dirichlet to Neumann maps for the Laplace and modified Helmholtz equations are given in Section 6.2. Numerical results are presented in Section 6.3. These results are discussed in Section 6.4.

6.2 The Dirichlet to Neumann map and its numerical implementation

The Laplace and modified Helmholtz equations are discussed in Sections 6.2.1 and 6.2.2 respectively.

6.2.1 The Laplace Equation

A system of $2n$ equations characterizing the Dirichlet to Neumann map is introduced in Section 6.2.1.1. For a sufficiently smooth solution q of the Dirichlet problem, the Neumann boundary values at the corners of the polygon are found explicitly in Section 6.2.1.2. The system of $2n$ equations characterizing the Dirichlet to Neumann map is reformulated in terms of functions that vanish at the corners in Section 6.2.1.3. Finally, the numerical method is introduced in Section 6.2.1.4.

6.2.1.1 The Dirichlet to Neumann map

Proposition 6.2.1 *Let the complex-valued function $q(z,\bar{z})$ satisfy the Laplace equation in the interior of a convex polygon Ω with corners $\{z_j\}_1^n$ (indexed anticlockwise, modulo n), and let S_j denote the side (z_j, z_{j+1}), see Figure 6.1. Let q satisfy Dirichlet boundary conditions on each side:*

$$q_j(s) = d_j(s), \quad j = 1, \cdots, n, \tag{6.2.1}$$

where s parametrizes the side S_j and q_j denotes q on this side Let $u_j(s)$ denote the unknown Neumann boundary data on S_j, and assume that $d_j, u_j \in C^1[-\pi,\pi]$.

The n unknown complex-valued functions $\{u_j\}_1^n$ satisfy the following $2n$ equations for $l \in \mathbb{R}^+$ and $p = 1, \cdots, n$:

$$\int_{-\pi}^{\pi} e^{ils} u_p(s)ds = -\sum_{\substack{j=1\\ j\neq p}}^{n} E_{jp}(l) \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} u_j(s)ds + G_p(l), \tag{6.2.2a}$$

$$\int_{-\pi}^{\pi} e^{-ils} u_p(s)ds = -\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{E}_{jp}(l) \int_{-\pi}^{\pi} e^{-il\frac{\bar{h}_j}{\bar{h}_p}s} u_j(s)ds + \tilde{G}_p(l); \tag{6.2.2b}$$

where the known functions $E_{jp}(l)$, $G_p(l)$ and $\tilde{G}_p(l)$, $j = 1, \cdots, n$, $p = 1, \cdots, n$, are defined by

$$E_{jp}(l) = \exp\left[\frac{il}{h_p}(m_j - m_p)\right], \quad l \in \mathbb{R}^+, \tag{6.2.3a}$$

$$G_p(l) = \sum_{j=1}^{n} l\frac{h_j}{h_p} E_{jp}(l) \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} d_j(s)ds, \quad \tilde{G}_p(l) = \sum_{j=1}^{n} l\frac{\bar{h}_j}{\bar{h}_p} \bar{E}_{jp}(l) \int_{-\pi}^{\pi} e^{-il\frac{\bar{h}_j}{\bar{h}_p}s} d_j(s)ds, \quad l \in \mathbb{R}^+, \tag{6.2.3b}$$

with

$$h_j = \frac{1}{2\pi}\left(z_{j+1} - z_j\right), \quad m_j = \frac{1}{2}\left(z_{j+1} + z_j\right), \quad j = 1, \cdots, n. \tag{6.2.3c}$$

Furthermore, each of the terms appearing in the four sums on the right-hand sides of equations (6.2.2) *decays exponentially as* $l \to \infty$*, except for the terms with* $j = p$ *which oscillate and for those with* $j = p \pm 1$ *which decay linearly.*

Proof Define $\hat{q}_j(k)$ by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz}\left[(q_z + ikq)dz - q_{\bar{z}}d\bar{z}\right], \quad j = 1, \cdots, n, \quad k \in \mathbb{C}. \tag{6.2.4}$$

Parametrizing the side S_j with respect to its midpoint m_j,

$$z(s) = m_j + sh_j, \quad -\pi < s < \pi, \tag{6.2.5}$$

and using

$$q_z dz = \frac{1}{2}\left[\frac{dq_j}{ds}(s) + iu_j(s)\right] ds \quad \text{and} \quad q_{\bar{z}} d\bar{z} = \frac{1}{2}\left[\frac{dq_j}{ds}(s) - iu_j(s)\right] ds \tag{6.2.6}$$

we find

$$\hat{q}_j(k) = e^{-ikm_j}\int_{-\pi}^{\pi} e^{-ikh_j s}\left[iu_j(s) + ikh_j d_j(s)\right] ds, \quad j = 1, \cdots, n, \quad k \in \mathbb{C}. \tag{6.2.7}$$

Writing the first global relation (6.1.3a) in the form

$$\hat{q}_p(k) = -\sum_{\substack{j=1 \\ j \neq p}}^{n} \hat{q}_j(k), \quad p = 1, \cdots, n, \quad k \in \mathbb{C}, \tag{6.2.8}$$

substituting for $\{\hat{q}_j(k)\}_1^n$ from (6.2.7), multiplying by $-ie^{im_p k}$, and evaluating the resulting equation at

$$k = -\frac{l}{h_p}, \quad l \in \mathbb{R}^+ \tag{6.2.9}$$

yields (6.2.2a). The second global relation (6.1.3b) can be obtained from the first (6.1.3a) by taking the Schwartz conjugate of all terms except q. Taking the complex conjugate of all terms in (6.2.2a) except u_j and d_j yields (6.2.2b).

Convexity implies the estimate

$$\left[\arg(m_j - m_p) - \arg(h_p)\right] \in (0, \pi), \quad j \neq p, \tag{6.2.10}$$

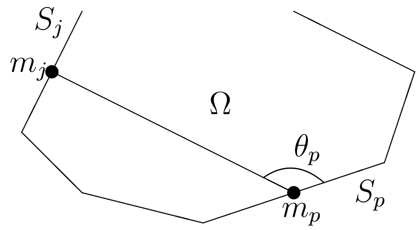


Figure 6.2: The angle $\theta_p := \arg(m_j - m_p) - \arg(h_p)$

see Figure 6.2. Each of the terms appearing on the right-hand side of (6.2.2a) involves

$$\int_{-\pi}^{\pi} e^{\frac{il}{h_p}(m_j+sh_j-m_p)} a_j(s)ds \quad \text{with} \quad a_j = u_j \quad \text{or} \quad a_j = d_j. \tag{6.2.11}$$

Convexity also implies that for $s \in (-\pi, \pi)$,

$$0 < [\arg(m_j + sh_j - m_p) - \arg h_p] < \pi \quad \text{if} \quad j \neq p,\, p \pm 1. \tag{6.2.12}$$

If $j = p - 1$, then

$$m_p - \pi h_p = z_p = m_j + \pi h_j \tag{6.2.13}$$

and hence

$$\arg(m_j + \pi h_j - m_p) - \arg h_p = \pi. \tag{6.2.14}$$

If $j = p + 1$, then

$$m_p + \pi h_p = z_{p+1} = m_j - \pi h_j \tag{6.2.15}$$

and hence

$$\arg(m_j - \pi h_j - m_p) - \arg h_p = 0. \tag{6.2.16}$$

Integration by parts (allowed since $d_j, u_j \in C^1[-\pi, \pi]$) implies that the left-hand side of equation (6.2.11) equals

$$\frac{h_p}{ilh_j}\left[e^{\frac{il}{h_p}(m_j+\pi h_j-m_p)} a_j(\pi) - e^{\frac{il}{h_p}(m_j-\pi h_j-m_p)} a_j(-\pi)\right] + O\left(\frac{1}{l^2}\right). \tag{6.2.17}$$

Hence each of the terms in the two sums on the right-hand side of (6.2.2a) decays exponentially if $j \neq p$ and $j = p \pm 1$. If $j = p \pm 1$ then there is linear decay, and the term in G_p with $j = p$ oscillates. Clearly, if $a_j(\pm\pi) = 0$ then the decay is quadratic. Similar considerations are valid for the right-hand side of (6.2.2b). □

Remark 6.2.2 *Since $l \in \mathbb{R}^+$, it follows that the expressions on the left-hand sides of equations* (6.2.2) *taken together define the Fourier transform of $u_p(s)$.*

Remark 6.2.3 *Regarding the choice of k in equation* (6.2.9)*, we note that $\hat{q}_p(k)$ involves the exponential function* $\exp[-ikh_ps]$. *To obtain the Fourier transform of $u_p(s)$, k must be chosen such that $kh_p \in \mathbb{R}$, i.e.*

$$\arg k = -\arg(h_p) \quad \text{or} \quad \arg k = \pi - \arg(h_p). \tag{6.2.18}$$

Of these two choices, it is only the second one that makes the right-hand sides of equations (6.2.2) *bounded as $l \to \infty$. We also note that in order to maximize the decay of the right-hand sides of equations* (6.2.2) *as $l \to \infty$, we use the parametrization in* (6.2.5) *instead of the parametrization*

$$z(s) = z_j + 2sh_j, \quad 0 < s < \pi. \tag{6.2.19}$$

6.2.1.2 The unknown values at the corners, $u_j(\pm\pi)$

Suppose that the given functions $d_j(s)$ satisfy appropriate compatibility conditions so that q_z and $q_{\bar{z}}$ are continuous at the corners. Then $u_j(\pm\pi)$, $j = 1, \cdots, n$ can be determined explicitly:

$$u_j(\pi) = \frac{|h_{j+1}|\cos(\alpha_{j+1} - \alpha_j)\frac{d}{ds}d_j(\pi) - |h_j|\frac{d}{ds}d_{j+1}(-\pi)}{|h_{j+1}|\sin(\alpha_{j+1} - \alpha_j)} \tag{6.2.20a}$$

and

$$u_j(-\pi) = \frac{|h_j|\frac{d}{ds}d_{j-1}(\pi) - |h_{j-1}|\cos(\alpha_j - \alpha_{j-1})\frac{d}{ds}d_j(-\pi)}{|h_{j-1}|\sin(\alpha_j - \alpha_{j-1})}, \tag{6.2.20b}$$

where

$$\alpha_j = \arg(h_j). \tag{6.2.21}$$

Since the polygon is convex,

$$\alpha_{j+1} \neq \alpha_j + \pi \tag{6.2.22}$$

and thus

$$\sin(\alpha_{j+1} - \alpha_j) \neq 0. \tag{6.2.23}$$

In order to derive (6.2.20), we note that (6.2.6) implies

$$q_z = \frac{e^{-i\alpha_j}}{2|h_j|}\left[\frac{dq_j}{ds}(s) + iu_j(s)\right], \quad s \in S_j. \tag{6.2.24}$$

The continuity of q_z at the corner z_j implies that the expression in (6.2.24) with j replaced by $j-1$ evaluated at $s=\pi$ (the right end of the side S_{j-1}) equals the expression in (6.2.24) evaluated at $s=-\pi$ (the left end of the side S_j), i.e.

$$\frac{e^{-i\alpha_{j-1}}}{|h_{j-1}|}\left[\frac{dq_{j-1}}{ds}(\pi)+iu_{j-1}(\pi)\right]=\frac{e^{-i\alpha_j}}{|h_j|}\left[\frac{dq_j}{ds}(-\pi)+iu_j(-\pi)\right]. \tag{6.2.25a}$$

Similarly, the continuity of $q_{\bar z}$ at z_j implies that

$$\frac{e^{i\alpha_{j-1}}}{|h_{j-1}|}\left[\frac{dq_{j-1}}{ds}(\pi)-iu_{j-1}(\pi)\right]=\frac{e^{i\alpha_j}}{|h_j|}\left[\frac{dq_j}{ds}(-\pi)-iu_j(-\pi)\right]. \tag{6.2.25b}$$

Solving equations (6.2.25) for $u_{j-1}(\pi)$ and $u_j(-\pi)$ and then letting $j \to j+1$ in the expression for $u_{j-1}(\pi)$, we find equations (6.2.20).

6.2.1.3 Unknown functions that vanish at the corners

Since the values of $u_p(s)$ are known at the two corners, it is possible to express the unknown function $u_p(s)$ in terms of a new unknown function, denoted by $\breve{u}_p(s)$, which *vanishes* at the corners:

$$u_p(s)=\breve{u}_p(s)+u_{\star p}(s), \quad p=1,\cdots,n, \quad -\pi<s<\pi, \tag{6.2.26a}$$

with

$$u_{\star p}(s)=\frac{1}{2\pi}\left[(s+\pi)u_p(\pi)-(s-\pi)u_p(-\pi)\right], \quad p=1,\cdots,n, \quad -\pi<s<\pi. \tag{6.2.26b}$$

The unknown functions $\{\breve{u}_j\}_1^n$ satisfy equations similar to (6.2.2) but with G_p and $\tilde G_p$ replaced by $G_p-U_{\star p}$ and $\tilde G_p-\tilde U_{\star p}$ respectively, where the known functions $U_{\star p}$ and $\tilde U_{\star p}$ are defined for $p=1,\cdots,n$ by

$$U_{\star p}(l)=\sum_{j=1}^n E_{jp}(l)\int_{-\pi}^{\pi}e^{il\frac{h_j}{h_p}s}u_{\star j}(s)ds, \quad \tilde U_{\star p}(l)=\sum_{j=1}^n \bar E_{jp}(l)\int_{-\pi}^{\pi}e^{-il\frac{\bar h_j}{\bar h_p}s}u_{\star j}(s)ds, \quad l\in\mathbb{R}^+. \tag{6.2.27}$$

By computing the integrals involving $u_{\star j}(s)$ using (6.2.26b), we find that for $p=1,\cdots,n$ and $l\in\mathbb{R}^+$,

$$\begin{aligned}U_{\star p}(l)=\sum_{j=1}^n E_{jp}(l)\Bigg\{&[u_j(\pi)-u_j(-\pi)]\left[\frac{h_p}{ilh_j}\cos\left(\frac{l\pi h_j}{h_p}\right)+i\frac{h_p^2}{\pi l^2h_j^2}\sin\left(\frac{l\pi h_j}{h_p}\right)\right]\\&+\frac{h_p}{lh_j}[u_j(\pi)+u_j(-\pi)]\sin\left(\frac{l\pi h_j}{h_p}\right)\Bigg\}\end{aligned} \tag{6.2.28a}$$

and

$$\tilde{U}_{\star p}(l) = \sum_{j=1}^{n} \bar{E}_{jp}(l) \left\{ [u_j(\pi) - u_j(-\pi)] \left[\frac{\bar{h}_p}{-il\bar{h}_j} \cos\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right) - i\frac{\bar{h}_p^2}{\pi l^2 \bar{h}_j^2} \sin\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right) \right] + \frac{\bar{h}_p}{l\bar{h}_j} [u_j(\pi) + u_j(-\pi)] \sin\left(\frac{l\pi\bar{h}_j}{\bar{h}_p}\right) \right\}. \quad (6.2.28b)$$

Remark 6.2.4 *The function $\check{u}_p(s)$, which is defined for $-\pi < s < \pi$, vanishes at the end points. As explained in the introduction, a convenient representation for such a function is a modified sine-Fourier series expansion:*

$$\check{u}_p(s) = \sum_{m=1}^{\infty} \left[s_m \sin ms + c_m \cos\left(m - \frac{1}{2}\right) s \right], \quad (6.2.29a)$$

$$s_m^p = \frac{1}{\pi} \int_{-\pi}^{\pi} \check{u}_p(s) \sin(ms) ds, \; c_m^p = \frac{1}{\pi} \int_{-\pi}^{\pi} \check{u}_p(s) \cos\left(m - \frac{1}{2}\right) s\, ds, \; m = 1, 2, \cdots, \; p = 1, \cdots, n. \quad (6.2.29b)$$

Using the techniques of [Olv09], it is possible to prove that if $\check{u} \in C^3(-\pi, \pi)$ and

$$\check{u}_p^N(s) = \sum_{m=1}^{N} \left[s_m^p \sin ms + c_m^p \cos\left(m - \frac{1}{2}\right) s \right], \quad p = 1, \cdots, n \quad (6.2.30)$$

then

$$\|\check{u}_p - \check{u}_p^N\|_\infty = O\left(\frac{1}{N^2}\right). \quad (6.2.31)$$

6.2.1.4 The numerical method

Proposition 6.2.5 *Let q satisfy the boundary value problem specified in Proposition 6.2.1. Assume that the values of the unknown functions $\{u_j\}_1^n$ at the corners $\{z_j\}_1^n$ are given by equations (6.2.20). Express $\{u_j\}_1^n$ in terms of the unknown functions $\{\check{u}_j\}_1^n$ defined in equation (6.2.26) and approximate the latter functions by*

$$\check{u}_j^N(s) = \sum_{m=1}^{N} \left[s_m^j \sin ms + c_m^j \cos\left(m - \frac{1}{2}\right) s \right], \quad j = 1, \cdots, n. \quad (6.2.32)$$

Then the constants s_m^p and c_m^p, $m = 1, \cdots, N$, $p = 1, \cdots, n$ satisfy the following $2Nn$ algebraic equations:

$$
\begin{aligned}
2\pi s_m^p =& i \sum_{\substack{j=1 \\ j\neq p}}^{n} \left\{ \sum_{r=1}^{N} s_r^j \left[E_{jp}(m) S_{jp}^r(m) - \bar{E}_{jp}(m) \bar{S}_{jp}^r(m) \right] \right. \\
& \left. + \sum_{r=1}^{N} c_r^j \left[E_{jp}(m) C_{jp}^r(m) - \bar{E}_{jp}(m) \bar{C}_{jp}^r(m) \right] \right\} \\
& - iG_p(m) + i\tilde{G}_p(m) + iU_{\star p}(m) - i\tilde{U}_{\star p}(m)
\end{aligned}
\tag{6.2.33a}
$$

and

$$
\begin{aligned}
2\pi c_m^p =& - \sum_{\substack{j=1 \\ j\neq p}}^{N} \left\{ \sum_{r=1}^{N} s_r^j \left[E_{jp}\left(m - \frac{1}{2}\right) S_{jp}^r\left(m - \frac{1}{2}\right) + \bar{E}_{jp}\left(m - \frac{1}{2}\right) \bar{S}_{jp}^r\left(m - \frac{1}{2}\right) \right] \right. \\
& \left. + \sum_{r=1}^{N} c_r^j \left[E_{jp}\left(m - \frac{1}{2}\right) C_{jp}^r\left(m - \frac{1}{2}\right) + \bar{E}_{jp}\left(m - \frac{1}{2}\right) \bar{C}_{jp}^r\left(m - \frac{1}{2}\right) \right] \right\} \\
& + G_p\left(m - \frac{1}{2}\right) + \bar{G}_p\left(m - \frac{1}{2}\right) - U_{\star p}\left(m - \frac{1}{2}\right) - \tilde{U}_{\star p}\left(m - \frac{1}{2}\right),
\end{aligned}
\tag{6.2.33b}
$$

where the known functions G_p, $\tilde{G}_p$, $U_{\star p}$, $\tilde{U}_{\star p}$ are defined by equations (6.2.3) *and* (6.2.28), *and*

$$
S_{jp}^r(m) = \frac{2ir(-1)^{r-1} \sin\left(\frac{m\pi h_j}{h_p}\right)}{r^2 - \frac{m^2 h_j^2}{h_p^2}}, \quad C_{jp}^r(m) = \frac{2\left(r - \frac{1}{2}\right)(-1)^{r-1} \cos\left(\frac{m\pi h_j}{h_p}\right)}{\left(r - \frac{1}{2}\right)^2 - \frac{m^2 h_j^2}{h_p^2}},
$$
$$
j = 1, \cdots, n, \quad p = 1, \cdots, n, \quad r = 1, \cdots, n, \quad m = 1, \cdots, N. \tag{6.2.33c}
$$

Proof Equation (6.2.32) implies

$$
s_m^p = \frac{1}{\pi} \int_{-\pi}^{\pi} \breve{u}_p^N(s) \sin(ms) ds \quad m = 1, \cdots, N, \quad p = 1, \cdots, n. \tag{6.2.34}
$$

Recall that the functions $\breve{u}_p$ satisfy equations similar to (6.2.2) but with G_p and $\tilde{G}_p$ replaced by $G_p - U_{\star p}$ and $\tilde{G}_p - \tilde{U}_{\star p}$ respectively. Taking the equations satisfied by $\breve{u}_p$, replacing $\breve{u}_p$ by $\breve{u}_p^N$ (defined in equation (6.2.32)), taking the difference of these equations and evaluating the resulting equation at $l = m$ implies equation (6.2.33a). In this respect we note that the left-hand side of the resulting equation immediately yields s_m^p, whereas to evaluate the right-hand side we use

$$\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}s}\check{u}_j^N(s)ds = \sum_{r=1}^{N}\left\{s_r^j\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}s}\sin rsds + c_r^j\int_{-\pi}^{\pi} e^{im\frac{h_j}{h_p}s}\cos\left(r-\frac{1}{2}\right)s\,ds\right\} \tag{6.2.35}$$

and evaluate the integrals on the right-hand side of this equation explicitly.

Proceeding as earlier, but adding the equations satisfied by $\{\check{u}_j^N\}_1^n$ and then evaluating the resulting equation at $l = m - \frac{1}{2}$, we find (6.2.33b). □

Remark 6.2.6 *The left-hand sides of equations* (6.2.2) *involve integrals of* $u_p(s)$ *with respect to the exponential functions* $\exp(\pm ils)$. *This suggests that* $u_p(s)$ *should be represented by a Fourier type expansion. Indeed, ignoring for a moment the first terms on the right-hand sides of equations* (6.2.2), *if* $u_p(s)$ *is represented by a Fourier type series such as equation* (6.2.29a) *then the relevant Fourier coefficients can be obtained immediately in terms of* G_p *and* $\tilde{G}_p$. *Thus the choice of the representation* (6.2.32) *is consistent with the fact that* $\check{u}_p(s)$ *vanishes at the corners, and the evaluation of the global relations at* $l = m$ *and* $l = m - \frac{1}{2}$ *is consistent with the orthogonality conditions associated with this expansion (see equation* (6.2.29b)*). One could use an expansion in terms of Chebyshev polynomials to improve convergence but, since the associated orthogonality conditions do not involve* $\exp(\pm ils)$, *the block diagonal submatrices of the associated coefficient matrix would become full matrices rather than the identity matrix.*

6.2.2 The modified Helmholtz equation

The results for the Laplace equation introduced in Section 6.2.1 are now extended to the modified Helmholtz equation.

6.2.2.1 The Dirichlet to Neumann map

The following proposition is the analogue of Proposition 6.2.1.

Proposition 6.2.7 *Let the complex-valued function $q(z,\bar{z})$ satisfy the modified Helmholtz equation in the interior of the convex polygon Ω described in Proposition 6.2.1 and let q satisfy the Dirichlet boundary conditions* (6.2.1) *on each side. Let $u_j(s)$ denote the unknown Neumann boundary data on S_j, and assume that $d_j, u_j \in C^1[-\pi,\pi]$.*

The n unknown complex-valued functions $\{u_j\}_1^n$ satisfy the following $2n$ equations for $l \in \mathbb{R}^+$ and $p = 1,\cdots,n$:

$$\int_{-\pi}^{\pi} e^{ils} u_p(s)ds = -\sum_{\substack{j=1\\ j\neq p}}^{n} E_{jp}(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l),s)u_j(s)ds - G_p(l), \tag{6.2.36a}$$

$$\int_{-\pi}^{\pi} e^{-ils} u_p(s)ds = -\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{E}_{jp}(\bar{k}_p(l)) \int_{-\pi}^{\pi} \bar{e}_j(\bar{k}_p(l),s)u_j(s)ds - \tilde{G}_p(l); \tag{6.2.36b}$$

where the known functions $E_{jp}(k)$, $e_j(k,s)$, $G_p(l)$ and $\tilde{G}_p(l)$ ($j = 1,\cdots,n$, $p = 1,\cdots,n$) are defined by

$$E_{jp}(k) = e^{-i(m_j-m_p)k+\frac{i\lambda}{k}(\bar{m}_j-\bar{m}_p)}, \quad e_j(k,s) = e^{-i\left(kh_j-\frac{\lambda\bar{h}_j}{k}\right)s}, \ k\in\mathbb{C}, \ -\pi<s<\pi, \tag{6.2.37a}$$

$$G_p(l) = \sum_{j=1}^{n} E_{jp}(k_p(l))\rho_j(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l),s)d_j(s)ds, \quad l\in\mathbb{R}^+, \tag{6.2.37b}$$

$$\tilde{G}_p(l) = \sum_{j=1}^{n} \bar{E}_{jp}(\bar{k}_p(l))\bar{\rho}_j(\bar{k}_p(l)) \int_{-\pi}^{\pi} \bar{e}_j(\bar{k}_p(l),s)d_j(s)ds, \quad l\in\mathbb{R}^+, \tag{6.2.37c}$$

and

$$\rho_j(k) = \frac{\lambda\bar{h}_j}{k} + kh_j, \quad j=1,\cdots,n, \quad k\in\mathbb{C}; \tag{6.2.37d}$$

while

$$k_p(l) = -\frac{l+\sqrt{l^2+4\lambda|h_p|^2}}{2h_p}, \quad p=1,\cdots,n, \quad l\in\mathbb{R}^+. \tag{6.2.37e}$$

Furthermore, each of the terms appearing in the four sums on the right-hand sides of equations (6.2.36) *decays exponentially as $l\to\infty$, except for the terms with $j=p$ which oscillate and for those with $j=p\pm1$ which decay linearly.*

Proof Define $\hat{q}_j(k)$ by

$$\hat{q}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz-\frac{\lambda}{ik}\bar{z}} \left[(q_z+ikq)dz - \left(q_{\bar{z}}+\frac{\lambda}{ik}q\right)d\bar{z}\right], \quad j=1,\cdots,n, \quad k\in\mathbb{C}. \tag{6.2.38}$$

Parametrizing the side S_j with respect to its midpoint m_j, see (6.2.5), and using (6.2.6) yields

$$\hat{q}_j(k) = ie^{-i\left(m_j k - \frac{\lambda \bar{m}_j}{k}\right)} \int_{-\pi}^{\pi} e_j(k,s) \left[u_j(s) + \rho_j(k) d_j(s)\right] ds, \quad j = 1, \cdots, n, \quad k \in \mathbb{C}. \tag{6.2.39}$$

Writing the first global relation (6.1.3a) in the form (6.2.8), substituting for $\{\hat{q}_j(k)\}_1^n$ from (6.2.39), multiplying by $-ie^{im_p k - \frac{i\lambda}{k}\bar{m}_p}$, and evaluating the resulting equation at $k = k_p(l)$ yields (6.2.36a). Since the second global relation (6.1.3b) can be obtained from the first (6.1.3a) by taking the Schwartz conjugate of all terms except q, equation (6.2.36b) follows from (6.2.36a).

For $j = 1, \cdots, n$ and $p = 1, \cdots, n$,

$$E_{jp}(k_p(l)) e_j(k_p(l), s) = e^{iL_p(l)\left(\frac{m_j - m_p + h_j s}{h_p}\right)} e^{-\frac{i\lambda |h_p|^2}{L_p(l)}\left(\frac{\bar{m}_j - \bar{m}_p + \bar{h}_j s}{\bar{h}_p}\right)}, \quad l \in \mathbb{R}^+, \tag{6.2.40}$$

where $L_p(l) > 0$ is defined for $p = 1, \cdots, n$ by

$$L_p(l) = -h_p k_p(l) = \frac{l + \sqrt{l^2 + 4\lambda |h_p|^2}}{2}, \quad l \in \mathbb{R}^+. \tag{6.2.41}$$

Using the estimate in equation (6.2.10) and treating the right-hand side of equation (6.2.40) in a similar way to the exponential term in (6.2.11) we find that the right-hand sides of equations (6.2.36) behave similarly to those of (6.2.2) as $l \to \infty$, where now for $j = p \pm 1$ the decay is $O(1/L_p(l))$ which equals $O(1/l)$. $\square$

Remark 6.2.8 *The reason for the choice $k_p(l)$ in (6.2.37e) is similar to that given in Remark 6.2.3. Indeed, in order to obtain the Fourier transform of $u_p(s)$, $k_p(l)$ is chosen so that*

$$e_p(k_p(l), s) \equiv \exp(ils), \quad -\pi < s < \pi. \tag{6.2.42}$$

This yields two possible choices:

$$k_p^{\pm}(l) = -\frac{l \pm \sqrt{l^2 + 4\lambda |h_p|^2}}{2h_p} \tag{6.2.43}$$

and $k_p(l) = k_p^+(l)$ is chosen so that $\arg k_p(l) = \pi - \arg h_j$.

6.2.2.2 Unknown functions that vanish at the corners

As for the Laplace equation, the unknown functions $\{\breve{u}_j\}_1^n$ are defined by (6.2.26) with $u_j(\pm\pi)$ given by (6.2.20).

The functions $\{\breve{u}_j\}_1^n$ satisfy equations similar to (6.2.36) but with G_p and $\tilde{G}_p$ replaced by $G_p + U_{\star p}$ and $\tilde{G}_p + \tilde{U}_{\star p}$ respectively, where the known functions $U_{\star p}(l)$ and $\tilde{U}_{\star p}(l)$ are defined for $p = 1, \cdots, n$ and $l \in \mathbb{R}^+$ by

$$U_{\star p}(l) = \sum_{j=1}^{n} E_{jp}(k_p(l)) \int_{-\pi}^{\pi} e_j(k_p(l), s) u_{\star j}(s) ds, \tag{6.2.44a}$$

$$\tilde{U}_{\star p}(l) = \sum_{j=1}^{n} \bar{E}_{jp}(\bar{k}_p(l)) \int_{-\pi}^{\pi} \bar{e}_j(\bar{k}_p(l), s) u_{\star p}(s) ds. \tag{6.2.44b}$$

By computing the integrals involving $u_{\star j}(s)$ using (6.2.26b) we find that for $p = 1, \cdots, n$ and $l \in \mathbb{R}^+$,

$$U_{\star p}(l) = \sum_{j=1}^{n} E_{jp}(k_p(l)) \left\{ [u_j(\pi) - u_j(-\pi)] \left[-\frac{i}{H_{jp}(l)} \cos(\pi H_{jp}(l)) + \frac{i}{\pi (H_{jp}(l))^2} \sin(\pi H_{jp}(l)) \right] \right.$$
$$\left. + \frac{1}{H_{jp}(l)} [u_j(\pi) + u_j(-\pi)] \sin(\pi H_{jp}(l)) \right\} \tag{6.2.45a}$$

and

$$\tilde{U}_{\star p}(l) = \sum_{j=1}^{n} \bar{E}_{jp}(\bar{k}_p(l)) \left\{ [u_j(\pi) - u_j(-\pi)] \left[\frac{i}{\bar{H}_{jp}(l)} \cos(\pi \bar{H}_{jp}(l)) - \frac{i}{\pi (\bar{H}_{jp}(l))^2} \sin(\pi \bar{H}_{jp}(l)) \right] \right.$$
$$\left. + \frac{1}{\bar{H}_{jp}(l)} [u_j(\pi) + u_j(-\pi)] \sin\left(\pi \bar{H}_{jp}(l)\right) \right\}, \tag{6.2.45b}$$

with

$$H_{jp}(l) = \frac{1}{2} \frac{h_j}{h_p} \left(l + \sqrt{l^2 + 4\lambda |h_p|^2} \right) - \frac{2\lambda \bar{h}_j h_p}{l + \sqrt{l^2 + 4\lambda |h_p|^2}}, \; j = 1, \cdots, n, \; p = 1, \cdots, n, \quad l \in \mathbb{R}^+. \tag{6.2.45c}$$

6.2.2.3 The Numerical Method

Proposition 6.2.9 *Let q satisfy the boundary value problem specified in Proposition 6.2.7. Assume that the values of the unknown functions $\{u_j\}_1^n$ at the corners $\{z_j\}_1^n$ are* *given by equations* (6.2.20). *Express* $\{u_j\}_1^n$ *in terms of the unknown functions* $\{\breve{u}_j\}_1^n$ *defined in equation* (6.2.26) *and approximate the latter functions by the functions* $\{\breve{u}_j^N\}_1^n$ *defined in equation* (6.2.32). *Then the constants* s_m^p *and* c_m^p, $m = 1, \cdots, N$, $p = 1, \cdots, n$ *satisfy the* $2Nn$ *algebraic equations*

$$\begin{aligned}2\pi s_m^p =& i \sum_{\substack{j=1\\ j\neq p}}^{n} \left\{ \sum_{r=1}^{N} s_r^j \left[E_{jp}(k_p(m)) S_{jp}^r(m) - \bar{E}_{jp}(\bar{k}_p(m)) \bar{S}_{jp}^r(m) \right] \right. \\ & \left. + \sum_{r=1}^{N} c_r^j \left[E_{jp}(k_p(m)) C_{jp}^r(m) - \bar{E}_{jp}(\bar{k}_p(m) \bar{C}_{jp}^r(m) \right] \right\} \\ & + iG_p(m) - i\tilde{G}_p(m) + iU_{\star p}(m) - i\tilde{U}_{\star p}(m) \end{aligned} \tag{6.2.46a}$$

and

$$\begin{aligned}&2\pi c_m^p \\ &= -\sum_{\substack{j=1\\ j\neq p}}^{n} \left\{ \sum_{r=1}^{N} s_r^j \left[E_{jp}\left(k_p\left(m-\frac{1}{2}\right)\right) S_{jp}^r\left(m-\frac{1}{2}\right) + \bar{E}_{jp}\left(\bar{k}_p\left(m-\frac{1}{2}\right)\right) \bar{S}_{jp}^r\left(m-\frac{1}{2}\right) \right] \right. \\ &\quad \left. + \sum_{r=1}^{N} c_r^j \left[E_{jp}\left(k_p\left(m-\frac{1}{2}\right)\right) C_{jp}^r\left(m-\frac{1}{2}\right) + \bar{E}_{jp}\left(\bar{k}_p\left(m-\frac{1}{2}\right)\right) \bar{C}_{jp}^r\left(m-\frac{1}{2}\right) \right] \right\} \\ &- G_p\left(m-\frac{1}{2}\right) - \tilde{G}_p\left(m-\frac{1}{2}\right) - U_{\star p}\left(m-\frac{1}{2}\right) - \tilde{U}_{\star p}\left(m-\frac{1}{2}\right), \end{aligned} \tag{6.2.46b}$$

where the known functions G_p, $\tilde{G}_p$, $U_{\star p}$, $\tilde{U}_{\star p}$ *are defined by equations* (6.2.37) *and* (6.2.45), *and*

$$S_{jp}^r(m) = \frac{2ir(-1)^{r-1}\sin\left(\pi H_{jp}(m)\right)}{r^2 - \left(H_{jp}(m)\right)^2}, \quad C_{jp}^r(m) = \frac{2\left(r-\frac{1}{2}\right)(-1)^{r-1}\cos\left(\pi H_{jp}(m)\right)}{\left(r-\frac{1}{2}\right)^2 - \left(H_{jp}(m)\right)^2},$$
$$j = 1, \cdots, n, \quad p = 1, \cdots, n, \quad r = 1, \cdots, n, \quad m = 1, \cdots, N. \tag{6.2.46c}$$

Proof The proof is similar to that of Proposition 6.2.5. □

6.3 Numerical results

In order to illustrate the numerical implementation of the new collocation method for the Laplace and modified Helmholtz equations, we will consider a variety of regular and irregular polygons.

We will study the Laplace equation with the exact solution

$$q(z,\bar{z}) = e^{3z} + 2e^{3\bar{z}}, \tag{6.3.1}$$

the modified Helmholtz equation with $\lambda = 100$ and the exact solution

$$q(z,\bar{z}) = e^{11z+\frac{100}{11}\bar{z}}, \tag{6.3.2}$$

and the modified Helmholtz equation with $\lambda = 5$ and the exact solution

$$q(z,\bar{z}) = e^{z+5\bar{z}}, \tag{6.3.3}$$

Analytic expressions for the known boundary functions $\{d_j(s)\}_{j=1}^n$, and the unknown boundary data $\{u_j(s)\}_{j=1}^n$ can be computed easily from (6.3.1), (6.3.2) and (6.3.3).

To demonstrate the performance of the method, we consider the discrete maximum relative error

$$E_\infty := \frac{||u - u^N||_\infty}{||u||_\infty}, \tag{6.3.4}$$

where

$$u_j^N(s) = \check{u}_j^N(s) + u_{*j}(s), \quad -\pi \le s \le \pi, \quad j = 1, \cdots, n, \tag{6.3.5}$$

$$||u||_\infty = \max_{1\le j\le n}\left\{\max_{s\in S}|u_j(s)|\right\}. \tag{6.3.6}$$

We consider 10001 evenly spaced points

$$S = \{s_i\}_{i=1}^{10001} \subset [-\pi,\, \pi]\,, \tag{6.3.7}$$

with the points s_i, $-\pi = s_1 < s_2 < \ldots < s_{10000} < s_{10001} = \pi$, given by

$$s_i = \pi\left[-1 + \frac{2(i-1)}{10000}\right], \quad 1 \le i \le 10001. \tag{6.3.8}$$

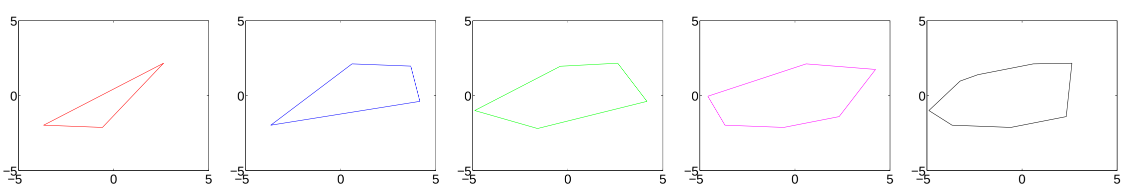
We consider regular polygons with $n = 3,\, 4,\, 5,\, 6,\, 8$ sides whose vertices lie on the circle centered at the origin with radius $\sqrt{2}$ in the complex plane (with a vertex on the positive real axis). These polygons are then rotated by an angle of $-\frac{1}{5}$ about the origin to avoid certain non-generic results that occur when the polygons are aligned with the coordinate axes. Thus, we consider the polygons with vertices

$$z(j) = \sqrt{2}e^{i\left[2(j-1)\frac{\pi}{n}-\frac{1}{5}\right]}, \quad j = 1, \cdots, n. \tag{6.3.9}$$

We also consider irregular polygons with $n = 3,\ 4,\ 5,\ 6,\ 8$ sides whose vertices lie on the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ in the complex plane rotated through an angle of $\frac{1}{5}$ about the origin. The x and y coordinates of the vertices of the polygons before rotation are given (in an anticlockwise direction) in Table 6.1 and the polygons are shown in Figure 6.3.

Table 6.1: Vertices of Irregular Polygons Prior to Rotation

Triangle	$\left(-4, -\frac{6}{5}\right)$, $\left(-1, -\frac{2\sqrt{24}}{25}\right)$, $\left(3, \frac{8}{5}\right)$
Square	$\left(1, \frac{2\sqrt{24}}{25}\right)$, $\left(-4, -\frac{6}{5}\right)$, $\left(4, -\frac{6}{5}\right)$, $\left(4, \frac{6}{5}\right)$
Pentagon	$(0, 2)$, $(-5, 0)$, $\left(-2, -\frac{2\sqrt{21}}{25}\right)$, $\left(4, -\frac{6}{5}\right)$, $\left(3, \frac{8}{5}\right)$
Hexagon	$\left(1, \frac{2\sqrt{24}}{25}\right)$, $\left(-\frac{9}{2}, \frac{2\sqrt{19}}{10}\right)$, $\left(-4, -\frac{6}{5}\right)$, $\left(-1, -\frac{2\sqrt{24}}{25}\right)$, $\left(2, -\frac{2\sqrt{21}}{25}\right)$, $\left(\frac{9}{2}, \frac{2\sqrt{19}}{10}\right)$
Octagon	$\left(1, \frac{2\sqrt{24}}{25}\right)$, $\left(-2, \frac{2\sqrt{21}}{25}\right)$, $\left(-3, \frac{8}{5}\right)$, $(-5, 0)$, $\left(-4, -\frac{6}{5}\right)$, $\left(-1, -\frac{2\sqrt{24}}{25}\right)$, $\left(2, -\frac{2\sqrt{21}}{25}\right)$, $\left(3, \frac{8}{5}\right)$



$n = 3$ $\qquad$ $n = 4$ $\qquad$ $n = 5$ $\qquad$ $n = 6$ $\qquad$ $n = 8$

Figure 6.3: Irregular Polygons

For the four different cases below, the following are displayed:

- graphs of E_∞ against N for the regular and irregular polygons;
- graphs of the condition number against N for the regular and irregular polygons;
- A table estimating the order of convergence (O.o.C.) for various values of N for the equilateral and the irregular triangle.

The cases considered are:

1. Fourier basis, the Laplace equation, Figures 6.4 & 6.5, table 6.2;
2. Fourier basis, the modified Helmholtz equation with $\lambda = 100$, Figures 6.6 & 6.7, table 6.3;
3. Chebyshev basis, the modified Helmholtz equation with $\lambda = 100$, Figures 6.8 & 6.9, table 6.4;
4. Chebyshev basis, the modified Helmholtz equation with $\lambda = 5$, Figures 6.10 & 6.11, table 6.5.

In the examples using the Chebyshev basis, the global relations are evaluated at the same points as for the Fourier basis, and the basis functions are constructed so that they vanish at the endpoints of the interval, see also [SFFS08, Section 4].

The graphs show that for the Fourier basis,

- the error lines appear to be asymptotically parallel to the $\frac{1}{N^2}$ line, meaning that the method has inherited the order of convergence of the underlying Fourier-sine series expansion (6.2.31);
- the condition numbers of the associated matrices are small and grow approximately linearly with N;

and for the Chebyshev basis,

- for both the $\lambda = 100$ and $\lambda = 5$ cases the convergence appears to be exponential, but the errors are much smaller for $\lambda = 5$;
- the conditions numbers for both cases are comparable and appear to grow exponentially with N.

6.4 Conclusions

A spectral collocation method for computing the Dirichlet to Neumann map has been implemented for the Laplace and modified Helmholtz equations in the interior of a convex polygon. This method has its origin in [FFX04]; however, although the values of k were chosen in [FFX04] to be those in (6.2.9), both the difference and the sum of the global relations were evaluated at $l = m$ instead of at $l = m$ and $l = m - \frac{1}{2}$ respectively. As a result, the relevant coefficient matrix possessed a large condition number and numerical computations performed in [FFX04] suggested linear convergence. In [SFFS08], computing the sum of the global relations at $l = m - \frac{1}{2}$ led to a coefficient matrix with a small condition number. Also, the numerical computations in [SFFS08] suggested quadratic convergence for the modified sine-Fourier series (6.2.32).

The method presented here differs from [SFFS08] in the following respects:

1. the solution q is allowed to be complex valued;
2. the Dirichlet problem for the Laplace equation has been solved by employing the global relation (6.1.5) instead of the global relation (6.1.7) (in (6.1.7), one must take a derivative of the given Dirichlet data in order to solve the Dirichlet problem);
3. the method has been implemented for the modified Helmholtz equation.

In addition, the reasons for choosing the particular collocation points $k_p(l)$ for the Fourier basis in (6.2.37e) have been further clarified. Because of the non-orthogonality of the Chebyshev polynomials to the exponentials in (6.2.2) and (6.2.36), it is not clear what the optimal choice of collocation points in the complex k-plane is for a Chebyshev basis. For the computations presented here we have used the same points as those used for the Fourier basis; whether this choice can be improved is work in progress.

The method presented in this chapter requires that the given boundary conditions are sufficiently compatible at the corners of the polygon so the solution has a continuous first derivative. This requirement makes it possible to compute the values of the unknown Neumann data at the corners of the polygon; then, by subtracting these known values, one obtains unknown functions that vanish at the boundary. If these unknown functions are sufficiently smooth, then a Chebyshev basis gives exponential convergence, although the condition number of the matrix appears to grow exponentially with the number of basis elements. On the other hand if the unknown functions do not have sufficient smoothness, the Fourier sine basis can be used, which gives second order convergence, and which has the advantage that the relevant condition number appears to grow only linearly.

If the boundary conditions are not sufficiently compatible then the solution does has have a continuous first derivative. In this case one cannot subtract the contribution from the corners, so that the method presented in this chapter cannot be applied. However it is still possible to use a slight variation of our approach, namely to use the Fourier cosine basis instead of the Fourier sine basis.

6.4.1 Discussion in the context of other methods

Owing to its immense applicability, there has been wide and continued interest in designing and analysing numerical methods for solving the PDE (1.1.1). As discussed in Section 1.2.5, for the Poisson equation in two dimensions, conformal mapping is an extremely powerful and widely applicable method, see [DT02] in particular for the state of the art implementation in polygonal domains using the Schwarz–Christoffel transformation. Finite element and boundary element methods are perhaps the most popular choices for solving (1.1.1) in general, since, unlike conformal mapping, they are applicable in both two and three dimensions. A particularly noteworthy addition to the boundary element method (and a milestone in computational mathematics in its own right) is the Fast Multipole Method [GR87]. This algorithm accelerates the numerical solution of boundary integral equations, see [Rok85, Rok90], but can also be used whenever the effects of groups of sources must be calculated at groups of observation points.

Much more work must be done on the numerical method described in this Chapter, including a detailed comparison with the existing methods described above, before its potential utility as a numerical method for solving (1.1.1) can be properly assessed.

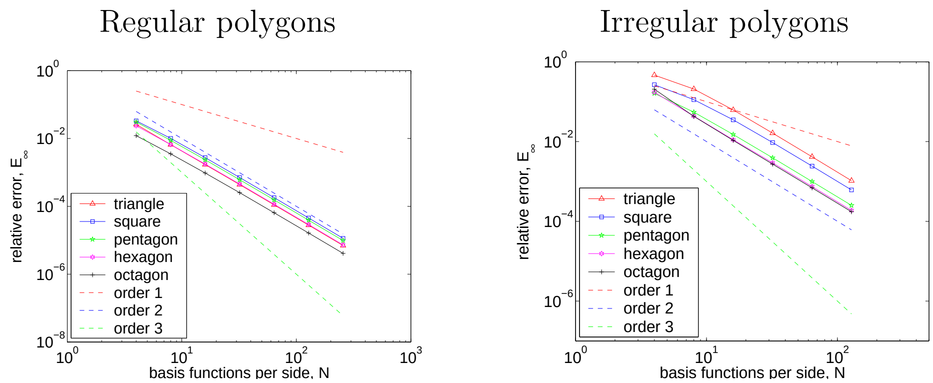


Figure 6.4: Fourier basis, the Laplace equation, E_∞ as a Function of N

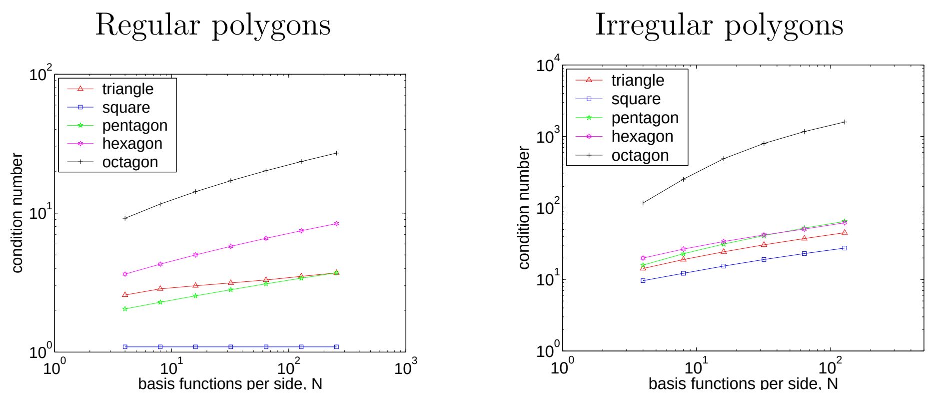


Figure 6.5: Fourier basis, the Laplace equation, the condition number of the coefficient matrix as a function of N

Equilateral Triangle

N	E_∞	O.o.c.
4	2.6019e-02	–
8	6.5908e-03	1.9811
16	1.7026e-03	1.9527
32	4.3592e-04	1.9656
64	1.1048e-04	1.9802
128	2.7823e-05	1.9895

Irregular Triangle

N	E_∞	O.o.c.
4	4.6405e-01	–
8	2.0730e-01	1.1626
16	6.2140e-02	1.7381
32	1.6293e-02	1.9313
64	4.1470e-03	1.9741
128	1.0453e-03	1.9881

Table 6.2: Fourier basis, order of convergence (o.o.c.) for the Laplace Equation in the Triangles

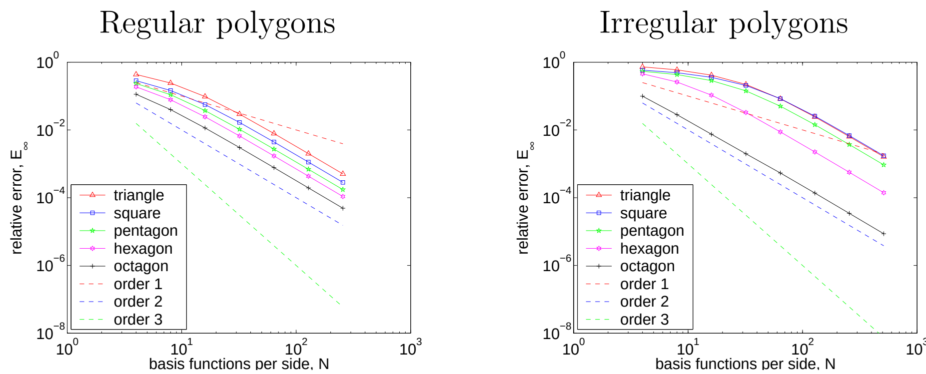


Figure 6.6: Fourier basis, the modified Helmholtz equation with $\lambda = 100$, E_∞ as a function of N

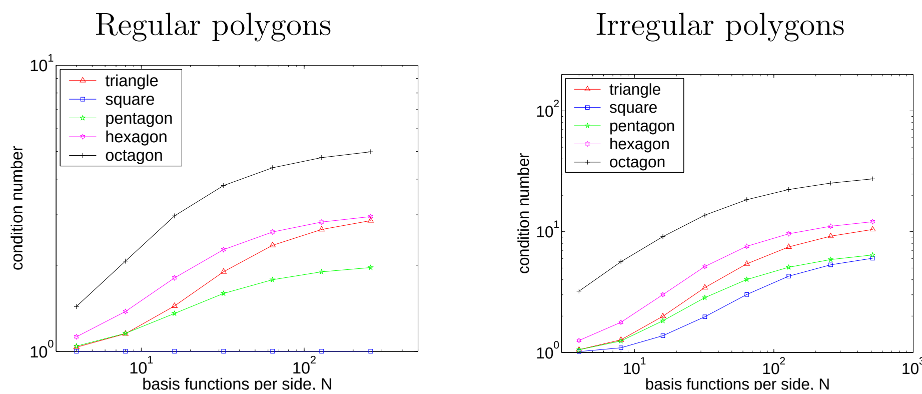


Figure 6.7: Fourier basis, the modified Helmholtz equation with $\lambda = 100$, the condition number of the coefficient matrix as a function of N

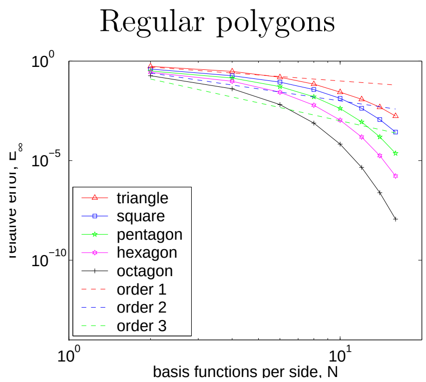
Equilateral Triangle

N	E_∞	O.o.c.
4	4.3435e-01	–
8	2.4280e-01	0.8391
16	9.6685e-02	1.3284
32	2.9334e-02	1.7207
64	7.8274e-03	1.9060
128	1.9983e-03	1.9697
256	5.0311e-04	1.9898

Irregular Triangle

N	E_∞	O.o.c.
4	7.2627e-01	–
8	5.9449e-01	0.2889
16	4.1420e-01	0.5213
32	2.2008e-01	0.9123
64	8.3167e-02	1.4039
128	2.4321e-02	1.7738
256	6.3678e-03	1.9334
512	1.6117e-03	1.9822

Table 6.3: Fourier basis, order of convergence (o.o.c.) for the modified Helmholtz equation in the triangles



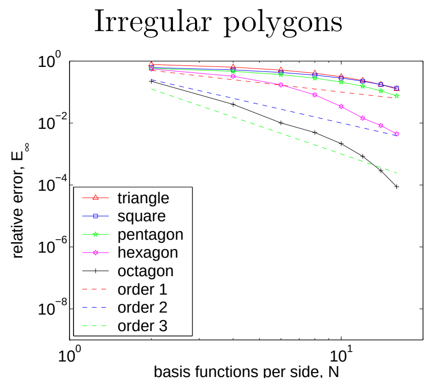
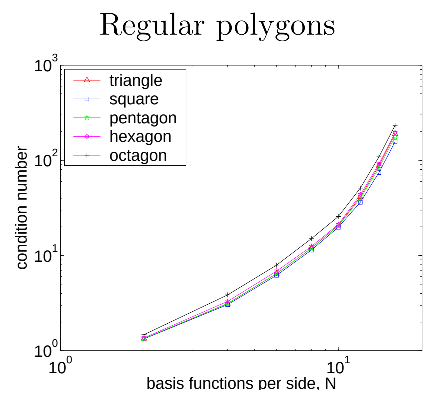


Figure 6.8: Chebyshev basis, the modified Helmholtz equation with $\lambda = 100$, E_∞ as a function of N



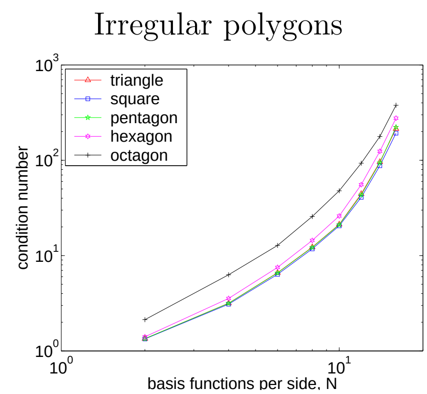


Figure 6.9: Chebyshev basis, the modified Helmholtz equation with $\lambda = 100$, the condition number of the coefficient matrix as a function of N

Equilateral Triangle

N	E_∞	O.o.c.
2	5.4095e-01	–
4	3.0398e-01	0.8315
6	1.5421e-01	0.9791
8	6.9589e-02	1.1480
10	2.7604e-02	1.3340
12	1.2099e-02	1.1900
14	4.8311e-03	1.3245
16	1.7322e-03	1.4797

Irregular Triangle

N	E_∞	O.o.c.
2	7.7750e-01	–
4	6.3583e-01	0.2902
6	5.1146e-01	0.3140
8	4.0402e-01	0.3402
10	3.1263e-01	0.3700
12	2.3646e-01	0.4028
14	1.7451e-01	0.4383
16	1.2543e-01	0.4764

Table 6.4: Chebychev basis, $\lambda = 100$, order of convergence (o.o.c.) for the modified Helmholtz equation in the triangles

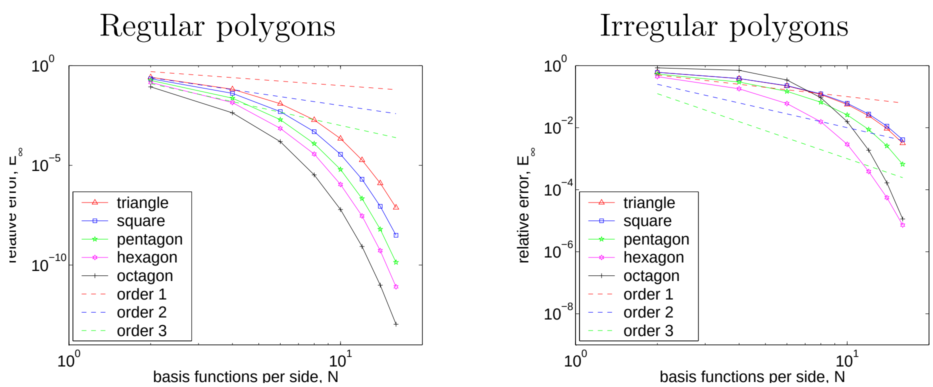


Figure 6.10: Chebyshev basis, the modified Helmholtz equation with $\lambda = 5$, E_∞ as a function of N

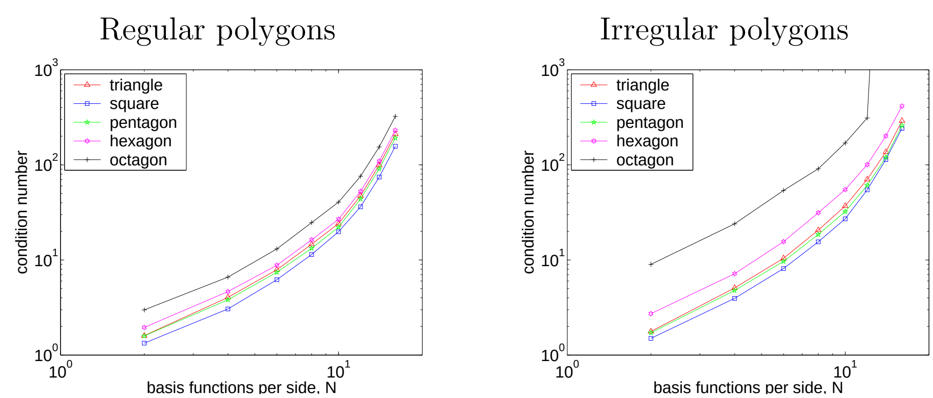


Figure 6.11: Chebyshev basis, the modified Helmholtz equation with $\lambda = 5$, the condition number of the coefficient matrix as a function of N

Equilateral Triangle

N	E_∞	O.o.c.
2	2.6324e-01	–
4	6.5756e-02	2.0012
6	1.2393e-02	2.4076
8	1.9007e-03	2.7049
10	2.1928e-04	3.1157
12	1.8413e-05	3.5739
14	1.2662e-06	3.8622
16	7.6152e-08	4.0554

Irregular Triangle

N	E_∞	O.o.c.
2	6.1165e-01	–
4	3.8320e-01	0.6746
6	2.2207e-01	0.7871
8	1.1717e-01	0.9224
10	5.6056e-02	1.0636
12	2.4242e-02	1.2094
14	9.3544e-03	1.3738
16	3.1766e-03	1.5581

Table 6.5: Chebychev basis, $\lambda = 5$, order of convergence (o.o.c.) for the modified Helmholtz equation in the triangles

Chapter 7

Future work

The most outstanding open problems to be tackled with the Fokas method applied to the PDE (1.1.1) are the following.

Regarding the analytical method (Chapters 2–5):

- **Solution in the interior of more complicated polygons.** Although Corollary 3.1.13 and Remark 3.1.15 indicate this may not be possible, if a domain has symmetry this property can be used to transform a boundary value problem into certain boundary value problems in subdomains. The price one pays is that the boundary value problems in the subdomains have more complicated boundary conditions. It is not yet clear whether these boundary value problems in the subdomains can be solved by the Fokas method.

- **Solution in the exterior of polygons.** This is substantially more difficult than the interior due to the fact that we only have the GR in subdomains of $\Omega^{(e)}$ which are convex at infinity (by Corollary 3.1.7).

- The investigation of **corner singularities**. Preliminary investigation has shown that the large k asymptotics of the GR yields information about the local behaviour of the solution at the corners.

- The study of both the **Wiener–Hopf technique**, a powerful method for solving boundary value problems with mixed boundary conditions [Nob88], and the **Sommerfeld-Malyuzhinets technique** [BLG08] in the context of the Fokas method. A first step would be careful study of the works of Shanin [Sha97], [Sha00], who treated the GR as a functional equation of the Malyuzhinets type (see Remark 1.6.1).

- The extension of the method to **three dimensions**; the first step should be to find the analogues of the coordinate dependent fundamental solutions in §2.1.1 and §2.2.1 in 3-d. We emphasise that, just like the classical transform method, the new method is applicable in three dimensions. For example, it has been applied to evolution PDEs in two space dimensions in [Fok02] and [KF10]. The GR now contains two complex variables; however these variables are not coupled, and this avoids the subtleties of the theory of several complex variables.

Regarding the numerical method of Chapter 6

- The extension of the method of Chapter 6 to the **Helmholtz** equation. Now the appropriate collocation points in the complex k-plane lie on circular arcs as well as rays.

- **The evaluation of the solution in the interior of the domain**, using the integral representations of Chapter 2, after the Dirichlet to Neumann map has been computed by the method of Chapter 6.

- The investigation of BVPs involving **corner singularities**.

- The design of a method to compute the Dirichlet to Neumann map for the **exterior of a polygon**; a first attempt at this could proceed by splitting $\Omega^{(e)}$ into subdomains.

- The coupling of the method with domain decomposition methods.

- The investigation of the use of polyharmonic basis functions [IN06], instead of trigonometric basis functions.

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