Coercivity of Combined Boundary Integral Equations in High-Frequency Scattering

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Dedicated to Cathleen S. Morawetz on the occasion of her 90th birthday

Abstract

We prove that the standard second-kind integral equation formulation of the exterior Dirichlet problem for the Helmholtz equation is coercive (i.e. sign-definite) for all smooth convex domains when the wavenumber k is sufficient large. (This integral equation involves the so-called "combined potential" or "combined field" operator.) This coercivity result yields k-explicit error estimates when the integral equation is solved using the Galerkin method, regardless of the particular approximation space used (and thus these error estimates apply to several hybrid numerical-asymptotic methods developed recently). Coercivity also gives k-explicit bounds on the number of GMRES iterations needed to achieve a prescribed accuracy when the integral equation is solved using the Galerkin method with standard piecewise-polynomial subspaces. The coercivity result is obtained by using identities for the Helmholtz equation originally introduced by Morawetz in her work on the local energy decay of solutions to the wave equation.

1 Introduction

The Helmholtz equation,

$$\Delta u + k^2 u = 0,$$

with wavenumber $k > 0$, posed in the domain exterior to a bounded obstacle is arguably the simplest possible model of wave scattering, and thus has been the subject of vast amounts of research.

On the one hand, much effort has gone into constructing the asymptotics as $k \to \infty$ of solutions of the Helmholtz equation in exterior domains using Geometrical Optics and Keller's Geometrical Theory of Diffraction, and then proving error bounds that justify these asymptotics (often via proving bounds on the inverse of the Helmholtz operator).

On the other hand, much research effort has gone into solving the Helmholtz equation numerically. For example, one popular method is the finite element method, which is based on the weak form of the PDE. Alternatively, if the wavenumber k is constant, then an explicit expression for the fundamental solution of the Helmholtz equation is available, and this allows the problem of finding u in the exterior domain to be reduced to solving an integral equation on the boundary of the obstacle (the so-called boundary integral method). The resulting integral equation can then be solved numerically in a variety of ways (e.g. using Galerkin, collocation, or Nyström methods).

Over the last two decades, there has been a lot of interest in

(a) determining how the conventional numerical methods for solving the Helmholtz equation behave as k increases, and

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(b) designing new methods that perform better as k increases than the conventional ones.

Regarding (a): the standard numerical analysis approach to numerical methods for the Helmholtz equation is to prove results about the convergence and conditioning of the methods as the number of degrees of freedom, N, increases with k fixed. In particular, the constants in the classical error estimates as $N \to \infty$ are *not* explicit in k. More recent work has sought to determine the dependence of these constants on k, and, more generally, determine how these methods perform as k increases with N either fixed or a prescribed function of k (see, e.g., the recent review articles [22], [10, §5, §6]).

Regarding (b): the engineering rule of thumb is that conventional numerical methods for the Helmholtz equation need a fixed number of degrees of freedom per wavelength to maintain accuracy as k increases (see, e.g., [38]); therefore, if the problem is d-dimensional, N must grow like k^d as $k \to \infty$ for volume discretisations and like k^{d-1} as $k \to \infty$ for discretisations on the boundary of the domain. (The investigations addressing (a) discussed above actually show that in some cases N must increase faster than k^d to maintain accuracy due to the so-called *pollution effect*; see, e.g., [1], [22].) This growth of N with k puts many high-frequency problems out of the range of standard numerical methods, and thus there has been much recent interest in designing methods that reduce this growth (see, e.g, [10, §1] and the references therein).

The classical results about the asymptotics of solutions to the Helmholtz equation and the associated bounds on the inverse of the Helmholtz operator have played an essential role in many of the attempts at tackling one or other of the tasks (a) and (b) above. Indeed, very roughly speaking, the knowledge of the large k asymptotics of the Helmholtz equation can be used for task (b) (leading to so-called *hybrid numerical-asymptotic methods*), and knowledge of the bounds on the inverse of the Helmholtz operator can be used for task (a) (for more details see [10, §3] and [10, §5, §6] respectively).

This paper considers a standard integral operator associated with the Helmholtz equation posed in the exterior of a bounded obstacle with Dirichlet boundary conditions (physically this corresponds to sound-soft acoustic scattering), and seeks to address a question that is relevant to both tasks (a) and (b) above. This oscillatory integral operator is often called the "combined potential operator" or "combined field operator", and we denote it by $A'_{k,\eta}$, where k is the wavenumber and η is a parameter that is usually chosen to be proportional to k (we define $A'_{k,\eta}$ and derive the associated integral equation in §1.1 below). We seek to prove that $A'_{k,\eta}$ is coercive as an operator on $L^2(\Gamma)$, where Γ is the boundary of the obstacle, i.e. that there exists an $\alpha_{k,\eta} > 0$ such that

$$\left|(A'_{k,\eta}\phi, \phi)_{L^2(\Gamma)}\right| \geq \alpha_{k,\eta} \left\|\phi\right\|^2_{L^2(\Gamma)} \qquad \text{for all } \phi \in L^2(\Gamma), \tag{1.1}$$

at least for k sufficiently large. The k- and η-subscripts on the coercivity constant, $\alpha_{k,\eta}$, indicate that, if this constant exists, it might depend on k and η; we see below that in some cases it can be independent of both. (Note that if $A'_{k,\eta}$ is coercive then the Lax-Milgram theorem implies that $A'_{k,\eta}$ is invertible with $\|(A'_{k,\eta})^{-1}\| \leq 1/\alpha_{k,\eta}$, but the converse is not true; i.e. $A'_{k,\eta}$ can be invertible but not coercive.) Establishing coercivity has two important consequences:

(i) It allows one to prove k-explicit error estimates when the integral equation involving $A'_{k,\eta}$ is solved numerically using the Galerkin method *both* for conventional methods (which use approximation spaces consisting of piecewise polynomials) *and* for hybrid numerical-asymptotic methods (where the approximation space is designed using knowledge of the large k asymptotics). Note that establishing coercivity is currently the only known way to prove these error estimates for the hybrid methods.

(ii) It proves that the numerical range (also known as the field of values) of the operator is bounded away from zero. This fact, along with a k-explicit bound on $\|A'_{k,\eta}\|$, then gives a k-explicit bound on the number of GMRES iterations needed to achieve a prescribed accuracy when the integral equation is solved using the Galerkin method with piecewise-polynomial approximation spaces (and where GMRES is the *generalised minimal residual method*). Note that no such bounds are currently available in the literature.

(Both of these consequences of coercivity are discussed in more detail in §1.3 below.)

Although it is well-known that $A'_{k,\eta}$ is invertible for every $k > 0$, it is not a priori clear that $A'_{k,\eta}$ will be coercive. Indeed, the usual numerical analysis of Helmholtz problems (posed either in the domain or on the boundary) seeks to prove that the relevant operator is coercive up to a compact perturbation (i.e. satisfies a Gårding inequality). However, in [19] $A'_{k,\eta}$ was proved to be coercive, with $\alpha_{k,\eta}$ independent of k, when Γ is the circle (in 2-d) or the sphere (in 3-d), $\eta = k$, and k is sufficiently large. Furthermore, numerical experiments conducted in [5] suggest that $A'_{k,\eta}$ is coercive for a wide variety of 2-d domains, with $\alpha_{k,\eta}$ independent of k, when $\eta = k$ and k is sufficiently large (in particular, domains that are *nontrapping*).

When considering the question of whether or not $A'_{k,\eta}$ is coercive, it is instructive to also consider two other questions about $A'_{k,\eta}$, namely, how do $\|A'_{k,\eta}\|$ and $\|(A'_{k,\eta})^{-1}\|$ depend on k (where $\|\cdot\|$ denotes the operator norm on $L^2(\Gamma)$)? Bounds on $\|A'_{k,\eta}\|$ that are sharp in their k-dependence for a wide variety of domains can be obtained just by using general techniques for bounding the norms of oscillatory integral operators; see [9], [10, §5.5], [56, §1.2]. In contrast, to obtain k-explicit bounds on $\|(A'_{k,\eta})^{-1}\|$ it is necessary to use the fact that $A'_{k,\eta}$ arises from solving boundary value problems (BVPs) for the Helmholtz equation, and convert the problem of finding a bound on $\|(A'_{k,\eta})^{-1}\|$ into bounding the exterior Dirichlet-to-Neumann map and the interior impedance-to-Dirichlet map for the Helmholtz equation. (Although it might seem strange that $\|(A'_{k,\eta})^{-1}\|$ depends on the solution operator to the interior impedance problem, it turns out that this interior problem can also be formulated as an integral equation involving $A'_{k,\eta}$, thus this dependence is natural.) Appropriate bounds on these interior and exterior Helmholtz problems can then be used to bound $\|(A'_{k,\eta})^{-1}\|$; see [10, Theorem 2.33 and §5.6.1], [11], [57, §1.3].

In constrast to the task of bounding $\|(A'_{k,\eta})^{-1}\|$, the task of proving that $A'_{k,\eta}$ is coercive apparently *cannot* be reformulated in terms of bounding the solutions of Helmholtz BVPs. In this paper, however, we show that this task can be tackled using identities for solutions of the Helmholtz equation originally introduced by Morawetz. (This builds on the earlier work of two of the authors and their collaborators in [58].) Recall that Morawetz showed in [46] that bounding the solution of the exterior Dirichlet problem could be converted (via her identities) into constructing an appropriate vector field in the exterior of the obstacle, and then such a vector field was constructed by Morawetz, Ralston, and Strauss for 2-d nontrapping domains in [48, §4]. (This bound on the solution is equivalent to bounding the exterior Dirichlet-to-Neumann map, and can also be used to show local energy decay of solutions of the wave equation.)

Here we convert the problem of proving that $A'_{k,\eta}$ is coercive into that of constructing a suitable vector field in *both* the exterior *and* the interior of the obstacle. In addition to needing a vector field in the interior as well as the exterior, the conditions that the vector field must satisfy for coercivity are stronger than those in [46] and [48]. Indeed, we prove that the conditions for coercivity cannot be satisfied if the obstacle is nonconvex, and then we construct a vector field satisfying these conditions for smooth, convex obstacles with strictly positive curvature in both 2- and 3-d.

1.1 Formulation of the problem

Let $\Omega_- \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded Lipschitz open set such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. In what follows we use *domain* to mean a connected open set, and thus Ω_+ is a Lipschitz domain. Let $\Gamma := \partial\Omega_-$ (so $\Gamma = \partial\Omega_+$ too). Let $H^1_{\rm loc}(\Omega_+)$ denote the set of functions, v, such that v is locally integrable on Ω_+ and $\psi v \in H^1(\Omega_+)$ for every compactly supported $\psi \in C^\infty(\overline{\Omega_+}) := \{\psi|_{\overline{\Omega_+}} : \psi \in C^\infty(\mathbb{R}^d)\}$.

Definition 1.1 (Sound-soft scattering problem) *Given $k > 0$ and an incident plane wave $u^I(\mathbf{x}) = \exp({\rm i}k\mathbf{x}\cdot\widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $|\widehat{\mathbf{a}}| = 1$, find $u^S \in C^2(\Omega_+) \cap H^1_{\rm loc}(\Omega_+)$ such that the total field $u := u^I + u^S$ satisfies*

$$\begin{aligned} \mathcal{L}u := \Delta u + k^2 u = 0 \quad &\text{in } \Omega_+, \\ u = 0 \quad &\text{on } \Gamma, \end{aligned}$$

and u^S satisfies the Sommerfeld radiation condition,

$$\frac{\partial u^S}{\partial r}(\mathbf{x}) - {\rm i}k\, u^S(\mathbf{x}) = o\left(\frac{1}{r^{(d-1)/2}}\right) \tag{1.2}$$

as $r := |\mathbf{x}| \to \infty$, uniformly in $\widehat{\mathbf{x}} := \mathbf{x}/r$.

It is well-known that the solution to this problem exists and is unique; see, e.g., [10, Theorem 2.12].

Note that, although we are restricting our attention to the case where the incident field is a plane wave, the results of this paper also apply to scattering by other incident fields, for example those satisfying [10, Definition 2.11], and also to the general exterior Dirichlet problem, i.e. given a function g_D on Γ (with suitable regularity), find u^S satisfying both the Helmholtz equation in Ω_+ and the Sommerfeld radiation condition, and also such that $u^S = g_D$ on Γ.

The BVP in Definition 1.1 can be reformulated as an integral equation on Γ in two different ways. The first, the so-called *direct* method, uses Green's integral representation for the solution u, i.e.

$$u(\mathbf{x}) = u^I(\mathbf{x}) - \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y}) \frac{\partial u}{\partial n}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Omega_+, \tag{1.3}$$

where $\partial/\partial n$ is the derivative in the normal direction, with the unit normal $\mathbf{n}$ directed into Ω_+, and $\Phi_k(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4} H_0^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big),\ d=2, \qquad \Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},\ d=3$$

(note that to obtain (1.3) from the usual form of Green's integral representation one must use the fact that u^I is a solution of the Helmholtz equation in Ω_-; see, e.g., [10, Theorem 2.43]).

Taking the Dirichlet and Neumann traces of (1.3) on Γ one obtains two integral equations for the unknown Neumann boundary value $\partial u/\partial n$:

$$S_k \frac{\partial u}{\partial n} = u^I, \tag{1.4}$$

$$\left(\frac{1}{2}I + D_k'\right)\frac{\partial u}{\partial n} = \frac{\partial u^I}{\partial n}, \tag{1.5}$$

where the integral operators S_k and D_k', the single-layer operator and its normal derivative respectively, are defined for $\psi \in L^2(\Gamma)$ by

$$S_k\psi(\mathbf{x}) = \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{1.6}$$

$$D_k'\psi(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma. \tag{1.7}$$

Both integral equations (1.4) and (1.5) fail to be uniquely solvable for certain values of k (for (1.4) these are the k such that k^2 is a Dirichlet eigenvalue of the Laplacian in Ω_-, and for (1.5) these are the k such that k^2 is a Neumann eigenvalue). The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation

$$A_{k,\eta}' \frac{\partial u}{\partial n} = f, \tag{1.8}$$

where

$$A_{k,\eta}' := \frac{1}{2}I + D_k' - \mathrm{i}\eta S_k \tag{1.9}$$

is the *combined potential* or *combined field* operator, with $\eta \in \mathbb{R}\setminus\{0\}$ the so-called coupling parameter, and

$$f(\mathbf{x}) = \frac{\partial u^I}{\partial n}(\mathbf{x}) - \mathrm{i}\eta u^I(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

When Ω_- is Lipschitz, standard trace results imply that the unknown Neumann boundary value $\partial u/\partial n$ is in $H^{-1/2}(\Gamma)$. When Ω_- is C^2, elliptic regularity implies that $\partial u/\partial n \in L^2(\Gamma)$ (see, e.g., [23, §6.3.2, Theorem 4]), but this is true even when Ω_- is Lipschitz via a regularity result of Nečas [49, §5.1.2], [40, Theorem 4.24 (ii)]. Therefore, even for Lipschitz Ω_- we can consider the integral equation (1.8) as an operator equation in $L^2(\Gamma)$, which is a natural space for the practical solution of second-kind integral equations since it is self-dual. It is well-known that, when $\eta \neq 0$, $A'_{k,\eta}$ is a bounded and invertible operator on $L^2(\Gamma)$ (see [10, Theorem 2.27]).

Instead of using Green's integral representation to formulate the BVP as an integral equation, one can pose the ansatz

$$u^S(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \mathrm{i}\eta \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

with the sought density $\phi \in L^2(\Gamma)$ and $\eta \in \mathbb{R}\setminus\{0\}$; this is the so-called *indirect* method. Imposing the boundary condition $u^S = -u^I$ on Γ leads to the integral equation

$$A_{k,\eta}\phi = -u^I, \tag{1.10}$$

where

$$A_{k,\eta} := \frac{1}{2}I + D_k - \mathrm{i}\eta S_k$$

and D_k is the double-layer operator, which is defined for $\psi \in L^2(\Gamma)$ by

$$D_k\psi(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

The operators $A_{k,\eta}$ and $A'_{k,\eta}$ are adjoint with respect to the real-valued $L^2(\Gamma)$ inner product, and then it is straightforward to show that, firstly, $\|A_{k,\eta}\| = \|A'_{k,\eta}\|$ and $\|(A_{k,\eta})^{-1}\| = \|(A'_{k,\eta})^{-1}\|$, where $\|\cdot\|$ denotes the operator norm from (complex-valued) $L^2(\Gamma)$ to itself, and, secondly, if one of $A_{k,\eta}$ or $A'_{k,\eta}$ is coercive then so is the other (with the same coercivity constant); see [10, Equations 2.37–2.40 and Remark 2.24] for more details.

The main difference between the direct and indirect integral equations, (1.8) and (1.10) respectively, is that the physical meaning of the unknown is clear in the direct equation (it is the normal derivative of the total field) but not in the indirect equation (it turns out that ϕ is the difference of traces of certain exterior and interior Helmholtz BVPs; see [10, p.132]).

Both the operators $A'_{k,\eta}$ and $A_{k,\eta}$ involve the arbitrary coupling parameter η. By proving bounds on $\|A'_{k,\eta}\|$ and $\|(A'_{k,\eta})^{-1}\|$, one can show that, when k is large, the choice $|\eta| \sim k$ is optimal, in that it minimises the condition number of $A'_{k,\eta}$ (and hence also of $A_{k,\eta}$); see [10, Remark 5.1].

There are several different ways of solving integral equations such as (1.8) and (1.10), but in this paper we focus on the Galerkin method. Concentrating on the direct equation (1.8) and denoting $\partial u/\partial n$ by v, we have that solving (1.8) is equivalent to the variational problem

$$\text{find } v \in L^2(\Gamma) \text{ such that } \big(A'_{k,\eta}v, \phi\big)_{L^2(\Gamma)} = \big(f, \phi\big)_{L^2(\Gamma)} \text{ for all } \phi \in L^2(\Gamma)$$

(where $(\psi,\phi)_{L^2(\Gamma)} = \int_\Gamma \psi\,\overline{\phi}\,\mathrm{d}s$). Given a finite-dimensional approximation space $\mathcal{V}_N \subset L^2(\Gamma)$ (with N being the dimension), the Galerkin method is

$$\text{find } v_N \in \mathcal{V}_N \text{ such that } \big(A'_{k,\eta}v_N, \phi_N\big)_{L^2(\Gamma)} = \big(f, \phi_N\big)_{L^2(\Gamma)} \text{ for all } \phi_N \in \mathcal{V}_N. \tag{1.11}$$

If one can prove that $A'_{k,\eta}$ is coercive (i.e. (1.1) holds), then the Lax-Milgram theorem and Céa's lemma give the following error estimate,

$$\|v - v_N\|_{L^2(\Gamma)} \leq \left(\frac{\|A'_{k,\eta}\|}{\alpha_{k,\eta}}\right) \inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)}, \tag{1.12}$$

and the Galerkin method is then said to be *quasi-optimal*. (If the left-hand side of (1.12) were equal to the *best approximation error*, $\inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)}$, then the method would be optimal; instead we have optimality up to a constant.)

1.2 What is known about the coercivity of $A'_{k,\eta}$?

The only domains so far for which coercivity is completely understood are balls (i.e. Γ is a circle or sphere); this is because the operator $A'_{k,\eta}$ acts diagonally in the bases of trigonometric polynomials (in 2-d) and spherical harmonics (in 3-d). For the circle, Domínguez, Graham, and the third author showed in [19] that if $\eta = k$ then there exists a k_0 such that, for all $k \geq k_0$, (1.1) holds with $\alpha_{k,\eta} = 1/2$; for the sphere they proved that if $\eta = k$ then (1.1) holds for sufficiently large k with

$$\alpha_{k,\eta} \geq \frac{1}{2} - \mathcal{O}\left(\frac{1}{k^{2/3}}\right).$$

These proofs relied on bounding below the eigenvalues of $A'_{k,\eta}$, which are combinations of Bessel and Hankel functions. (Note that $A'_{k,\eta}$ is not invertible, and hence is not coercive, when both η and k equal zero. Therefore, if we take $\eta = k$ we cannot hope for $A'_{k,\eta}$ to be coercive for all $k \geq 0$, only for k sufficiently large.)

Although nothing has been proved until now about the coercivity of $A'_{k,\eta}$ on domains other than the circle or sphere, weaker results about the norm of $(A'_{k,\eta})^{-1}$ can be used to deduce information about possible values of the coercivity constant, $\alpha_{k,\eta}$, using the fact that if $A'_{k,\eta}$ is coercive then

$$\alpha_{k,\eta} \leq \left\|(A'_{k,\eta})^{-1}\right\|^{-1}$$

(this follows from (1.1) using the Cauchy-Schwarz inequality). Furthermore, if a part of Γ is C^1 then

$$\left\|(A'_{k,\eta})^{-1}\right\| \geq 2$$

in both 2- and 3-d [9, Lemma 4.1] (this follows from the fact that S_k and D'_k are compact operators when Γ is C^1) and hence, if $\alpha_{k,\eta}$ exists,

$$\alpha_{k,\eta} \leq \frac{1}{2}; \tag{1.13}$$

therefore the bound obtained on $\alpha_{k,\eta}$ for the circle in [19] is sharp. Examples of 2-d trapping domains where $\|(A'_{k,\eta})^{-1}\|$ grows either polynomially or exponentially in k through some increasing sequence of wavenumbers can be found in [9, Theorem 5.1] and [3, Theorem 2.8] (for a summary of these results, and an outline of the general argument, see [10, §5.6.2]). Therefore, if $A'_{k,\eta}$ is coercive for these domains, then $\alpha_{k,\eta}$ must decay either polynomially or exponentially as k increases.

Betcke and the first author undertook a numerical investigation of coercivity by computing the *numerical range* (also known as the *field of values*) of $A'_{k,\eta}$, $W(A'_{k,\eta})$, for various 2-d domains in [5]. Recall that

$$W(A'_{k,\eta}) := \left\{ \left(A'_{k,\eta}\phi, \phi\right)_{L^2(\Gamma)} : \phi \in L^2(\Gamma) \text{ with } \|\phi\|_{L^2(\Gamma)} = 1 \right\},$$

and thus if $A'_{k,\eta}$ is coercive, then $\alpha_{k,\eta} = \operatorname{dist}\left(W(A'_{k,\eta}), 0\right)$. These experiments (all conducted with $\eta = k$) indicated that if Ω_+ is trapping then $A'_{k,k}$ is not coercive at values of k close to the "wavenumber" of the cavity that traps waves, and if Ω_+ is nontrapping then $A'_{k,k}$ is coercive with $\alpha_{k,k}$ independent of k, as long as k is sufficiently large.

It is interesting to note that, although changing η from k to $-k$ does not affect the bounds on $\|(A'_{k,\eta})^{-1}\|$ (since they depend on $|\eta|$; see [10, §5.6.1], [57, §1.3]), it completely changes the coercivity properties of $A'_{k,\eta}$. Indeed, whereas $A'_{k,k}$ appears to be coercive when Ω_+ is nontrapping and k is sufficiently large, $A'_{k,-k}$ is not coercive when Γ is the unit circle and $k \geq 1$. (This can be seen by plotting the eigenvalues of $A'_{k,-k}$, which are given explicitly in terms of Bessel and Hankel functions by, e.g., [10, Equation 5.20c], and noting that they encircle the origin; thus the fact that $W(A'_{k,-k})$ is convex [5, Proposition 3.2] implies that $A'_{k,-k}$ is not coercive.)

Finally, to give some indication of why proving that $A'_{k,\eta}$ is coercive is difficult, we note that it appears that $A'_{k,\eta}$ is a normal operator if and only if Ω_- is a ball (i.e. Γ is a circle or sphere). Indeed, it is straightforward to prove that if Γ is the circle or sphere then $A'_{k,\eta}$ is normal (via the diagonalisation in trigonometric polynomials or spherical harmonics). The numerical experiments in [5] suggest that the converse is true, and the analogue of this result for the operator S_k was proved in [4, Theorem 3.1]. It is well-known that, although the spectrum determines the behaviour of normal operators, this is not the case for nonnormal operators; see, e.g., [62].

1.3 The main result of the paper and its consequences

In this paper we prove that $A'_{k,\eta}$ is coercive for smooth, convex domains in 2- or 3-d when $\eta \gtrsim k$ and k is sufficiently large. More precisely:

Theorem 1.2 *Let Ω_- be a convex domain in either 2- or 3-d whose boundary, Γ, has strictly positive curvature and is both C^3 and piecewise analytic. Then there exists a constant $\eta_0 > 0$ such that, given $\delta > 0$, there exists $k_0 > 0$ (depending on δ) such that, for $k \geq k_0$ and $\eta \geq \eta_0 k$,*

$$\Re\big(A'_{k,\eta}\phi, \phi\big)_{L^2(\Gamma)} \geq \left(\frac{1}{2} - \delta\right) \|\phi\|^2_{L^2(\Gamma)} \tag{1.14}$$

for all $\phi \in L^2(\Gamma)$. (By the remarks in §1.1, the bound also holds with $A'_{k,\eta}$ replaced by $A_{k,\eta}$.)

Note that the inequality (1.14) implies that $\alpha_{k,\eta} \geq (1/2 - \delta)$, and then this bound on the coercivity constant is effectively sharp by (1.13) above. In fact, the proof of Theorem 1.2 shows that, as $k \to \infty$,

$$\alpha_{k,\eta} \geq \frac{1}{2} - \mathcal{O}\left(\frac{1}{k^{1/2}}\right) \quad \text{when } d = 2, \text{ and} \tag{1.15a}$$

$$\alpha_{k,\eta} \geq \frac{1}{2} - \mathcal{O}\left(\frac{(\log k)^{1/2}}{k^{1/13}}\right) \quad \text{when } d = 3. \tag{1.15b}$$

In the rest of this paper, we call a convex domain with strictly positive curvature a *uniformly convex* domain (motivated by the fact that if a convex function has $D^2 f \geq \theta$, in the sense of quadratic forms, for some $\theta > 0$ then it is sometimes described as being uniformly convex; see, e.g., [23, p.621]). In 3-d, by strictly positive curvature we mean that both of the principal curvatures are strictly positive.

We now outline the two main consequences of Theorem 1.2. Both of these need an upper bound on the norm of $A'_{k,\eta}$ as an operator on $L^2(\Gamma)$. The currently best available bound when Ω_- is a uniformly convex domain satisfying the conditions of Theorem 1.2 is

$$\|A'_{k,\eta}\| \lesssim 1 + k^{(d-1)/2}\left(1 + \frac{|\eta|}{k}\right) \tag{1.16}$$

for all $k > 0$ and $\eta \in \mathbb{R}$; see [9, Theorem 3.6]. Note that we are using the notation $A \lesssim B$ if $A \leq cB$ with c independent of k and η. In fact, the bound (1.16) is valid when Ω_- is a general Lipschitz domain and appears not to be sharp when Ω_- is uniformly convex. Indeed, when Γ is the circle or sphere, $\|A'_{k,\eta}\| \lesssim k^{1/3}$ when $\eta \sim k$; see [10, §5.4–5.5] for more details.

k-explicit quasi-optimality of the Galerkin method for any finite-dimensional subspace. The main application of Theorem 1.2 is that it implies that the Galerkin method (1.11) is quasi-optimal for *any* finite-dimensional subspace. Indeed, combining the result (1.14) with the estimates (1.16) and (1.12), we see that if Ω_- satisfies the conditions of Theorem 1.2 and the direct integral equation (1.8) is solved via (1.11) with η chosen so that $\eta_0 k \leq \eta \lesssim k$, then, for all $k \geq k_0$,

$$\|v - v_N\|_{L^2(\Gamma)} \lesssim k^{(d-1)/2} \inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)}, \tag{1.17}$$

where k_0 and η_0 are as in Theorem 1.2 and $v := \partial u/\partial n$. An analogous result also holds for the indirect equation (1.10).

The key point is that the quasi-optimality (1.17) is established for *any* subspace $\mathcal{V}_N \subset L^2(\Gamma)$ without *any* constraint on the dimension N. In constrast, the usual approach to the numerical analysis of Helmholtz problems is to prove coercivity *up to a compact perturbation* (i.e. a Gårding inequality). Even when these arguments can be made explicit in k, they yield quasi-optimality only when N is larger than some k-dependent threshold. For example, if the integral equation (1.8) is solved using the Galerkin method with $\mathcal{V}_N$ consisting of piecewise polynomials of degree $\leq p$, for some fixed $p \geq 0$, on shape regular meshs of diameter at most h (so $N \sim h^{-(d-1)}$), then a k-explicit version of the classical compact perturbation argument shows that, if Ω_- is a C^2, star-shaped, 2- or 3-d domain, then

$$\|v - v_N\|_{L^2(\Gamma)} \lesssim \inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)} \qquad \text{provided that} \qquad hk^{(d+1)/2} \lesssim 1; \tag{1.18}$$

see [28, Theorem 1.6]. The fact that the mesh threshold in (1.18) is more stringent that the $hk \lesssim 1$ rule of thumb can be understood as a consequence of the pollution effect (see, e.g. [22, §1]).

To compare the error estimates (1.17) and (1.18) we need to understand how the best approximation error, $\inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)}$, depends on h and k. It is generally believed that this is $\lesssim \|v\|_{L^2(\Gamma)}$ if $hk \lesssim 1$, and this has been proved if Ω_- is a C^∞, uniformly convex, 2-d domain. Indeed, the results about the asymptotics of $v := \partial u/\partial n$ for this type of domain in, e.g., [42], adapted for a numerical analysis context in [19, Theorem 5.4, Corollary 5.5], imply that

$$\inf_{\phi_N \in \mathcal{V}_N} \|v - \phi_N\|_{L^2(\Gamma)} \lesssim hk\|v\|_{L^2(\Gamma)};$$

see [28, Theorem 1.2]. Using this bound in both (1.17) and (1.18), we see that *both* the estimate from coercivity *and* the estimate from the k-explicit compact perturbation argument show that, in the 2-d case, the relative error $\|v - v_N\|_{L^2(\Gamma)}/\|v\|_{L^2(\Gamma)}$ is bounded independently of k when $hk^{3/2} \lesssim 1$.

In summary, since any quasi-optimality estimate for piecewise-polynomial subspaces will ultimately be considered under some k-dependent threshold for N (coming from controlling the best approximation error), the advantage of the "no-threshold" quasi-optimality given by coercivity over the "threshold" quasi-optimality (usually called "asymptotic" quasi-optimality) of the compact perturbation arguments is *not* felt for these subspaces.

The advantage of coercivity is crucial, however, when seeking to establish quasi-optimality of *hybrid numerical-asymptotic methods*. Indeed, as discussed above, there has been much recent research in designing k-dependent approximation spaces that incorporate the oscillation of the solution, with the result that the best approximation error for these spaces either is bounded or grows mildly as k increases with N fixed. If one applies the standard compact-perturbation arguments to try to establish quasi-optimality of Galerkin methods using these subspaces, it is not at all clear how the threshold for quasi-optimality depends on k and whether N will ever be large enough to exceed this threshold (since the whole point of these methods is to keep N relatively small). Establishing coercivity, however, bypasses these difficulties.

For example, a k-dependent approximation space, $\mathcal{V}_{N,k}$, for sound-soft scattering by smooth, uniformly convex obstacles in 2-d was designed in [19] by using knowledge of the $k \to \infty$ asymptotics. The space $\mathcal{V}_{N,k}$ divides Γ into the illuminated zone, the shadow zone, and two shadow boundary zones. The solution to the integral equation $v := \partial u/\partial n$ is then approximated by an oscillatory factor multiplied by a polynomial of degree N in the illuminated zone and the two shadow boundary zones, and by zero in the shadow zone. Combining Theorem 1.2 with results about the best approximation error in $\mathcal{V}_{N,k}$ proved in [19, Theorem 6.7] (using results from [42] about the asymptotics of v), we obtain the following error estimate for the Galerkin method using $\mathcal{V}_{N,k}$.

Theorem 1.3 *Let Ω_- be a uniformly convex, 2-d domain whose boundary is C^∞ and piecewise analytic. Suppose that the sound-soft scattering problem of Definition 1.1 is solved with the Galerkin method using the combined potential integral equation (1.8) and the hybrid approximation space introduced in [19] (and denoted by $\mathcal{V}_{N,k}$ above). Let N be the degree of the polynomials used in each of the three zones (so N is proportional to the total number of degrees of freedom of the method). Then there exist η_0, k_0, δ, and c_0, all greater than zero, such that, if the coupling parameter η is chosen so that $\eta_0 k \leq \eta \lesssim k$, then*

$$\|v - v_N\|_{L^2(\Gamma)} \lesssim k^{19/18} \left\{ \left(\frac{k^{1/9}}{N} \right)^{N+1} + k^{4/9} \exp(-c_0\, k^\delta) \right\}$$

for all $k \geq k_0$. Therefore, provided that N grows like $k^{1/9+\varepsilon}$ for some $\varepsilon > 0$, the error is bounded as $k \to \infty$.

Using similar ideas, a k-dependent approximation space for scattering by smooth, uniformly convex obstacles in 3-d was designed in [26]. Theorem 1.2, along with a bound on the best approximation error for this subspace, can then also give rigorous error estimates for this method.

Bounding the numerical range of $A'_{k,\eta}$ and the associated k-explicit bounds on GMRES iterations. Whereas the first consequence of coercivity (k-explict quasi-optimality for any approximation space) is more relevant for the Galerkin method with hybrid, k-dependent subspaces, the second consequence is more applicable to the Galerkin method with conventional piecewise polynomial subspaces. In this case, the Galerkin matrices with be of size $N \times N$ and, with N having to grow at least like k^{d-1} to maintain accuracy, the associated linear systems will usually be solved using iterative methods such as GMRES. (Note that the hybrid subspaces are specifically designed so that N grows mildly with k, and thus, for geometries where these subspaces are available, the linear systems can be solved using direct, as opposed to iterative, methods.)

Although nothing has yet been proven about how GMRES behaves when applied to linear systems resulting from Galerkin discretisations of $A'_{k,\eta}$, it is usually believed that the number of iterations needed to achieve a prescribed accuracy must grow mildly with k, e.g. like k^a for some $0 < a < 1$. (We could not find any relevant numerical results for Galerkin discretisations of $A'_{k,\eta}$ in the literature, however results for Nyström discretisations of both the analogous operator for the Neumann problem, and modifications of this operator that make it a compact perturbation of the identity on smooth domains, can be found in [6] and [7]. These results show the number of GMRES iterations growing like k^a for a range of different $0 < a < 1$, depending on the geometry.)

It is well-known that a sufficient (but not necessary) condition for iterative methods to be well behaved is that the numerical range of the matrix is bounded away from zero. Furthermore, the following bound was proved in [21] (see also [20, Theorem 3.3]) and appears in this particular form in, e.g., [2, Equation 1.2].

Theorem 1.4 *If the matrix equation $\mathbf{A}\mathbf{v} = \mathbf{f}$ is solved using GMRES then, for $m \in \mathbb{N}$, the m-th GMRES residual, $\mathbf{r}_m := \mathbf{A}\mathbf{v}_m - \mathbf{f}$, satisfies*

$$\frac{\|\mathbf{r}_m\|_2}{\|\mathbf{r}_0\|_2} \leq \sin^m \beta, \qquad \text{where} \qquad \cos\beta = \frac{\mathrm{dist}\big(0, W(\mathbf{A})\big)}{\|\mathbf{A}\|_2}, \tag{1.19}$$

and where $W(\mathbf{A}) := \big\{(\mathbf{A}\mathbf{v}, \mathbf{v}) : \mathbf{v} \in \mathbb{C}^N, \|\mathbf{v}\|_2 = 1\big\}$ is the numerical range *of $\mathbf{A}$ and $\|\cdot\|_2$ denotes the l_2 (i.e. Euclidean) vector norm.*

Coercivity of the operator $A'_{k,\eta}$ implies that the numerical range of the associated Galerkin matrix, $\mathbf{A}$, is bounded away from zero, and thus allows us to obtain k-explicit bounds on the number of GMRES iterations needed to solve $\mathbf{A}\mathbf{v} = \mathbf{f}$. Indeed, consider the h-version of the Galerkin method, i.e. $\mathcal{V}_N \subset L^2(\Gamma)$ is the space of piecewise polynomials of degree $\leq p$ for some fixed $p \geq 0$ on quasi-uniform meshes of diameter h, with h decreasing to zero (thus $N \sim h^{-(d-1)}$). Let $\mathcal{V}_N = \mathrm{span}\{\phi_i : i = 1, \ldots, N\}$, let $v_N \in \mathcal{V}_N$ be equal to $\sum_{j=1}^N V_j \phi_j$, and define $\mathbf{v} \in \mathbb{C}^N$ by $\mathbf{v} := (V_j)_{j=1}^N$. Then, with $\mathbf{A}_{ij} := (A'_{k,\eta}\phi_j, \phi_i)_{L^2(\Gamma)}$ and $\mathbf{f}_i := (f, \phi_i)_{L^2(\Gamma)}$, the Galerkin method (1.11) is equivalent to solving the linear system $\mathbf{A}\mathbf{v} = \mathbf{f}$.

If $A'_{k,\eta}$ is coercive with coercivity constant $\alpha_{k,\eta}$, then, combining this property with the boundedness of $A'_{k,\eta}$, we have that

$$\big|(\mathbf{A}\mathbf{u}, \mathbf{v})_2\big| \lesssim \|A'_{k,\eta}\|\, h^{d-1}\, \|\mathbf{u}\|_2\, \|\mathbf{v}\|_2 \qquad \text{and} \qquad \big|(\mathbf{A}\mathbf{v}, \mathbf{v})_2\big| \gtrsim \alpha_{k,\eta}\, h^{d-1}\, \|\mathbf{v}\|_2^2 \tag{1.20}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{C}^N$, where we have used the bound $\|v_N\|^2_{L^2(\Gamma)} \sim h^{d-1}\, \|\mathbf{v}\|_2^2$ (see, e.g., [55, Corollary 5.3.28]). The two bounds in (1.20) imply that the ratio $\cos\beta$ in (1.19) satisfies

$$\cos\beta \gtrsim \frac{\alpha_{k,\eta}}{\|A'_{k,\eta}\|},$$

and then Theorem 1.4 implies that, given $\varepsilon > 0$, there exists a C, independent of k, η, and ε, such that

$$\text{if} \quad m \geq C\left(\frac{\|A'_{k,\eta}\|}{\alpha_{k,\eta}}\right)^2 \log\left(\frac{1}{\varepsilon}\right) \qquad \text{then} \qquad \frac{\|\mathbf{r}_m\|}{\|\mathbf{r}_0\|} \leq \varepsilon. \tag{1.21}$$

If Ω_- satisfies the conditions of Theorem 1.2 and we take η as prescribed in that theorem, then $\alpha_{k,\eta} \gtrsim 1$ (and we know that this bound is sharp in its k-dependence from (1.13)). Whether or not the bound (1.21) tells us anything practical about m then rests on the k-explicit bounds for $\|A'_{k,\eta}\|$ and their sharpness. (In the rest of this discussion we assume that η is taken so that $\eta_0 k \leq \eta \lesssim k$.)

Using the upper bound on $\|A'_{k,\eta}\|$ (1.16) in (1.21), we find that choosing m so that $m \gtrsim k^{d-1}$ is sufficient for $\|\mathbf{r}_m\|/\|\mathbf{r}_0\|$ to be bounded independently of k as k increases. However, N will be either k^{d-1} (if a fixed number of degrees of freedom per wavelength are chosen, i.e. $hk \lesssim 1$) or $k^{(d+1)(d-1)/2}$ (if we take $hk^{(d+1)/2} \lesssim 1$ to be sure of eliminating the pollution effect by (1.18)). Therefore, since GMRES always converges in at most N steps (in exact arithmetic), the bound $m \gtrsim k^{d-1}$ either doesn't tell us anything about the k-dependence of the number of iterations, or is very pessimistic.

Nevertheless, since the bound (1.16) on $\|A'_{k,\eta}\|$ appears not to be sharp when Ω_- is smooth and uniformly convex, there is hope that more practical bounds on m can be obtained. Indeed, if Γ is a sphere then $\|A'_{k,\eta}\| \lesssim k^{1/3}$, and therefore (1.21) gives $m \gtrsim k^{2/3}$. Since N will be at least proportional to k^2 in this case, this bound on m is now non-trivial, and comes much closer to proving the mild growth observed in practice.

1.4 The classical method of "transferring" coercivity properties of the PDE to boundary integral operators

The method used to prove the main result (Theorem 1.2) is closely linked to a well-established idea in the theory of boundary integral equations, namely that coercivity properties of the weak form of the PDE can be "transferred" to the associated boundary integral operators. This idea was introduced for first-kind integral equations independently by Nédélec and Planchard [51], Le Roux [34], and Hsiao and Wendland [31], and for second-kind equations by Steinbach and Wendland [60]. We briefly recap this idea here, and then explain in §1.5 how it can be modified to prove Theorem 1.2.

For the Helmholtz equation posed in a bounded domain D (with outward-pointing unit normal vector $\boldsymbol{\nu}$), the weak form of the PDE is based on Green's identity integrated over D:

$$-\int_D \overline{v}\mathcal{L}u \,\mathrm{d}\mathbf{x} = \int_D \left(\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v}\right) \mathrm{d}\mathbf{x} - \int_{\partial D} \overline{v}\frac{\partial u}{\partial \nu}\,\mathrm{d}s \tag{1.22}$$

(recall that $\mathcal{L}u := \Delta u + k^2 u$). The fact that the volume terms on the right-hand side of (1.22) are single-signed when $k = 0$ and $v = u$ means that the standard variational formulations of Laplace's equation are coercive. The fact that the volume terms are *not* single-signed when $k > 0$ and $v = u$ means that the standard variational formulations of the Helmholtz equation are *not* coercive when k is large, only coercive up to a compact perturbation (see, e.g., [44, §1.1]).

Let Ω_- be as in §1.1 (i.e. Ω_- is bounded and $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected); the following argument is valid when Ω_- is Lipschitz, but we ignore the technicalities needed in this case. Given $\phi \in L^2(\Gamma)$, let u be the single-layer potential $\mathcal{S}_k$ with density $\phi \in L^2(\Gamma)$, that is,

$$u(\mathbf{x}) = \mathcal{S}_k\phi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \Gamma.$$

Then $\mathcal{L}u = 0$ in $\Omega_- \cup \Omega_+$, and the following jump relations hold on Γ:

$$u_\pm(\mathbf{x}) = S_k\phi(\mathbf{x}) \quad \text{and} \quad \frac{\partial u_\pm}{\partial n}(\mathbf{x}) = \left(\mp\frac{1}{2}I + D'_k\right)\phi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma \tag{1.23}$$

(where S_k and D'_k are defined by (1.6) and (1.7) respectively). Let $B_R := \{|\mathbf{x}| < R\}$ and apply Green's identity (1.22) with $v = u$, first with $D = \Omega_-$, then with $D = \Omega_+ \cap B_R$ (with $R > 0$ chosen large enough so that $\overline{\Omega_-} \subset B_R$), and then add the resulting two equations. Using the jump relations (1.23), we find that

$$\overline{\left(S_k\phi, \phi\right)_{L^2(\Gamma)}} = \int_{(\Omega_+ \cap B_R)\cup\Omega_-} \left(|\nabla u|^2 - k^2|u|^2\right)\mathrm{d}\mathbf{x} - \int_{\partial B_R} \bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s. \tag{1.24}$$

This last equation holds when $\phi \in H^{-1/2}(\Gamma)$ if the left-hand side is replaced by $\overline{\langle S_k\phi, \phi\rangle_\Gamma}$, where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

We now seek to relate the terms on the right-hand side of (1.24) to $\|\phi\|^2_{H^{-1/2}(\Gamma)}$, ideally proving that they are $\gtrsim \|\phi\|^2_{H^{-1/2}(\Gamma)}$, which would show that S_k is coercive as a mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$.

First consider the case when $k = 0$, i.e. the PDE is Laplace's equation, and $d = 3$ (the case $d = 2$ for Laplace's equation is more complicated because the fundamental solution does not decay at infinity). In this case $u(\mathbf{x}) = \mathcal{O}(1/r)$ and $\nabla u(\mathbf{x}) = \mathcal{O}(1/r^2)$ as $r := |\mathbf{x}| \to \infty$, and thus

$$\int_{\partial B_R} \bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \to 0 \text{ as } R \to \infty. \tag{1.25}$$

The definition of $\partial u/\partial n$ in $H^{-1/2}(\Gamma)$ (which is essentially Green's identity; see, e.g., [40, Lemma 4.3]) implies that

$$\int_{\Omega_\pm} |\nabla u|^2\,\mathrm{d}\mathbf{x} \gtrsim \left\|\frac{\partial u_\pm}{\partial n}\right\|^2_{H^{-1/2}(\Gamma)}; \tag{1.26}$$

see, e.g., [59, Corollary 4.5]. The second jump relation in (1.23) implies that

$$\|\phi\|^2_{H^{-1/2}(\Gamma)} \lesssim \left\|\frac{\partial u_+}{\partial n}\right\|^2_{H^{-1/2}(\Gamma)} + \left\|\frac{\partial u_-}{\partial n}\right\|^2_{H^{-1/2}(\Gamma)}, \tag{1.27}$$

and so, using (1.25), (1.26), and (1.27) in (1.24), we obtain that

$$\big\langle S_0\phi, \phi\big\rangle_\Gamma \gtrsim \|\phi\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } \phi \in H^{-1/2}(\Gamma).$$

In summary, we have just used Green's identity to prove that the Laplace single-layer operator in 3-d is coercive as a mapping from $H^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$ (i.e. we "transferred" the coercivity of the weak form of the PDE to the integral operator). A slightly more complicated argument yields the analogous result in 2-d [59, Theorem 6.23], [40, Theorem 8.16], and repeating the same argument with u equal to the double-layer potential yields an analogous result for the hypersingular operator as a mapping from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ (after its nonzero kernel is quotiented out) [59, Theorem 6.24], [55, Theorem 3.5.3], [40, Theorem 8.21]. Furthermore, using these results Steinbach and Wendland showed that $\frac{1}{2}I - D_0'$ is coercive on $H^{-1/2}(\Gamma)$, in the sense that

$$\left\langle \left(\frac{1}{2}I - D_0'\right)\phi, S_0\phi \right\rangle_\Gamma \gtrsim \|\phi\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } \phi \in H^{-1/2}(\Gamma),$$

and that $\frac{1}{2}I - D_0$ is coercive on $H^{1/2}(\Gamma)$; analogous results also hold for $\frac{1}{2}I + D_0'$ and $\frac{1}{2}I + D_0$ after their nonzero kernels are quotiented out [60, Theorems 3.1 and 3.2], [32, Theorem 5.6.11]. See [17] (in particular [17, Theorems 1 and 2]) for an insightful overview of all these results.

When we try to repeat this argument for $k > 0$, we run into two difficulties:

(i) the integral over ∂B_R does not tend to zero as $R \to \infty$, and

(ii) the volume terms in Green's identity (1.22) are not single-signed when $v = u$.

Indeed, if $u = \mathcal{S}_k\phi$ then u satisfies the radiation condition (1.2) and one can then show that, as $R \to \infty$,

$$\Re \int_{\partial B_R} \bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \to 0 \quad \text{and} \quad \Im \int_{\partial B_R} \bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \to k\int_{\mathbb{S}^{d-1}} |f_1(\widehat{\mathbf{x}})|^2\,\mathrm{d}s, \tag{1.28}$$

where $f_1(\widehat{\mathbf{x}})$ is the far-field pattern of u and $\mathbb{S}^{d-1}$ is the d-dimensional unit sphere. Letting $R \to \infty$ in (1.24) and using these limits we find that

$$\Re\big\langle S_k\phi, \phi\big\rangle_\Gamma = \int_{\Omega_+\cup\Omega_-} \big(|\nabla u|^2 - k^2|u|^2\big)\,\mathrm{d}\mathbf{x} \quad \text{and} \tag{1.29}$$

$$\Im\big\langle S_k\phi, \phi\big\rangle_\Gamma = k\int_{\mathbb{S}^{d-1}} |f_1(\widehat{\mathbf{x}})|^2\,\mathrm{d}s \qquad \text{for all } \phi \in H^{-1/2}(\Gamma). \tag{1.30}$$

Therefore, considering only $\Re\langle S_k\phi,\phi\rangle_\Gamma$ bypasses (for now) the difficulty (i) above. The jump relations (1.23) again imply the bound (1.27), so all we need to do is bound the volume terms in (1.29) below by $\|\partial u_\pm/\partial n\|^2_{H^{-1/2}(\Gamma)}$. However, the analogue for $k>0$ of the bound (1.26) in Ω_- now contains $k^2\int_{\Omega_-}|u|^2\,\mathrm{d}\mathbf{x}$ on the left-hand side, so the sign-indefiniteness of the volume terms in (1.29) means that they cannot be bounded below by $\|\partial u_-/\partial n\|^2_{H^{-1/2}(\Gamma)}$. The analogue for $k>0$ of the bound (1.26) in Ω_+ is more complicated; it shares the problem of the bound in Ω_- just described, and, additionally, the fact that the integral over ∂B_R in Green's identity does not tend to zero as $R\to\infty$ means that the left-hand side of the bound must contain a contribution from dealing with this term.

Ultimately, all one can prove in the Helmholtz case is that there exists a compact operator $T_k: H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ such that

$$\Re\big\langle (S_k+T_k)\phi,\phi\big\rangle_\Gamma \gtrsim \|\phi\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } \phi\in H^{-1/2}(\Gamma),$$

that is, S_k is coercive up to a compact perturbation (i.e. satisfies a Gårding inequality); see, e.g, [16, Theorem 2], [32, Theorem 5.6.8], or [40, Theorem 7.6] for the details.

In summary, using the ideas sketched above, the coercivity properties of the weak form of the PDE, i.e. coercivity for Laplace's equation and coercivity up to a compact perturbation for the Helmholtz equation, can be "transferred" to the first- and second-kind boundary integral operators for these PDEs.

1.5 Modifying the classical method using Morawetz's identities

The previous subsection showed that there are two reasons why the classical "transfer of coercivity" method only proves coercivity up to a compact perturbation for the Helmholtz boundary integral operators, as opposed to proving coercivity for the Laplace ones:

1. the volume terms of Green's identity (1.22) are not single-signed when $v=u$,
2. when Green's identity is applied in $\Omega_+\cap B_R$ with $v=u$ and u satisfying the radiation condition (1.2), the integral over ∂B_R does not tend to zero as $R\to\infty$.

This paper uses the idea, first introduced in [58], to replace Green's identity in the argument of the previous section with another identity for solutions of the Helmholtz equation for which the problems outlined in 1. and 2. above do not apply.

Recall that Green's identity arises from multiplying $\mathcal{L}u$ by $\overline{v}$. The multiplier $\overline{r\mathcal{M}u}$, where

$$\mathcal{M}u := \frac{\mathbf{x}}{r}\cdot\nabla u - \mathrm{i}ku + \frac{d-1}{2r}u,$$

and $r:=|\mathbf{x}|$, was introduced by Morawetz and Ludwig in [47]. In that paper, the resulting identity,

$$2\Re\left(\overline{r\mathcal{M}u}\mathcal{L}u\right) = \nabla\cdot\Big[2\Re\left(r\overline{\mathcal{M}u}\nabla u\right) + \left(k^2|u|^2-|\nabla u|^2\right)\mathbf{x}\Big] - \Big(|\nabla u|^2-|u_r|^2\Big) - |u_r-\mathrm{i}ku|^2, \quad (1.31)$$

was used to bound the Dirichlet-to-Neumann map for the Helmholtz equation in the exterior of a star-shaped domain (and it can also be used to bound the energy norm of the solution of the exterior Dirichlet problem in this class of domains). This is possible because

1. the non-divergence terms on the right-hand side of (1.31) are single-signed, and
2. when the identity (1.31) is integrated over $\Omega_+\cap B_R$, the integral over ∂B_R tends to zero as $R\to\infty$ if u satisfies the radiation condition (1.2).

(To understand where the star-shapedness requirement comes from, note that when we integrate (1.31) over $\Omega_+\cap B_R$ we get a surface integral on Γ involving $\mathbf{x}\cdot\mathbf{n}(\mathbf{x})$, where $\mathbf{n}(\mathbf{x})$ is the unit normal vector on Γ pointing into Ω_+. It turns out that we need $\mathbf{x}\cdot\mathbf{n}(\mathbf{x})>0$ for all $\mathbf{x}\in\Gamma$ for the bounds to hold, and this means that Ω_- must be star-shaped.)

Repeating the "transfer of coercivity" argument reviewed in §1.4 with Green's identity (1.22) replaced by the integrated form of the Morawetz-Ludwig identity (1.31), we obtain that

$$\Re\Big(\big((\mathbf{x}\cdot\mathbf{n})\,D_k' + \mathbf{x}\cdot\nabla_\Gamma S_k - \mathrm{i}\eta S_k\big)\phi,\phi\Big)_{L^2(\Gamma)} \geq 0 \tag{1.32}$$

for all $k > 0$ and $\phi \in L^2(\Gamma)$ if $\eta = kr + \mathrm{i}(d-1)/2$ (where ∇_Γ in (1.32) is the surface gradient on Γ) [58]. This inequality shows that the integral operator

$$\mathscr{A}_k := (\mathbf{x}\cdot\mathbf{n})\left(\frac{1}{2}I + D_k'\right) + \mathbf{x}\cdot\nabla_\Gamma S_k - \mathrm{i}\eta S_k \tag{1.33}$$

is coercive as an operator on $L^2(\Gamma)$ if Ω_- is a star-shaped Lipschitz domain and η is chosen as above. Using Green's integral representation, one can show that

$$\mathscr{A}_k \frac{\partial u}{\partial n} = \mathbf{x}\cdot\nabla u^I - \mathrm{i}\eta u^I, \tag{1.34}$$

and so the operator $\mathscr{A}_k$ can be used to solve the exterior Dirichlet problem. Note that if Γ is the unit circle or sphere then, on Γ, $\mathbf{x} = \mathbf{n}(\mathbf{x})$, and so $\mathscr{A}_k = A'_{k,\eta}$ (with the particular choice of η above). Therefore, the coercivity of the so-called "star-combined" operator $\mathscr{A}_k$ gives an alternative proof of the coercivity of $A'_{k,\eta}$ on the circle and sphere (see [58, Corollary 4.8]).

The main idea of this paper is to use the more general multiplier

$$\mathcal{Z}u = \mathbf{Z}\cdot\nabla u - \mathrm{i}k\beta u + \alpha u, \tag{1.35}$$

essentially introduced by Morawetz in [46], where $\mathbf{Z}(\mathbf{x})$ is a vector field, and $\beta(\mathbf{x}), \alpha(\mathbf{x})$ are scalar fields. Replacing the identity (1.31), coming from the multiplier $r\mathcal{M}u$, by the more general identity coming from the multiplier $\mathcal{Z}u$, and repeating the argument that led to (1.32), we find in §3 that, if $\mathbf{Z}$ is continuous across Γ, $\Re(\partial_i \mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\overline{\boldsymbol{\xi}_j}) \geq 0$ for all $\boldsymbol{\xi}\in\mathbb{C}^d$ and $\mathbf{x}\in\Omega_-\cup(\Omega_+\cap B_R)$, and $\eta \gtrsim k$, then

$$\Re\Big(\big((\mathbf{Z}\cdot\mathbf{n})\,D_k' + \mathbf{Z}\cdot\nabla_\Gamma S_k - \mathrm{i}\eta S_k\big)\phi,\phi\Big)_{L^2(\Gamma)} \geq -o(1)\,\|\phi\|^2_{L^2(\Gamma)} \quad \text{as } k\to\infty. \tag{1.36}$$

Since $\nabla_\Gamma S_k$ is a vector-valued operator that is tangent to Γ, if $\mathbf{Z}$ is a constant multiple of $\mathbf{n}$ on Γ (and the condition on the derivative of $\mathbf{Z}$ in the domain is satisfied) then the inequality (1.36) proves that $A'_{k,\eta}$ is coercive.

1.6 Vector-field conditions for coercivity

The method outlined in §1.5 above shows that $A'_{k,\eta}$ is coercive, for $\eta \gtrsim k$ and k sufficiently large, if there exists a vector field, $\mathbf{Z}$, defined in Ω_- and $\Omega_+\cap B_R$ for some $R>0$ such that

1. $\mathbf{Z}$ and $\nabla\cdot\mathbf{Z}$ are continuous across Γ,
2. $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ for some constant $C_\Gamma > 0$,
3. $\mathbf{Z}(\mathbf{x}) = \mathbf{x}$ in a neighbourhood of ∂B_R, and
4. $\Re\big(\partial_i\mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\overline{\boldsymbol{\xi}_j}\big) \geq 0$ for all $\boldsymbol{\xi}\in\mathbb{C}^d$ and $\mathbf{x}\in\Omega_-\cup(\Omega_+\cap B_R)$

(for simplicity, we have ignored the smoothness requirements on $\mathbf{Z}$ at this stage; see §3.1 for the details). These vector-field conditions are similar to those obtained by Morawetz to prove a bound on the energy norm of the solution to the Dirichlet problem for the Helmholtz equation in Ω_+ (i.e., a local resolvent estimate) [46, Equation 1.3], [48, Equation 4.2]. However, Morawetz needed a vector field *only in* $\Omega_+\cap B_R$, satisfying the conditions 3 and 4 above, and satisfying the weaker condition than 2 that $\mathbf{Z}\cdot\mathbf{n} > 0$ on Γ. Morawetz, Ralston, and Strauss then showed in [48, §4] that such a vector field exists when Ω_+ is a 2-d nontrapping domain.

From one perspective it is clear why we need a vector field in both $\Omega_+\cap B_R$ and Ω_- to prove that $A'_{k,\eta}$ is coercive: following the method outlined in §1.5 we applied the identity coming from the $\mathcal{Z}u$ multiplier (1.35) in both $\Omega_+\cap B_R$ and Ω_-. From another perspective, however, it is natural to ask the question: since the scattering problem that we are trying to solve is posed only in Ω_+, why should Ω_- be involved? The fact that we need a vector field in Ω_- as well as in $\Omega_+ \cap B_R$ becomes clear when we recall that the integral operators $A'_{k,\eta}$ and $A_{k,\eta}$, in addition to being able to solve the exterior Dirichlet problem, can also be used to solve the interior impedance problem (i.e. the Helmholtz equation posed in Ω_- with boundary condition $\partial u/\partial n - \mathrm{i}\eta u = g$ on Γ for some $g \in L^2(\Gamma)$ and $\eta \in \mathbb{R}\setminus\{0\}$). Indeed, the operator $A_{k,\eta}$ arises from the direct formulation of the interior impedance problem (see [10, Theorem 2.30]), and the operator $A'_{k,\eta}$ arises from an indirect formulation of the interior impedance problem (assuming that $u = \mathcal{S}_k\phi$ for some $\phi \in L^2(\Gamma)$).

Returning to the conditions for coercivity, 1–4 above, we show in §5 below that if Ω_- is nonconvex then there does *not* exist a $\mathbf{Z}$ satisfying these conditions; indeed for these geometries one can reach a contradiction between the nonnegativity condition on $\partial_i \mathbf{Z}_j$ in Ω_+ and the condition that $\mathbf{Z} = C_\Gamma \mathbf{n}$ on Γ.

If $\mathbf{Z} = \nabla\phi$ for some ϕ then the nonnegativity condition, 4, becomes the requirement that ϕ is convex. In §4 we construct a $\mathbf{Z}$ satisfying conditions 1–4 above (by constructing a suitable ϕ) when Ω_- is a uniformly convex, 2- or 3-d domain with Γ both C^3 and piecewise analytic; thus proving Theorem 1.2. The main idea of the construction is that $\pm\operatorname{dist}(\mathbf{x},\Gamma)$ is convex in $\Omega_\pm$ if Ω_- is convex (see, e.g., [54, p.28, 34]) and its gradient is the normal vector, $\mathbf{n}$, on Γ. There are then three issues:

(a) the derivative of $\operatorname{dist}(\mathbf{x},\Gamma)$ is not defined on the set of points in Ω_- that do not have a unique closest point to Γ (this set is called the *medial axis* or *ridge* of Ω_-),

(b) we need ϕ to be equal to $\frac{1}{2}r^2$ in a neighbourhood of ∂B_R (so that $\mathbf{Z} = \mathbf{x}$), and

(c) it turns out that if we have *uniform convexity* of ϕ, i.e. $D^2\phi \geq \theta$ for some $\theta > 0$, then we need less smoothness of Γ (C^3 instead of C^4).

The idea is to then use

$$\phi(\mathbf{x}) = \pm C_\Gamma \operatorname{dist}(\mathbf{x},\Gamma) + \frac{1}{2}\operatorname{dist}(\mathbf{x},\Gamma)^2,$$

smooth it in Ω_- (to deal with (a) above), smoothly change it to $\frac{1}{2}r^2$ in $\Omega_+\cap B_R$ (to deal with (b)), and choose C_Γ and R large enough to maintain the uniform convexity in (c). The condition that Γ must be piecewise analytic is needed to control the geometry of the medial axis, since without analyticity this set can behave very strangely (see the paragraph below Theorem 4.2 for more details).

1.7 Outline of the paper

In Section 2 we recall the identities introduced for solutions of the Helmholtz equation by Morawetz in [46]. In Section 3 we translate the problem of proving that $A'_{k,\eta}$ is coercive into that of constructing an appropriate vector field $\mathbf{Z}$ in the multiplier $\mathcal{Z}u$. In Section 4 we construct such a vector field for uniformly convex, 2- and 3-d domains that are C^3 and piecewise analytic. (The main result, Theorem 1.2, is then proved by combining Parts 1 and 2 of Theorem 3.2, Part 1 of Theorem 3.4, and Lemma 4.1.) In Section 5 we show that the vector-field conditions for coercivity obtained in Section 3 cannot be satisfied if Ω_- is nonconvex. In Section 6 we conclude by placing this paper's use of Morawetz's identities into a wider context.

2 Morawetz's identities for the Helmholtz equation

In this section, we state and prove two identities for solutions of the Helmholtz equation that arise from the multiplier $\mathcal{Z}u$ (1.35). (In the rest of the paper we refer to these as "Morawetz 1" and "Morawetz 2" respectively.)

Lemma 2.1 (First Morawetz identity for Helmholtz ("Morawetz 1")) *Let v be a complex-valued C^2 function on some set $D \subset \mathbb{R}^d$. Let $\mathcal{L}v := \Delta v + k^2 v$ with $k \in \mathbb{R}$. Let $\mathbf{Z} \in (C^1(D))^d$ and*

$\beta, \alpha \in C^1(D)$ *(i.e.* $\mathbf{Z}$ *is a vector and* β *and* α *are scalars) and let all three be* real-valued. *Then, with the summation convention,*

$$2\Re\big(\overline{\mathcal{Z}v}\,\mathcal{L}v\big) = \nabla\cdot\Big[2\Re\big(\overline{\mathcal{Z}v}\,\nabla v\big) + \big(k^2|v|^2 - |\nabla v|^2\big)\mathbf{Z}\Big] + \big(2\alpha - \nabla\cdot\mathbf{Z}\big)\big(k^2|v|^2 - |\nabla v|^2\big)$$
$$- 2\Re\big(\partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v}\big) - 2\Re\big(\overline{v}\,(\mathrm{i}k\nabla\beta + \nabla\alpha)\cdot\nabla v\big), \tag{2.1}$$

where

$$\mathcal{Z}v := \mathbf{Z}\cdot\nabla v - \mathrm{i}k\beta v + \alpha v. \tag{2.2}$$

Lemma 2.2 (Second Morawetz identity for Helmholtz ("Morawetz 2")) *If the assumptions of Lemma* (2.1) *hold and, additionally,* $\alpha \in C^2(D)$ *then*

$$2\Re\big(\overline{\mathcal{Z}v}\,\mathcal{L}v\big) = \nabla\cdot\Big[2\Re\big(\overline{\mathcal{Z}v}\,\nabla v\big) + \big(k^2|v|^2 - |\nabla v|^2\big)\mathbf{Z} - \nabla\alpha|v|^2\Big] + \big(2\alpha - \nabla\cdot\mathbf{Z}\big)\big(k^2|v|^2 - |\nabla v|^2\big)$$
$$- 2\Re\big(\partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v}\big) - 2\Re\big(\mathrm{i}k\,\overline{v}\,\nabla\beta\cdot\nabla v\big) + \Delta\alpha|v|^2 \tag{2.3}$$

where $\mathcal{Z}v$ *is given by* (2.2).

Note that Lemma 2.2 follows from Lemma 2.1 by using

$$2\Re\big(\overline{v}\,\nabla\alpha\cdot\nabla v\big) = \nabla\cdot\big[\nabla\alpha|v|^2\big] - \Delta\alpha|v|^2.$$

Proof of Lemma 2.1. Splitting $\mathcal{Z}v$ up into its component parts we see that the identity (2.1) is the sum of the following three identities:

$$2\Re\big(\mathbf{Z}\cdot\overline{\nabla v}\,\mathcal{L}v\big) = \nabla\cdot\Big[2\Re\big(\mathbf{Z}\cdot\overline{\nabla v}\,\nabla v\big) + \big(k^2|v|^2 - |\nabla v|^2\big)\mathbf{Z}\Big] + (\nabla\cdot\mathbf{Z})\,\big(|\nabla v|^2 - k^2|v|^2\big) - 2\Re\big(\partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v}\big), \tag{2.4}$$

$$2\Re\big(\mathrm{i}k\beta\overline{v}\,\mathcal{L}v\big) = \nabla\cdot\big[2\Re\big(\mathrm{i}k\beta\overline{v}\,\nabla v\big)\big] - 2\Re\big(\mathrm{i}k\,\overline{v}\,\nabla\beta\cdot\nabla v\big), \tag{2.5}$$

and

$$2\Re\big(\alpha\overline{v}\,\mathcal{L}v\big) = \nabla\cdot\big[2\Re(\alpha\overline{v}\,\nabla v)\big] + 2\alpha\big(k^2|v|^2 - |\nabla v|^2\big) - 2\Re\big(\overline{v}\,\nabla\alpha\cdot\nabla v\big). \tag{2.6}$$

To prove (2.5) and (2.6), expand the divergences on the right-hand sides (remembering that α and β are real). The basic ingredient of (2.4) is the identity

$$\overline{\mathbf{Z}\cdot\nabla v}\,\Delta v = \nabla\cdot\big[\mathbf{Z}\cdot\overline{\nabla v}\,\nabla v\big] - \partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v} - \nabla v\cdot(\mathbf{Z}\cdot\nabla)\overline{\nabla v}; \tag{2.7}$$

to prove this, expand the divergence on the right-hand side and use the fact that the second derivatives of v commute. We would like each term on the right-hand side of (2.7) to either be single-signed or be the divergence of something. We cannot do anything at this stage about the $\partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v}$ term (and making this single-signed will be one of the key requirements later). To deal with the final term we use the identity

$$2\Re\big(\nabla v\cdot(\mathbf{Z}\cdot\nabla)\,\overline{\nabla v}\big) = \nabla\cdot\big[|\nabla v|^2\mathbf{Z}\big] - (\nabla\cdot\mathbf{Z})|\nabla v|^2 \tag{2.8}$$

(which can be proved by expanding the divergence on the right-hand side). Indeed, taking twice the real part of (2.7) and using (2.8) yields

$$2\Re\big(\mathbf{Z}\cdot\overline{\nabla v}\,\Delta v\big) = \nabla\cdot\Big[2\Re\big(\mathbf{Z}\cdot\overline{\nabla v}\,\nabla v\big) - |\nabla v|^2\mathbf{Z}\Big] + (\nabla\cdot\mathbf{Z})\,|\nabla v|^2 - 2\Re\big(\partial_i\mathbf{Z}_j\partial_i v\overline{\partial_j v}\big). \tag{2.9}$$

Now add k^2 times

$$2\Re\big[v\,\mathbf{Z}\cdot\overline{\nabla v}\big] = \nabla\cdot\big[|v|^2\mathbf{Z}\big] - (\nabla\cdot\mathbf{Z})|v|^2$$

(which is the analogue of (2.8) with the vector ∇v replaced by the scalar v) to (2.9) to obtain (2.4). ∎

A particular special case of the identity (2.1) is obtained by taking $\mathbf{Z} = \mathbf{x}, \beta = r$, and α a constant. Then $\mathcal{Z}v = r\mathcal{M}_\alpha v$, where

$$\mathcal{M}_\alpha v := v_r - \mathrm{i}kv + \frac{\alpha}{r}v, \tag{2.10}$$

and (2.1) becomes the following identity.

Lemma 2.3 (Morawetz-Ludwig identity, [47, Equation 1.2]) *Let v and $\mathcal{L}v$ be as in Lemma 2.1. Define the operator $\mathcal{M}_\alpha$ by (2.10) where $\alpha \in \mathbb{R}$ and $v_r = \mathbf{x}\cdot\nabla v/r$. Then*

$$2\Re(r\overline{\mathcal{M}_\alpha v}\mathcal{L}v) = \nabla\cdot\Big[2\Re\left(r\overline{\mathcal{M}_\alpha v}\nabla v\right) + \left(k^2|v|^2 - |\nabla v|^2\right)\mathbf{x}\Big]$$
$$+\big(2\alpha-(d-1)\big)\big(k^2|v|^2-|\nabla v|^2\big) - \big(|\nabla v|^2 - |v_r|^2\big) - \Big|\mathcal{M}_\alpha v - \frac{\alpha}{r}v\Big|^2. \qquad (2.11)$$

Proof. To see that the non-divergence terms of (2.1) and (2.11) are equivalent when $\mathbf{Z} = \mathbf{x}$, $\beta = r$ and α is a constant, note that in this case $\mathbf{Z} = \beta\nabla\beta$, and thus one can express $2\Re(ik\,\overline{v}\,\nabla\beta\cdot\nabla v)$ in terms of $|\mathcal{Z}v - \alpha v|^2/\beta^2$. ■

As discussed in §1.5, the Morawetz-Ludwig identity (2.11) has two important features:

1. If $\alpha = (d-1)/2$ then all the non-divergence terms on the right-hand side are ≤ 0.
2. If v is a solution of the Helmholtz equation outside a ball of radius R_0 satisfying the Sommerfeld radiation condition (1.2), then when (2.11) is integrated over $\{R_0 \leq |\mathbf{x}| \leq R\}$ the surface integral on $|\mathbf{x}| = R$ tends to zero as $R \to \infty$ (independently of the value of α in $\mathcal{M}_\alpha$) [47, Proof of Lemma 5], [58, Lemma 2.4].

When we apply the identities Morawetz 1 (2.1) and Morawetz 2 (2.3) in $\Omega_+ \cap B_R$ we also want the non-divergence terms to be ≤ 0 and for there to be no contribution from the surface integral at infinity. One way to ensure the latter condition is to make $\mathbf{Z} = \mathbf{x}, \beta = r$, and $2\alpha = (d-1)$ when $r \geq R_0$ for some $R_0 > 0$. In fact, the next lemma implies that there is no contribution from infinity when $\mathbf{Z} = \mathbf{x}$, $\beta = C_1 r$, and $2\alpha = C_2$ for $C_1, C_2 \geq 1$, which gives us a bit more flexibility.

Lemma 2.4 (Inequality on ∂B_R used to deal with the contribution from infinity) *Let u be a solution of the homogeneous Helmholtz equation in $\mathbb{R}^d \setminus B_{R_0}$, for some $R_0 > 0$, satisfying the Sommerfeld radiation condition. If C_1 and C_2 are both constants ≥ 1, then, for $R > R_0$,*

$$\int_{\partial B_R} R\left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 - |\nabla_S u|^2\right)\mathrm{d}s - 2C_1 kR\,\Im\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s + C_2\,\Re\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \leq 0, \quad (2.12)$$

where ∇_S is the surface gradient on $r = R$ (recall that this is such that $\nabla v = \nabla_S v + \widehat{\mathbf{x}}v_r$ on $r = R$).

Sketch proof including references. The inequality (2.12) follows from the combining the following three inequalities:

$$\Re\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \leq 0, \quad \Im\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \geq 0, \qquad (2.13)$$

and

$$\int_{\partial B_R} R\left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 - |\nabla_S u|^2\right)\mathrm{d}s - 2kR\,\Im\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s + \Re\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s \leq 0. \qquad (2.14)$$

The two inequalities (2.13) are well known but (2.14) not so. All three can be proved using the explicit expression for the solution of the Helmholtz equation in the exterior of a ball (i.e. an expansion in either trigonometric polynomials, for $d = 2$, or spherical harmonics, for $d = 3$, with coefficients given in terms of Bessel and Hankel functions) and then proving bounds on the particular combinations of Bessel and Hankel functions. For proofs of (2.13) via this method see [50, Theorems 2.6.1 and 2.6.4] or [11, Lemma 2.1], with the latter reference also proving (2.14). (Note that the second inequality in (2.13) can also be obtained from applying Green's identity in $\mathbb{R}^d \setminus B_R$ and using the second equation in (1.28).)

The Morawetz-Ludwig identity (2.11) can be used to prove the inequality (2.14) for $d = 2$, and a slightly weaker inequality for $d = 3$. Indeed, integrating (2.11) with $v = u$ and $2\alpha = d-1$ over $B_{R_1} \setminus B_R$, using the divergence theorem, and then letting $R_1 \to \infty$ (using the fact mentioned above that the surface integral on $|\mathbf{x}| = R_1$ tends to zero as $R_1 \to \infty$ [58, Lemma 2.4]), yields

$$\int_{\partial B_R} R\left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 - |\nabla_S u|^2\right)\mathrm{d}s - 2kR\,\Im\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s + (d-1)\Re\int_{\partial B_R}\bar{u}\frac{\partial u}{\partial r}\,\mathrm{d}s$$

$$= -\int_{\mathbb{R}^d\setminus B_R}\left(\left(|\nabla v|^2 - |v_r|^2\right) + \left|\mathcal{M}_{(d-1)/2}v - \frac{d-1}{2r}v\right|^2\right)\mathrm{d}\mathbf{x} \leq 0. \tag{2.15}$$

By looking at the coefficient of the final term on the left-hand side, we see that the inequality (2.15) is weaker than (2.14) when $d = 3$. (See [10, §5.3.1] for more discussion on both the inequality (2.14) and its proof in [11, Lemma 2.1].) ∎

In what follows we need the identities Morawetz 1 and Morawetz 2 integrated over domains. We make these into lemmas here in order to keep track of how smooth $\mathbf{Z}, \beta, \alpha$, and u need to be (later we make α a function of $\mathbf{Z}$, so outlining these conditions now will be helpful).

Lemma 2.5 (Integrated version of Morawetz 1) *Let D be a bounded Lipschitz domain with outward-pointing unit normal vector $\boldsymbol{\nu}$, and let $u \in C^2(D) \cap C^1(\overline{D})$ be a solution of the Helmholtz equation in D. If $\mathbf{Z} \in (C^1(D))^d \cap (C(\overline{D}))^d$, $\partial_i \mathbf{Z}_j \in L^1(D)$, for $i, j = 1, \dots, d$, $\beta \in C^1(D) \cap C(\overline{D})$, $\nabla\beta \in (L^1(D))^d$, $\alpha \in C^1(D) \cap C(\overline{D})$, and $\nabla\alpha \in (L^1(D))^d$, then*

$$\int_{\partial D}\left[2\Re\left(\overline{\mathcal{Z}u}\frac{\partial u}{\partial \nu}\right) + \left(k^2|u|^2 - |\nabla u|^2\right)(\mathbf{Z}\cdot\boldsymbol{\nu})\right]\mathrm{d}s$$
$$= \int_D\left(2\Re\big(\partial_i\mathbf{Z}_j\partial_i u\overline{\partial_j u}\big) + 2\Re\big(\overline{u}\,(\mathrm{i}k\nabla\beta + \nabla\alpha)\cdot\nabla u\big) - \big(2\alpha - \nabla\cdot\mathbf{Z}\big)\big(k^2|u|^2 - |\nabla u|^2\big)\right)\mathrm{d}\mathbf{x}. \tag{2.16}$$

Proof. The divergence theorem

$$\int_D \nabla\cdot\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial D}\mathbf{F}\cdot\boldsymbol{\nu}\,\mathrm{d}s \tag{2.17}$$

is valid if D is Lipschitz and $\mathbf{F} \in (C^1(\overline{D}))^d$ [40, Theorem 3.34]. Limiting arguments involving approximating either $\mathbf{F}$ or D show that (2.17) is in fact valid when

$$\mathbf{F} \in \left(C^1(D)\right)^d \cap \left(C(\overline{D})\right)^d \quad \text{and} \quad \nabla\cdot\mathbf{F} \in L^1(D). \tag{2.18}$$

When we apply the divergence theorem to the integrated Morawetz identity we take

$$\mathbf{F} = 2\Re\big((\mathbf{Z}\cdot\overline{\nabla u} + \mathrm{i}k\beta\overline{u} + \alpha\overline{u})\nabla u\big) + \big(k^2|u|^2 - |\nabla u|^2\big)\mathbf{Z}. \tag{2.19}$$

Therefore, if the conditions on $\mathbf{Z}$, β, and α in the assertion hold, then (2.19) satisfies (2.18), and (2.16) follows from integrating (2.1) over D and applying (2.17). ∎

Lemma 2.6 (Integrated version of Morawetz 2) *The integrated version of Morawetz 2 (2.3) holds if the conditions of Lemma 2.5 are satisfied and, in addition, $\alpha \in C^2(D) \cap C^1(\overline{D})$ and $\Delta\alpha \in L^1(D)$.*

Proof. Almost identical to that of Lemma 2.5. ∎

Remark 2.7 (Bibliographic remarks) *The multiplier $\mathbf{Z}\cdot\nabla v$ is associated with the name Rellich, due to Rellich's introduction of the multiplier $\mathbf{x}\cdot\nabla v$ for the Helmholtz equation in [52]. Rellich identities have been well-used in the study of the Laplace, Helmholtz, and other elliptic equations, see, e.g., the references in [10, §5.3], [44, §1.4].*

The idea of using a multiplier that is a linear combination of derivatives of v and v itself, such as $\mathcal{Z}v$, is attributed by Morawetz in [45] to Friedrichs. The multiplier $r\mathcal{M}_\alpha v$ for the Helmholtz equation was introduced by Morawetz and Ludwig in [47] and the multiplier $\mathcal{Z}v$ (2.2) is implicit in Morawetz's paper [46]. Indeed, using the multiplier $\mathcal{Z}v$ is discussed informally at the beginning of [46, §I.2], but the resulting identity (essentially equation (2.3)) is only written down with $\mathbf{Z} = \phi\nabla\chi$, $\beta = \phi\psi$, and $\alpha = \phi\Delta\chi/2$, for arbitrary χ and particular ϕ and ψ [46, Lemma 3]. Finally, we note that the multiplier $\mathbf{Z}\cdot\nabla v + \alpha v$ was independently introduced by Maz'ya for Laplace's equation in the context of linear water waves in [39] (see also [33, Equation 2.28]).

3 Formulation of coercivity in terms of conditions on the vector field $\mathbf{Z}$

The main goal of this section is to prove Theorem 3.2 below, which gives sufficient conditions for $A'_{k,\eta}$ to be coercive in terms of the existence of an appropriate vector field $\mathbf{Z}$. We begin by defining exactly what we mean by "coercivity" in this section.

Condition 3.1 (Coercivity) *There exists an $\eta_0 > 0$ such that, given δ, there exists a $k_0(\delta) > 0$ such that, for any $k \geq k_0$ and $\eta \geq \eta_0 k$,*

$$\Re\big(A'_{k,\eta}\phi, \phi\big)_{L^2(\Gamma)} \geq \left(\frac{1}{2} - \delta\right) \|\phi\|^2_{L^2(\Gamma)} \tag{3.1}$$

for all $\phi \in L^2(\Gamma)$.

3.1 Statement of the two different formulations of coercivity

In this subsection we give two sufficient conditions for coercivity: Condition A and Condition B below. These conditions concern the existence of certain vector fields $\mathbf{Z}$ defined in both Ω_- and $\Omega_+ \cap B_R$ for some sufficiently large R (recall that $B_R := \{|\mathbf{x}| < R\}$). The two conditions are similar except that Condition B demands higher smoothness of $\mathbf{Z}$ (and thus ultimately of Γ) in exchange for a slightly less restrictive condition on $\partial_i \mathbf{Z}_j$ in the domain. We show in Section 4 below that Condition A is satisfied when Ω_- is a uniformly convex, 2- or 3-d domain that is C^3 and piecewise analytic. The advantage of Condition B is that it is closer to the vector-field condition obtained by Morawetz in [46, Equation 1.3] (see also [48, Equation 4.2] and [61, Equations 2–4]) for bounding the energy norm of the solution of the Helmholtz exterior Dirichlet problem (which can then be used to prove local energy decay of the wave equation).

Condition A (Concerning the vector field associated with Morawetz 1 (2.1))
Γ is C^2, there exists a constant R with $\overline{\Omega_-} \subset B_R$, a vector field $\mathbf{Z} : B_R \to \mathbb{R}^d$, and a constant $C_\Gamma > 0$ such that the following hold:

A1. *$\mathbf{Z}$ is piecewise C^2 up to the boundary, i.e.,* $\mathbf{Z} \in \big(C^2(\overline{\Omega_-})\big)^d \cap \big(C^2(\overline{\Omega_+ \cap B_R})\big)^d$.

A2. $\mathbf{Z}_+ = \mathbf{Z}_- = C_\Gamma \mathbf{n}$ *and* $(\nabla \cdot \mathbf{Z})_+ = (\nabla \cdot \mathbf{Z})_-$ *on* Γ.

A3. $\mathbf{Z} = \mathbf{x}$ *in a neighbourhood of* ∂B_R.

A4. *There exists a $\theta > 0$ such that* $\Re\big(\partial_i \mathbf{Z}_j(\mathbf{x}) \boldsymbol{\xi}_i \overline{\boldsymbol{\xi}_j}\big) \geq \theta |\boldsymbol{\xi}|^2$ *for all* $\boldsymbol{\xi} \in \mathbb{C}^d$ *and* $\mathbf{x} \in \Omega_- \cup (\Omega_+ \cap B_R)$.

(Note that both here and in the rest of the paper, we use $+$ and $-$ subscripts to denote the limit of a function, here $\mathbf{Z}(\mathbf{x})$, as $\mathbf{x} \to \Gamma$ from Ω_+ and Ω_- respectively.)

By using the identity Morawetz 2, (2.3), instead of the identity Morawetz 1, (2.1), we can make Condition A4 less restrictive (i.e. $\partial_i \mathbf{Z}_j$ only needs to be nonnegative rather than uniformly positive) if Condition A1 is made more restrictive (i.e. increased smoothness of $\mathbf{Z}$).

Condition B (Concerning the vector field associated with Morawetz 2 (2.3))
Γ is C^2, there exists a constant R with $\overline{\Omega_-} \subset B_R$, a vector field $\mathbf{Z} : B_R \to \mathbb{R}^d$, and a constant $C_\Gamma > 0$ such that the following hold:

B1. *$\mathbf{Z}$ is piecewise C^3 up to the boundary, i.e.,* $\mathbf{Z} \in \big(C^3(\overline{\Omega_-})\big)^d \cap \big(C^3(\overline{\Omega_+ \cap B_R})\big)^d$.

B2. $\mathbf{Z}_+ = \mathbf{Z}_- = C_\Gamma \mathbf{n}$ *and* $(\nabla \cdot \mathbf{Z})_+ = (\nabla \cdot \mathbf{Z})_-$ *on* Γ.

B3. $\mathbf{Z} = \mathbf{x}$ *in a neighbourhood of* ∂B_R,

B4. $\Re\big(\partial_i \mathbf{Z}_j(\mathbf{x}) \boldsymbol{\xi}_i \overline{\boldsymbol{\xi}_j}\big) \geq 0$ *for all* $\boldsymbol{\xi} \in \mathbb{C}^d$ *and* $\mathbf{x} \in \Omega_- \cup (\Omega_+ \cap B_R)$.

The extra smoothness of $\mathbf{Z}$ in Condition B comes from the fact that, in formulating these conditions, the function α in the multiplier $\mathcal{Z}v$ is defined in terms of $\mathbf{Z}$ (it turns out to involve $\nabla\cdot\mathbf{Z}$). Therefore, if we use the identity Morawetz 2 (2.3) instead of Morawetz 1 (2.1), the additional smoothness of α needed for (2.3) to hold entails additional smoothness of $\mathbf{Z}$.

The next theorem shows how Conditions A and B (along with some constraints on the norm of the single-layer potential when $d = 3$) are sufficient for coercivity.

Theorem 3.2 (Sufficient conditions for coercivity) *Coercivity (i.e. Condition 3.1) holds if one of the following four criteria is met.*

1. *$d = 2$, Condition A holds.*
2. *$d = 3$, Condition A holds and $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = o(1)$ as $k \to \infty$.*
3. *$d = 2$, Condition B holds.*
4. *$d = 3$, Condition B holds and $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = o(1)$ as $k \to \infty$.*

We prove this theorem in §3.2 below, but first we make some remarks.

Remark 3.3 (Asymptotics of the coercivity constant) *Theorem 3.2 gives sufficient conditions for coercivity (in the sense of Condition 3.1) to hold, however it is also interesting to then ask how the coercivity constant depends on k.*

The proof of Theorem 3.2 below shows that if Condition A holds then there exist $\eta_0 > 0$ and $k_1 > 0$ such that, if $k \geq k_1$ and $\eta \geq \eta_0$ then $A'_{k,\eta}$ is coercive (i.e. (1.1) holds) with

$$\alpha_{k,\eta} \geq \frac{1}{2} - \mathcal{O}\Big(\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\Big). \tag{3.2}$$

Similarly, the proof of Theorem 3.2 shows that if Condition B holds then the asymptotics (3.2) hold with $-\mathcal{O}\big(\|\chi\mathcal{S}_k\|^2_{L^2(\Gamma)\to L^2(\mathbb{R}^d)}\big)$ and $-\mathcal{O}\big(\|S_k\|^2_{L^2(\Gamma)\to L^2(\Gamma)}\big)$ added to the right-hand side.

The previous remark shows us that, in order to prove coercivity via this method, we need to have $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ tending to zero as $k \to \infty$ (and if we use Condition B then we also need $\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)}$ tending to zero). The follow theorem recaps bounds on these two quantities, which we then use in the proofs of Theorems 3.2 and 1.2.

Theorem 3.4 (Bounds on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)}$)
1. If Ω_- is a bounded Lipschitz domain in 2- or 3-d and $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^d)$ then, given $k_0 > 0$,

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-3)/2} \quad \text{and} \quad \|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} \lesssim k^{-1/2} \tag{3.3}$$

for all $k \geq k_0$.
2. If Ω_- is a C^2, uniformly convex, 3-d domain then, given $k_0 > 0$,

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \frac{(\log k)^{1/2}}{k^{1/13}} \tag{3.4}$$

for all $k \geq k_0$.

Proof. The first bound in (3.3) was proved in [9, Theorem 3.3] and the second bound was proved in [57, Lemma 4.3]. The bound (3.4) was proved in [56, Theorem 1.5]. ∎

Remark 3.5 (Smoothness of Γ and $\mathbf{Z}$) *To keep things simple, we have assumed in Conditions A and B that Γ is C^2. The Conditions A2 and B2 then imply that Γ must additionally be C^3. Indeed, if $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ and $\mathbf{Z}$ is piecewise C^2 up to the boundary, then $\mathbf{n}$ must be C^2, which implies that Γ must be C^3.*

An important feature of Rellich and Morawetz identities is that they can be applied when Γ is Lipschitz, but this requires extra technicalities such as the notion of non-tangential limits and, when $v = \mathcal{S}_k\phi$ for $\phi \in L^2(\Gamma)$, harmonic analysis results about the single-layer potential (see [58, Remark 4.7] and [10, Theorem 2.16] and the references therein). The paper [12] goes through the argument of Theorem 3.2 when Γ is Lipschitz and shows that requiring $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ means that Γ must be at least $C^{2,1}$ (so we have not lost much here by avoiding these technicalities).

Remark 3.6 (A third set of conditions for coercivity) *As discussed above, in the proof of Theorem 3.2 below, the scalar function α in the multiplier $\mathcal{Z}v$ involves $\nabla\cdot\mathbf{Z}$. The difference in smoothness of $\mathbf{Z}$ in the two conditions A and B is then due to the fact that Condition A uses the identity Morawetz 1 (2.1), which needs $\alpha\in C^1$, whereas Condition B uses Morawetz 2 (2.3), which needs $\alpha\in C^2$.*

There is an additional set of conditions for coercivity that arise from letting α be a constant. In this case $\mathbf{Z}$ need only be C^1 (and thus, from Remark 3.5, Γ need only be C^2), but these conditions are much more restrictive than Condition A, with $\nabla\cdot\mathbf{Z}$ needing to be bounded in terms of d and the constant θ in the positivity condition. The vector field $\mathbf{x}$ satisfies these conditions when Γ is a circle or sphere, but it is not at all clear whether they can be satisfied for more general domains.

Remark 3.7 (A modified integral operator) *The proof of Theorem 3.2 below shows that if the condition $\mathbf{Z}_+=\mathbf{Z}_-=C_\Gamma\mathbf{n}$ in B2 is replaced by $\mathbf{Z}_+=\mathbf{Z}_-$ and $\mathbf{Z}\cdot\mathbf{n}>0$ on Γ, this modified version of Condition B holds, and also $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}=o(1)$ as $k\to\infty$, then the integral operator*

$$A'_{k,\eta,\mathbf{Z}}:=(\mathbf{Z}\cdot\mathbf{n})\left(\frac{1}{2}I+D'_k\right)+\mathbf{Z}\cdot\nabla_\Gamma S_k-\mathrm{i}\eta S_k$$

is coercive for k sufficiently large. More precisely, we have that given $\delta>0$ there exists a $k_0>0$ such that if $\eta=kR+\mathrm{i}(\nabla\cdot\mathbf{Z})|_\Gamma/2$ then $A'_{k,\eta,\mathbf{Z}}$ is coercive for all $k\geq k_0$, with coercivity constant $\inf_{\mathbf{x}\in\Gamma}(\mathbf{Z}\cdot\mathbf{n})-\delta$. (Note that, firstly, the operator $A'_{k,\eta,\mathbf{Z}}$ can be used to solve the exterior Dirichlet problem for the Helmholtz equation, see [10, Theorem 2.36], and, secondly, if $\mathbf{Z}=\mathbf{x}$, then $A'_{k,\eta,\mathbf{Z}}$ becomes the star-combined operator, (1.33), of [58].) The vector field constructed by Morawetz, Ralston, and Strauss in [48, §4] satisfies this modified version of Condition B in Ω_+ if Ω_+ is nontrapping, but it is not clear how to construct a continuation of this vector field into Ω_- satisfying the nonnegativity condition B4.

3.2 Proof of Theorem 3.2

Proof of Theorem 3.2. We first prove Parts 1 and 2 (relating to Condition A) using Morawetz 1 (2.1), and then discuss the changes needed to prove Parts 3 and 4 (relating to Condition B) using Morawetz 2 (2.3).

Our strategy is to mimic the classical method of "transferring" the coercivity properties of the PDE formulation to the associated boundary integral operators (as discussed in §1.4), but with Green's identity (1.22) replaced by the identity Morawetz 1 (2.1). That is, we apply the integrated version of (2.1), namely (2.16), with v replaced by $u=\mathcal{S}_k\phi$ (with $\phi\in L^2(\Gamma)$), and D first equal to Ω_-, and then equal to $\Omega_+\cap B_R$. The multiplier in the identity (2.16) is given by (2.2) with $\mathbf{Z}$ the vector field in Condition A, $\beta=R$, and $2\alpha=(\nabla\cdot\mathbf{Z})-\theta$, where θ is the constant in Condition A4. As the proof develops, we see why we make these choices of β and α. We go through the majority of the proof without worrying about how smooth $\mathbf{Z}$ needs to be, and then return to this question at the end.

With the identity (2.1) written as $\nabla\cdot\mathbf{Q}=P$, integrating it over Ω_- and $\Omega_+\cap B_R$ yields

$$\int_\Gamma \mathbf{Q}_-\cdot\mathbf{n}\,\mathrm{d}s=\int_{\Omega_-}P\,\mathrm{d}\mathbf{x} \tag{3.5}$$

and

$$-\int_\Gamma \mathbf{Q}_+\cdot\mathbf{n}\,\mathrm{d}s+\int_{\partial B_R}Q_R\,\mathrm{d}s=\int_{\Omega_+\cap B_R}P\,\mathrm{d}\mathbf{x}, \tag{3.6}$$

where (remembering that $\mathcal{L}u=0$)

$$P=2\Re\big(\partial_i\mathbf{Z}_j\partial_i u\overline{\partial_j u}\big)-\big(2\alpha-\nabla\cdot\mathbf{Z}\big)\big(k^2|u|^2-|\nabla u|^2\big)+2\Re\big(\overline{u}(\mathrm{i}k\nabla\beta+\nabla\alpha)\cdot\nabla u\big), \tag{3.7}$$

$$\mathbf{Q}_\pm\cdot\mathbf{n}=(\mathbf{Z}_\pm\cdot\mathbf{n})\left(\left|\frac{\partial u_\pm}{\partial n}\right|^2+k^2|u_\pm|^2-|\nabla_\Gamma u_\pm|^2\right)+2\Re\left(\big(\mathbf{Z}_\pm\cdot\overline{\nabla_\Gamma u_\pm}+\mathrm{i}k\beta\overline{u_\pm}+\alpha\overline{u_\pm}\big)\frac{\partial u_\pm}{\partial n}\right)$$

for $\mathbf{x} \in \Gamma$ (where we have used that $\nabla u = \nabla_\Gamma u + \mathbf{n}\partial u/\partial n$ on Γ), and

$$Q_R = \mathbf{Q} \cdot \widehat{\mathbf{x}} = R\left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2|u|^2 - |\nabla_S u|^2\right) - 2k\beta\Im\left(\bar{u}\frac{\partial u}{\partial r}\right) + 2\alpha\Re\left(\bar{u}\frac{\partial u}{\partial r}\right)$$

for $\mathbf{x} \in \partial B_R$ (where we have used that $\mathbf{Z} = \mathbf{x}$ on ∂B_R, i.e. A3). Adding (3.5) and (3.6) yields

$$\int_\Gamma (\mathbf{Q}_- - \mathbf{Q}_+) \cdot \mathbf{n}\, \mathrm{d}s + \int_{\partial B_R} Q_R\, \mathrm{d}s = \int_{\Omega_-} P\, \mathrm{d}\mathbf{x} + \int_{\Omega_+ \cap B_R} P\, \mathrm{d}\mathbf{x}.$$

Dealing with the integral on ∂B_R. Using the inequality (2.12) from Lemma 2.4 we see that $\int_{\partial B_R} Q_R\, \mathrm{d}s \le 0$ if

$$\beta \ge R \quad \text{and} \quad 2\alpha \ge 1 \quad \text{on} \quad \partial B_R. \tag{3.8}$$

Since $\beta = R$, the first inequality is satisfied. Recall that we chose $2\alpha = (\nabla \cdot \mathbf{Z}) - \theta$. Since $\mathbf{Z} = \mathbf{x}$ in a neighbourhood of ∂B_R (Condition A3), $\nabla \cdot \mathbf{Z} = d$ in this neighbourhood, and thus the second inequality in (3.8) is satisfied if $\theta \le d - 1$. This is not restrictive, since if we have constructed a $\mathbf{Z}$ that satisfies the positivity condition A4 with a value of $\theta > d - 1$ we can just choose $\theta = d - 1$ for the remainder of this argument (we see later that all we need is $\theta > 0$). Therefore,

$$\int_\Gamma (\mathbf{Q}_- - \mathbf{Q}_+) \cdot \mathbf{n}\, \mathrm{d}s \ge \int_{\Omega_-} P\, \mathrm{d}\mathbf{x} + \int_{\Omega_+ \cap B_R} P\, \mathrm{d}\mathbf{x}. \tag{3.9}$$

Dealing with the integral on Γ. We now show that

$$\int_\Gamma (\mathbf{Q}_- - \mathbf{Q}_+) \cdot \mathbf{n}\, \mathrm{d}s = 2\Re\Big(\big(C_\Gamma D'_k - \mathrm{i}k\beta S_k + \alpha S_k\big)\phi, \phi\Big)_{L^2(\Gamma)}. \tag{3.10}$$

Indeed, we first note that, by Condition A2, $\mathbf{Z}_\pm = C_\Gamma \mathbf{n}$ and $(\nabla \cdot \mathbf{Z})_+ = (\nabla \cdot \mathbf{Z})_-$ on Γ. Therefore, $\mathbf{Z} \cdot \mathbf{n} = C_\Gamma$ and $\mathbf{Z} \cdot \nabla_\Gamma u = 0$ on Γ, and α is continuous across Γ. We next simplify $(\mathbf{Q}_- - \mathbf{Q}_+) \cdot \mathbf{n}$ using these facts along with the single-layer potential jump relations

$$u_\pm(\mathbf{x}) = S_k\phi(\mathbf{x}), \quad \nabla_\Gamma u_+(\mathbf{x}) = \nabla_\Gamma u_-(\mathbf{x}), \quad \frac{\partial u_\pm}{\partial n}(\mathbf{x}) = \left(\mp\frac{1}{2}I + D'_k\right)\phi(\mathbf{x}), \text{ for } \mathbf{x} \in \Gamma,$$

which are given for $\Gamma \in C^2$ by, e.g., [15, Theorems 2.12 and 2.17]. A key identity to help us do this is

$$\left|\frac{\partial u_-}{\partial n}(\mathbf{x})\right|^2 - \left|\frac{\partial u_+}{\partial n}(\mathbf{x})\right|^2 \quad = \quad 2\Re\big(D'_k\phi(\mathbf{x})\,\overline{\phi(\mathbf{x})}\big), \qquad \text{for } \mathbf{x} \in \Gamma,$$

which can be established using $|a|^2 - |b|^2 = \Re[(a+b)\overline{(a-b)}]$ and the jump relation for $\partial u_\pm/\partial n$.

Putting together the inequality (3.9) and the equality (3.10) yields

$$\Re\left(\left(D'_k - \mathrm{i}k\frac{R}{C_\Gamma}S_k + \frac{\alpha}{C_\Gamma}S_k\right)\phi, \phi\right)_{L^2(\Gamma)} \ge \frac{1}{2C_\Gamma}\left(\int_{\Omega_-} P\, \mathrm{d}\mathbf{x} + \int_{\Omega_+ \cap B_R} P\, \mathrm{d}\mathbf{x}\right). \tag{3.11}$$

The definition of $A'_{k,\eta}$, equation (1.9), implies that if we can show that

$$\Re\Big(\big(D'_k - \mathrm{i}\eta_0 k S_k\big)\phi, \phi\Big)_{L^2(\Gamma)} + o(1)\|\phi\|^2_{L^2(\Gamma)} \ge 0 \quad \text{as } k \to \infty, \tag{3.12}$$

then this establishes the inequality (3.1) in Condition 3.1 for $\eta = \eta_0 k$. Note that Condition 3.1 requires the inequality (3.1) to hold for $\eta \ge \eta_0 k$, and not just for $\eta = \eta_0 k$. However, the former case follows from the latter by first noting that

$$\Re\Big(\big(D'_k - \mathrm{i}\eta S_k\big)\phi, \phi\Big)_{L^2(\Gamma)} = \Re\Big(\big(D'_k - \mathrm{i}\eta_0 k S_k\big)\phi, \phi\Big)_{L^2(\Gamma)} + \Re\Big(-\mathrm{i}\big(\eta - \eta_0 k\big)S_k\phi, \phi\Big)_{L^2(\Gamma)},$$

and then using the fact that $\Re(-\mathrm{i}S_k\phi,\phi)_{L^2(\Gamma)} \geq 0$ from (1.30).

Choosing $\eta_0 = R/C_\Gamma$ we see that (3.11) gives us (3.12) if $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = o(1)$ and

$$\int_{\Omega_-} P\,\mathrm{d}\mathbf{x} + \int_{\Omega_+\cap B_R} P\,\mathrm{d}\mathbf{x} \geq -o(1)\|\phi\|^2_{L^2(\Gamma)} \quad \text{as } k\to\infty. \tag{3.13}$$

The decay $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = o(1)$ as $k\to\infty$ is given by the first bound in (3.3) when $d=2$, and is a hypothesis in the theorem when $d=3$.

Therefore, all that remains to prove coercivity (Condition 3.1) is to establish that the inequality (3.13) holds.

Dealing with the volume terms. We need to establish the inequality (3.13) with P given by (3.7). Using A4 (the positivity of $\partial_i\mathbf{Z}_j$), the fact that $2\alpha = \nabla\cdot\mathbf{Z} - \theta$, and also the fact that β is a constant, we have that

$$P \geq \theta\big(k^2|u|^2 + |\nabla u|^2\big) + 2\Re\big(\overline{u}\,\nabla\alpha\cdot\nabla u\big).$$

The inequality

$$2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon},$$

for all a, b, and $\varepsilon > 0$, implies that

$$2\left|\int_{\Omega_+\cap B_R} \overline{u}\,\nabla\alpha\cdot\nabla u\,\mathrm{d}\mathbf{x}\right| \leq \frac{\|\nabla\alpha\|_{L^\infty(\Omega_+\cap B_R)}}{k}\int_{\Omega_+\cap B_R}\big(|\nabla u|^2 + k^2|u|^2\big)\,\mathrm{d}\mathbf{x}, \tag{3.14}$$

(and similiarly for the integral over Ω_-). The bound (3.14) implies that choosing k large enough ensures that the left-hand side of (3.13) is ≥ 0; thus we have proved that $A'_{k,\eta}$ is coercive (Condition 3.1).

Smoothness of Z We now go back through the above argument and see what smoothness we need from $\mathbf{Z}$ (and this will give us the condition A1).

We first check the conditions on $u, \mathbf{Z}, \beta$, and α required by Lemma 2.5. A proof that $u = \mathcal{S}_k\phi$ is in $C^2(\Omega_\pm)\cap C^1(\overline{\Omega_\pm})$ when Γ is C^2 and $\phi\in L^2(\Gamma)$ is given in [15, Theorems 2.12 and 2.17] (this proof is for Hölder continuous ϕ, but since Hölder continous functions are dense in $L^2(\Gamma)$ this gives the result for $\phi\in L^2(\Gamma)$). Turning to the conditions on $\mathbf{Z}, \beta$, and α, those on β are satisfied since β is a constant. We need $\mathbf{Z}\in (C^1(\Omega_-))^d\cap (C(\overline{\Omega_-}))^d, \partial_i\mathbf{Z}_j \in L^1(\Omega_-)$ (and similarly in $\Omega_+\cap B_R$). Furthermore, the fact that $2\alpha = (\nabla\cdot\mathbf{Z}) - \theta$ means that we also need $\nabla\cdot\mathbf{Z}\in C^1(\Omega_-)\cap C(\overline{\Omega_-}), \nabla(\nabla\cdot\mathbf{Z})\in (L^1(\Omega_-))^d$ (and again in $\Omega_+\cap B_R$). If $\mathbf{Z}$ is piecewise C^2 up to the boundary (i.e. $\mathbf{Z}$ satisfies Condition A1) then all these conditions are satisfied.

After using Lemma 2.5 the proof needed (i) $\mathbf{Z}$ and α to be continuous across Γ, and (ii) $\nabla\alpha$ to be in both $L^\infty(\Omega_-)$ and $L^\infty(\Omega_+\cap B_R)$. Regarding (i): this leads to A2. It turns out that we could drop the restriction that α is continuous if we added the extra condition that $\|S_k\|\|D'_k\| = o(1)$ as $k\to\infty$ (to deal with the term on Γ resulting from the non-zero jump of α). However, at least in our construction of $\mathbf{Z}$ in §4, ensuring that $\nabla\cdot\mathbf{Z}$ is continuous across Γ is not the limiting factor, and so we retain the condition that α is continuous. Regarding (ii): this implies that we need $\nabla(\nabla\cdot\mathbf{Z})\in L^\infty$, which is ensured by $\mathbf{Z}$ being piecewise C^2 up to the boundary.

Changes to the above argument necessary to prove Parts 3 and 4. We now repeat the above argument using Morawetz 2 (2.3) instead of Morawetz 1 (2.1); the changes are as follows.

We choose $2\alpha = \nabla\cdot\mathbf{Z}$. To apply Lemma 2.6 (the analogue of Lemma 2.5 with Morawetz 1 replaced by Morawetz 2) we need $\alpha\in C^2(D)\cap C^1(\overline{D})$ and $\Delta\alpha\in (L^1(D))^d$; these conditions are satisfied if $\mathbf{Z}$ is piecewise C^3 up to the boundary (i.e. $\mathbf{Z}$ satisfies Condition B1). Similar to before, the fact that $\mathbf{Z}$ and α must be continuous across Γ leads to Condition B2.

Q_R now contains the extra term $-(\nabla\alpha\cdot\widehat{\mathbf{x}})|u|^2$. This is zero, however, since $2\alpha = \nabla\cdot\mathbf{Z} = d$ (i.e. a constant) in a neighbourhood of ∂B_R. The condition that $2\alpha\geq 1$ (necessary for controlling the integral on ∂B_R) is now satisfied automatically.

$\mathbf{Q}_\pm\cdot\mathbf{n}$ now contains the extra term $-(\partial\alpha_\pm/\partial n)|u_\pm|^2$. If we assume that $\nabla\alpha$ is continuous across Γ then this extra term does not contribute to (3.10) since there is no jump in u across Γ.

However, this would impose the extra condition that $\nabla(\nabla \cdot \mathbf{Z})$ is continuous across Γ. If we don't assume that $\nabla\alpha$ is continuous, then, if $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} = o(1)$ as $k \to \infty$, we obtain (3.10) with $o(1)\|\phi\|^2_{L^2(\Gamma)}$ added to the right-hand side. Since we assume this decay in $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ to go from (3.11) to (3.12) we choose this second option (i.e. $\nabla\alpha$ discontinous and no extra restriction on $\mathbf{Z}$).

Since we are using Morawetz 2 (2.3), P is now given by

$$P = 2\Re\big(\partial_i \mathbf{Z}_j \partial_i u \overline{\partial_j u}\big) - \big(2\alpha - \nabla\cdot\mathbf{Z}\big)\big(k^2|u|^2 - |\nabla u|^2\big) + 2\Re\big(\mathrm{i}k\,\overline{u}\,\nabla\beta\cdot\nabla u\big) - \Delta\alpha|u|^2.$$

Using A4, and the fact that $2\alpha = \nabla\cdot\mathbf{Z}$, we find that

$$P + \Delta\alpha|u|^2 \geq 0.$$

Taking the L^∞ norm of $\Delta\alpha$ out of the integrals (noting that $\mathbf{Z}$ being piecewise C^3 up to the boundary means that this is allowed) we see that, since $u = \mathcal{S}_k\phi$, the inequality needed for coercivity (3.13) will hold if $\|\chi\mathcal{S}_k\|_{L^2(\Gamma)\to L^2(\mathbb{R}^d)} = o(1)$ as $k\to\infty$, and this decay is ensured by the second bound in (3.3). ■

4 Construction of a vector field Z satisfying Condition A for uniformly convex, 2- and 3-d domains that are C^3 and piecewise analytic

This section proves the following result:

Lemma 4.1 *If Ω_- is a uniformly convex, 2- or 3-d domain with Γ both C^3 and piecewise analytic, then there exists a $\mathbf{Z}$ satisfying Condition A.*

The main result of this paper, Theorem 1.2, then follows by combining Lemma 4.1 with Parts 1 and 2 of Theorem 3.2 and Part 2 of Theorem 3.4. The asymptotics of $\alpha_{k,\eta}$ given in (1.15) then follow from using the first bound in (3.3) (for $d = 2$) and the bound (3.4) (for $d = 3$) in equation (3.2).

We first prove the result of Lemma 4.1 for the 2-d case (in §4.2-4.3), and then outline the small modifications needed to establish the result for the 3-d case (in §4.4).

4.1 Orthogonal curvilinear coordinates defined by Γ in 2-d

We are going to use the orthogonal curvilinear coordinate system defined by Γ and so it is convenient to recap some facts about this in an initial subsection. At this stage we only need that Ω_- is convex and Γ is C^2 (the conditions that Ω_- is uniformly convex and Γ is both C^3 and piecewise analytic will come later in connection with $\mathbf{Z}$).

Coordinate system in the exterior Let $\mathbf{r}_0(s)$ be the position vector of a point on Γ, parametrised by the arc length s. The fact that Γ is C^2 means that $\mathbf{r}_0(s)$ is C^2 as a function of s. Recall that $(\mathrm{d}\mathbf{r}_0/\mathrm{d}s)(s)$ is the unit tangent vector to Γ and denote the outward-pointing unit normal vector by $\mathbf{n}(s)$ (recall that this is proportional to $(\mathrm{d}^2\mathbf{r}_0/\mathrm{d}s^2)(s)$). Define the (signed) curvature $\kappa(s)$ by

$$\frac{\mathrm{d}^2\mathbf{r}_0}{\mathrm{d}s^2}(s) = -\kappa(s)\mathbf{n}(s), \tag{4.1}$$

and define κ_* and κ^* by

$$\kappa_* := \min_s \kappa(s) \qquad \text{and} \qquad \kappa^* := \max_s \kappa(s) \tag{4.2}$$

respectively. The fact that Ω_- is convex then implies that $\kappa_* \geq 0$. The fact that $\mathbf{n}$ is perpendicular to both $\mathrm{d}\mathbf{n}/\mathrm{d}s$ and $\mathrm{d}\mathbf{r}_0/\mathrm{d}s$ can then be used to show that

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}(s) = \kappa(s)\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s). \tag{4.3}$$

Given a point P in Ω_+, let $\mathbf{r}_0(s)$ be the position vector of the closest point on Γ to P (this closest point is unique since Ω_- is convex). The position vector of P, $\mathbf{r}$, can then be written as

$$\mathbf{r}(s,n) = \mathbf{r}_0(s) + n\,\mathbf{n}(s)$$

where $n := \operatorname{dist}(\mathbf{r},\Gamma)$.

The basis vectors in the (n,s)-coordinate system, $\mathbf{e}_n^+$ and $\mathbf{e}_s$ (where we use the + superscript on $\mathbf{e}_n$ to emphasise that we are in Ω_+) are then defined by

$$\mathbf{e}_n^+(n,s) := \frac{\partial \mathbf{r}}{\partial n}(n,s) = \mathbf{n}(s),$$

and

$$\begin{aligned}\mathbf{e}_s(n,s) := \frac{\partial \mathbf{r}}{\partial s}(n,s) &= \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s) + n\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}(s),\\ &= \big(1+n\,\kappa(s)\big)\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s) \quad \text{by (4.3)}.\end{aligned}$$

The scale factors, h_n and h_s, are then

$$h_n(n,s) := |\mathbf{e}_n^+(n,s)| = 1 \quad \text{and} \quad h_s(n,s) := |\mathbf{e}_s(n,s)| = 1 + n\,\kappa(s),$$

and thus

$$\widehat{\mathbf{e}}_n^+(n,s) := \frac{1}{h_n(n,s)}\mathbf{e}_n^+(n,s) = \mathbf{n}(s) \quad \text{and} \quad \widehat{\mathbf{e}}_s(n,s) := \frac{1}{h_s(n,s)}\mathbf{e}_s(n,s) = \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s).$$

The (n,s)-coordinate system with basis vectors $\mathbf{e}_n^+$ and $\mathbf{e}_s$ is orthogonal and, given a vector $\mathbf{v}$, we write

$$\mathbf{v} = v^n\mathbf{e}_n^+ + v^s\mathbf{e}_s.$$

If $\psi : \Omega_+ \to \mathbb{R}$ is differentiable then

$$\nabla\psi = \frac{1}{h_n}\frac{\partial\psi}{\partial n}\widehat{\mathbf{e}}_n^+ + \frac{1}{h_s}\frac{\partial\psi}{\partial s}\widehat{\mathbf{e}}_s = \frac{\partial\psi}{\partial n}\widehat{\mathbf{e}}_n^+ + \frac{1}{1+n\,\kappa(s)}\frac{\partial\psi}{\partial s}\widehat{\mathbf{e}}_s. \tag{4.4}$$

If $\mathbf{v}$ is a differentiable vector field in general curvilinear coordinates, u^i, with basis $\mathbf{e}_i := \partial\mathbf{r}/\partial u^i$, then

$$\left(\frac{\partial \mathbf{v}}{\partial u^j}\right)^i = \frac{\partial v^i}{\partial u^j} + \Gamma^i_{kj}v^k, \tag{4.5}$$

where Γ^i_{kj} are the Christoffel symbols; see, e.g., [53, Equation 21.85]. It is straightforward to check that the derivative of the vector $\mathbf{v}$ as a linear map from $\mathbb{R}^d$ with basis $\{\mathbf{e}_i\}$ to itself is given by $(D\mathbf{v})_{ij} = (\partial\mathbf{v}/\partial u^j)^i$. In what follows we consider vector fields, $\mathbf{v} : \Omega_+ \to \mathbb{R}^d$, with $v^s = 0$ and v^n a function of n only. For such vectors, after calculating the Christoffel symbols in (4.5) (using the fact that h_n is constant), we find that

$$\frac{\partial\mathbf{v}}{\partial n} = \frac{\partial v^n}{\partial n}\mathbf{e}_n^+, \quad \text{and} \quad \frac{\partial\mathbf{v}}{\partial s} = \frac{v^n}{h_s}\frac{\partial h_s}{\partial n}\mathbf{e}_s.$$

The derivative of the vector $\mathbf{v}$, as a linear map from $\mathbb{R}^2$ with basis $\{\mathbf{e}_n^+, \mathbf{e}_s\}$ to itself, is then

$$D\mathbf{v} = \begin{pmatrix} \frac{\partial v^n}{\partial n} & 0 \\ 0 & \frac{v^n}{h_s}\frac{\partial h_s}{\partial n}\end{pmatrix} = \begin{pmatrix} \frac{\partial v^n}{\partial n} & 0 \\ 0 & \frac{v^n\,\kappa(s)}{1+n\,\kappa(s)}\end{pmatrix}. \tag{4.6}$$

The vector field $\mathbf{Z}$ that we construct below to satisfy Condition A will be of the above form (i.e. $Z^s = 0$ and Z^n is only a function of n). To verify the positivity condition that $\Re(\partial_i \mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\overline{\boldsymbol{\xi}_j}) \geq \theta|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{C}^d$ and $\mathbf{x} \in \Omega_+ \cap B_R$, we claim that it is sufficient to prove that the matrix $D\mathbf{Z}$ (defined by the analogue of (4.6)) is $\geq \theta$ (in the sense of quadratic forms) for all n and s. Indeed, $D\mathbf{Z}$ defined by the analogue of (4.6) is the derivative of $\mathbf{Z}$ as a linear map both from $\mathbb{C}^d$ to $\mathbb{C}^d$ with basis $\{\mathbf{e}_n^+, \mathbf{e}_s\}$ and from $\mathbb{C}^d$ to $\mathbb{C}^d$ with basis $\{\widehat{\mathbf{e}}_n^+, \widehat{\mathbf{e}}_s\}$ (this is a consequence of the matrix being diagonal and the facts that $\mathbf{e}_n^+ = h_n\widehat{\mathbf{e}}_n^+$ and $\mathbf{e}_s = h_s\widehat{\mathbf{e}}_s$). Now, given an $\mathbf{x} \in \Omega_+ \cap B_R$, there exist n_1, s_1 such that $\mathbf{x} = (n_1, s_1)$ in the (n,s)-coordinate system defined by Γ. Since $\{\widehat{\mathbf{e}}_n^+(n_1,s_1), \widehat{\mathbf{e}}_s(n_1,s_1)\}$ form an orthonormal basis, there exists an orthogonal matrix $\mathbf{B}$ such that $(\mathbf{B}^T(D\mathbf{Z})(n_1,s_1)\mathbf{B})_{ij} = \partial_i\mathbf{Z}_j(\mathbf{x})$. It then follows that if $D\mathbf{Z}(n_1,s_1) \geq \theta$ then $\Re(\partial_i\mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\overline{\boldsymbol{\xi}_j}) \geq \theta|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{C}^d$.

Coordinate system in the interior Given a point P in Ω_- that has a unique closest point on Γ, let $\mathbf{r}_0(s)$ be the position vector of the closest point. (The set of points in Ω_- that do not have a unique closest point on Γ is called the *medial axis*, and we discuss this set below.) The position vector of P, $\mathbf{r}$, can then be written as

$$\mathbf{r}(s,n) = \mathbf{r}_0(s) - n\,\mathbf{n}(s)$$

where again $n = \mathrm{dist}(\mathbf{r},\Gamma)$. Proceeding in a similar manner to the exterior case, we have that

$$\mathbf{e}_n^-(n,s) = -\mathbf{n}(s) \quad \text{and} \quad \mathbf{e}_s(n,s) = \big(1 - n\,\kappa(s)\big)\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s).$$

Therefore,

$$h_n(n,s) = 1, \quad h_s(n,s) = 1 - n\,\kappa(s), \tag{4.7}$$

$\widehat{\mathbf{e}}_n^-(n,s) = -\mathbf{n}(s)$ and $\widehat{\mathbf{e}}_s(n,s) = (\mathrm{d}\mathbf{r}_0/\mathrm{d}s)(s)$. Equations analogous to (4.4) and (4.6) hold for the derivatives of scalar and vector fields.

For a given s, this coordinate system breaks down when $n = 1/\kappa(s)$, and thus the bounds on κ (4.2) imply that the earliest breakdown is at $n = 1/\kappa^*$. This corresponds to reaching an interior point that does not have a unique nearest point on Γ.

Following the notation in [13, §2.1], given $\mathbf{x} \in \Omega_-$, let

$$\mathcal{B}(\mathbf{x}) := \big\{\mathbf{y} \in \Gamma : |\mathbf{x}-\mathbf{y}| = \mathrm{dist}(\mathbf{x},\Gamma)\big\},$$

and let the *medial axis*, $\mathcal{M}_{\Omega_-}$, be defined by

$$\mathcal{M}_{\Omega_-} := \big\{\mathbf{x} \in \Omega_- : \mathrm{card}\,\mathcal{B}(\mathbf{x}) \geq 2\big\}$$

(note that, with this definition, the medial axis is not closed, and the closure of the medial axis is called the *cut locus*). Since $\mathrm{dist}(\mathbf{x},\Gamma)$ is differentiable at $\mathbf{x} \in \Omega_-$ if and only if $\mathrm{card}\,\mathcal{B}(\mathbf{x}) = 1$ [24, Theorem 3.3], $\mathcal{M}_{\Omega_-}$ is the set of points at which $\mathrm{dist}(\mathbf{x},\Gamma)$ is not differentiable.

There are several, slightly different, notions in the literature that go by the names of the medial axis or ridge. For example, the definition of the ridge in [24, Definition 3.6] allows it to contain points with $\mathrm{card}\,\mathcal{B}(\mathbf{x}) = 1$, and the definition of the ridge used by [35] is $\overline{\mathcal{M}_{\Omega_-}}$ in our notation.

The following theorem collects some geometric properties of $\mathcal{M}_{\Omega_-}$ that we need later.

Theorem 4.2 (Properties of the medial axis in 2-d)

(i) If Ω_- is a bounded, 2-d domain such that Γ is piecewise analytic (i.e. the finite union of analytic curves), then $\mathcal{M}_{\Omega_-}$ is a connected geometric graph with finitely many vertices and edges, and each edge is an analytic curve.

(ii) If Ω_- is as in (i) and is also simply connected, then $\mathcal{M}_{\Omega_-}$ is a tree.

(ii) If Ω_- is as in (i) and is also C^2, then there exists a constant $0 < n_0 \leq 1/\kappa^*$ such that $\mathrm{dist}(\mathcal{M}_{\Omega_-},\Gamma) \geq n_0$.

Proof. (i) This is proved in [14, Theorem 8.2], [37, Theorem 5.6], and [13, Theorem 2.1 and Corollary 2.1]. (ii) This is a consequence of the main result in [36].

(iii) If U is a bounded open set, then $\partial U \in C^k$ implies that $\mathrm{dist}(\cdot,\partial U)$ is C^k in a neighbourhood of ∂U for $k \geq 2$ [27, Lemma 14.16, Page 355], [25]. Therefore, $\mathrm{dist}(\mathbf{x},\Gamma)$ is differentiable in a neighbourhood of Γ, and then, since $\mathcal{M}_{\Omega_-}$ has finitely many vertices and edges (by (i)), $\mathrm{dist}(\mathcal{M}_{\Omega_-},\Gamma)$ is bounded below by a positive constant, which we denote by n_0. The inequality $n_0 \leq 1/\kappa^*$ follows from the facts that the osculating circle to a point on the boundary has radius $1/\kappa(s)$, and centres of osculating circles are in $\overline{\mathcal{M}_{\Omega_-}}$ [8, Lemma 2.2]. ■

Counterexamples to point (i) in the theorem above when Γ is only C^∞ and not analytic can be found in [14, §2], and an example of a $C^{1,1}$, convex domain such that $\overline{\mathcal{M}_{\Omega_-}}$ has positive Lebesgue 2-measure can be found in [37, §3]. These examples demonstrate how the "nice" behaviour of $\mathcal{M}_{\Omega_-}$ under piecewise analyticity can disappear for domains that are only C^∞.

4.2 Definition of a $\mathbf{Z}$ satisfying Condition A

For a fixed $R > 0$, we construct a $\phi : \Omega_- \cup (\Omega_+ \cap B_R) \to \mathbb{R}$ and then let $\mathbf{Z} = \nabla\phi$. (Note that we always assume that the origin from which B_R is defined is inside Ω_-.)

Under the assumption that $\mathbf{Z} = \nabla\phi$, the requirements of Condition A become

A1. ϕ is piecewise C^3 up to the boundary, i.e. $\phi \in C^3(\overline{\Omega_-}) \cap C^3(\overline{\Omega_+ \cap B_R})$.

A2. $(\nabla\phi)_+ = (\nabla\phi)_- = C_\Gamma \mathbf{n}$ and $(\Delta\phi)_+ = (\Delta\phi)_-$ on Γ.

A3. $\phi = \frac{1}{2}r^2$ in a neighbourhood of ∂B_R.

A4. There exists a $\theta > 0$ such that $D^2\phi(\mathbf{x}) \geq \theta$ (in the sense of quadratic forms) for all $\mathbf{x} \in \Omega_- \cup (\Omega_+ \cap B_R)$, where $(D^2\phi)_{ij} = \partial_i\partial_j\phi$. (Note that we have lost the $\Re$ that was in front of the original condition in terms of $\mathbf{Z}$ since ϕ is real and $D^2\phi$ is symmetric.)

Let ϕ be defined piecewise by $\phi := \phi^+$ in Ω_+ and $\phi := \phi^-$ in Ω_-. The overview of how $\phi^\pm$ are defined is as follows:

$$\phi^+ \text{is a smooth transition between} \left\{ \begin{array}{ll} \phi_{\rm ML}, & \text{which satisfies the requirement A3 on } \partial B_R, \text{ and} \\ \phi_\Gamma^+, & \text{which satisfies the requirement A2 on } \Gamma. \end{array} \right.$$

$$\phi^- \text{is a smooth transition between} \left\{ \begin{array}{ll} \phi_\Gamma^-, & \text{which satisfies the requirement A2 on } \Gamma, \text{ and} \\ \phi_\varepsilon, & \text{which is } \phi_\Gamma^- \text{ smoothed near } \mathcal{M}_{\Omega_-}. \end{array} \right.$$

The functions $\phi_{\rm ML}, \phi_\Gamma^+, \phi_\Gamma^-$, and ϕ_ε are all uniformly convex, and from this we are able to ensure that the positivity condition A4 on $D^2\phi$ is satisfied. Indeed, ϕ defined below depends on two parameters, $\mathcal{R}$ and ε ($\mathcal{R}$ is not quite R, the radius of B_R, but is closely related). We show in §4.3 below that A4 is satisfied if $\mathcal{R}$ is large enough and ε is small enough, and that taking $\mathcal{R}$ large enough is equivalent to taking R large enough.

Definition of ϕ^+. Let $n(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \Gamma)$ and let $\chi(n) \in C^\infty[0,\infty)$ be monotonically decreasing, equal to 1 in a neighbourhood of $n = 0$, equal to 0 in a neighbourhood of $n = 1$, and then identically zero for $n \geq 1$. For a fixed $\mathcal{R} > 0$, define $\chi_\mathcal{R}(n) = \chi(n/\mathcal{R})$.

Define ϕ^+ in terms of two other functions, ϕ_Γ^+ and $\phi_{\rm ML}$, by

$$\phi^+(\mathbf{x}) := \chi_\mathcal{R}\big(n(\mathbf{x})\big)\,\phi_\Gamma^+(\mathbf{x}) + \Big(1 - \chi_\mathcal{R}\big(n(\mathbf{x})\big)\Big)\phi_{\rm ML}(\mathbf{x}), \quad \mathbf{x} \in \Omega_+. \tag{4.8}$$

The function ϕ_Γ^+ is defined by

$$\phi_\Gamma^+(\mathbf{x}) := C_\Gamma n(\mathbf{x}) + \frac{1}{2}n(\mathbf{x})^2, \tag{4.9}$$

where $C_\Gamma = 1/\kappa_*$ (recall that Ω_- being uniformly convex implies that $\kappa_* > 0$). The function $\phi_{\rm ML}$ is defined by

$$\phi_{\rm ML}(\mathbf{x}) := \frac{1}{2}r^2 \tag{4.10}$$

where $r := |\mathbf{x}|$. (The subscript ML stands for "Morawetz-Ludwig", since the gradient of $\frac{1}{2}r^2$ is the vector field $\mathbf{x}$ that appears in the Morawetz-Ludwig identity (2.11).)

Definition of ϕ_-. Let n_0 be as in Theorem 4.2 (i.e. $n(\mathbf{x}) = \mathrm{dist}(\mathbf{x}, \Gamma)$ is differentiable when $0 < n < n_0$). Let $\chi_-(n) \in C^\infty[0,\infty)$ be monotonically decreasing, equal to one for $n \in [0, n_0/3]$, equal to zero for $n \in [2n_0/3, \infty)$, and such that all its derivatives are zero at $n = n_0/3$ and $n = 2n_0/3$.

Define ϕ_- in terms of ϕ_Γ^- and ϕ_ε by

$$\phi^-(\mathbf{x}) := \chi_-\big(n(\mathbf{x})\big)\,\phi_\Gamma^-(\mathbf{x}) + \Big(1 - \chi_-\big(n(\mathbf{x})\big)\Big)\phi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \Omega_- \tag{4.11}$$

(note that the definition of χ_- implies that $\phi^- = \phi_\Gamma^-$ for $0 \leq n \leq n_0/3$, and $\phi^- = \phi_\varepsilon$ for $n \geq 2n_0/3$).

The function ϕ_Γ^- is defined for all $\mathbf{x} \in \Omega_-$ by

$$\phi_\Gamma^-(\mathbf{x}) = -C_\Gamma n(\mathbf{x}) + \frac{1}{2}n(\mathbf{x})^2, \tag{4.12}$$

where (as above) $C_\Gamma = 1/\kappa_*$. To define ϕ_ε, first define the set D by

$$D := \big\{\mathbf{x} \in \Omega_- : \mathrm{dist}(\mathbf{x}, \Gamma) \geq n_0/3\big\} \tag{4.13}$$

and note that from the definitions of ϕ^- and χ_- we only need to define ϕ_ε on D. For $\mathbf{x} \in D$ and $\varepsilon < n_0/3$, $\phi_\varepsilon(\mathbf{x})$ is defined by

$$\phi_\varepsilon(\mathbf{x}) := \int_{B_\varepsilon(\mathbf{0})} \phi_\Gamma^-(\mathbf{x}-\mathbf{y})\eta_\varepsilon(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{B_\varepsilon(\mathbf{x})} \phi_\Gamma^-(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{4.14}$$

where (following, e.g., [23, §C.4])

$$\eta(\mathbf{x}) := \begin{cases} C\exp\left(\frac{1}{|\mathbf{x}|^2-1}\right) & \text{if } |\mathbf{x}| < 1, \\ 0 & \text{if } |\mathbf{x}| \geq 1, \end{cases}$$

C is selected so that $\int_{\mathbb{R}^d} \eta(\mathbf{x})\mathrm{d}\mathbf{x} = 1$, and $\eta_\varepsilon(\mathbf{x}) := \eta(\mathbf{x}/\varepsilon)/\varepsilon^d$.

4.3 Proof of Lemma 4.1 in 2-d (i.e. that Z defined in §4.2 satisfies Condition A)

We first check Condition A1 (the smoothness) for both ϕ^+ and ϕ^-, then Conditions A2–A4 for ϕ^+, and finally Conditions A2–A4 for ϕ^-.

Checking A1 for both ϕ^+ and ϕ^-. We need ϕ to be piecewise C^3 up to the boundary. Recall that ϕ is a smooth transition between ϕ_Γ^+ and ϕ_{ML} in Ω_+ and ϕ_Γ^- and ϕ_ε in Ω_-. Now $\phi_{\mathrm{ML}} \in C^\infty(\mathbb{R}^d)$ and, by properties of mollifiers (see, e.g., [23, §C.4, Theorem 6]), $\phi_\varepsilon \in C^\infty(D)$ (where D is the set on which ϕ_ε needs to be defined). Therefore, if $\phi_\Gamma^\pm$ are both C^3 up to the boundary then so is ϕ.

The functions $\phi_\Gamma^\pm$ are both defined in terms of the distance function. Since Γ is assumed to be C^3 in the statement of Lemma 4.1, the result about the differentiability of the distance function used in the proof of Theorem 4.2 above implies that $\phi_\Gamma^\pm$ are both C^3 up to the boundary.

Checking A2–A4 for ϕ^+. Using the expression for the gradient in (n,s)-coordinates, equation (4.4), and the definition of ϕ_Γ^+, equation (4.9), we find that

$$\nabla\phi_\Gamma^+(n,s) = (C_\Gamma + n)\widehat{\mathbf{e}}_n^+(s). \tag{4.15}$$

Therefore, on Γ (i.e. $n=0$), $\nabla\phi_\Gamma = C_\Gamma\widehat{\mathbf{e}}_n^+ = C_\Gamma\mathbf{n}$, which is part of the first requirement of A2. Next, noting that $\nabla\phi_\Gamma^+$ satisfies the conditions for its derivative to be given by (4.6), we have that

$$D^2\phi_\Gamma^+(n,s) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(C_\Gamma+n)\kappa(s)}{1+n\,\kappa(s)} \end{pmatrix}. \tag{4.16}$$

(We postpone checking the other requirements in A2, i.e. that $\nabla\phi$ and $\Delta\phi$ are continuous across Γ, to after we have found $\nabla\phi_\Gamma^-$ and $D^2\phi_\Gamma^-$.)

Turning to A3, we see that the definitions of ϕ_{ML}, (4.10), and ϕ^+, (4.8), imply that if $n \geq \mathcal{R}$ then $\phi^+ = \frac{1}{2}r^2$. Thus, for A3 to hold, we need to relate $\mathcal{R}$ to the radius of B_R. Let $d_{\Omega_-} := \max_{\mathbf{x},\mathbf{y}\in\Gamma}|\mathbf{x}-\mathbf{y}|$ (i.e. d_{Ω_-} is the diameter of Ω_-). Since we are assuming that the origin is inside Ω_-,

$$n(\mathbf{x}) < |\mathbf{x}| < n(\mathbf{x}) + d_{\Omega_-}, \tag{4.17}$$

and thus

$$B_{\mathcal{R}+d_{\Omega_-}} \supset \big\{\mathbf{x} : n(\mathbf{x}) \leq \mathcal{R}\big\}. \tag{4.18}$$

Therefore, if $R \geq \mathcal{R} + d_{\Omega_-}$, then $\phi^+ = \frac{1}{2}r^2$ in a neighbourhood of $|\mathbf{x}| = R$.

Moving to A4, we first prove that $D^2\phi_\Gamma^+$ and $D^2\phi_{\mathrm{ML}}$ are both ≥ 1 in Ω_+. Indeed, looking at the (2,2)-element of $D^2\phi_\Gamma^+$, given by (4.16), as a function of $n \in [0,\infty)$, and writing $(C_\Gamma + n)\kappa(s)$ as $C_\Gamma\kappa(s) - 1 + (1 + n\kappa(s))$, we see that if $C_\Gamma\kappa(s) \geq 1$ for all s then the (2,2)-element is smallest when $n = \infty$ and its value is one. If $C_\Gamma\kappa(s) < 1$ for some s then the (2,2)-element is smallest when $n = 0$ and its value is $C_\Gamma\kappa(s)$. Therefore,

$$D^2\phi_\Gamma^+ \geq \min(1, C_\Gamma\kappa_*),$$

and so the choice $C_\Gamma = 1/\kappa_*$ gives $D^2\phi_\Gamma^+(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_+$. The definition of ϕ_{ML}, (4.10), implies that $\nabla\phi_{\mathrm{ML}}(\mathbf{x}) = \mathbf{x}$ and hence $D^2\phi_{\mathrm{ML}} = I$.

Using the uniform convexity of ϕ_Γ^+ and ϕ_{ML}, we now show that ϕ^+ is uniformly convex if R is large enough,

Lemma 4.3 (ϕ^+ is uniformly convex if R is large enough) *Given $\delta > 0$ there exists an R_0 such that, for all $R \geq R_0$, $D^2\phi^+(\mathbf{x}) \geq (1-\delta)$ for all $\mathbf{x} \in \Omega_+ \cap B_R$*

Proof. From (4.18) above, we only need to show that, given $\delta > 0$ there exists an $\mathcal{R}_0$ such that, for all $\mathcal{R} \geq \mathcal{R}_0$, $D^2\phi^+ \geq (1-\delta)$ for all $\mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\}$, and then we set $R_0 = \mathcal{R}_0 + d_{\Omega_-}$.

Differentiating twice the definition of ϕ^+, equation (4.8), yields that

$$D^2\phi^+ = \chi_\mathcal{R} D^2\phi_\Gamma^+ + (1-\chi_\mathcal{R})D^2\phi_{\mathrm{ML}} + D^2\chi_\mathcal{R}(\phi_\Gamma^+ - \phi_{\mathrm{ML}}) + 2\nabla\chi_\mathcal{R} \otimes_s (\nabla\phi_\Gamma^+ - \nabla\phi_{\mathrm{ML}}), \quad (4.19)$$

where

$$(\mathbf{a} \otimes_s \mathbf{b})_{ij} := \frac{a_ib_j + a_jb_i}{2}.$$

From the fact that $D^2\phi_\Gamma^+(\mathbf{x})$ and $D^2\phi_{\mathrm{ML}}(\mathbf{x})$ are both ≥ 1 for all $\mathbf{x} \in \Omega_+$, we see that the first two terms of (4.19) are ≥ 1. We now need to show that the third and fourth terms are $o(1)$ as $\mathcal{R} \to \infty$, which gives the assertion. Equation (4.17) implies that

$$r = n + \mathcal{O}(1) \quad \text{as } n \to \infty$$

and simple geometry gives us that

$$\widehat{\mathbf{e}}_r = \widehat{\mathbf{e}}_n^+ + \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

Using these asymptotics in the definitions of ϕ_Γ^+ and ϕ_{ML} and the expressions for $\nabla\phi_\Gamma^+$ (4.15) and $\nabla\phi_{\mathrm{ML}}$, we find that

$$\phi_\Gamma^+(\mathbf{x}) - \phi_{\mathrm{ML}}(\mathbf{x}) = \mathcal{O}(n) \quad \text{as } n \to \infty, \quad (4.20)$$

$$\nabla\phi_\Gamma^+(\mathbf{x}) - \nabla\phi_{\mathrm{ML}}(\mathbf{x}) = \mathcal{O}(1) \quad \text{as } n \to \infty. \quad (4.21)$$

Using (4.20) and the fact that

$$D^2\chi_\mathcal{R}(n(\mathbf{x})) = \frac{1}{\mathcal{R}^2}D^2\chi\left(\frac{n(\mathbf{x})}{\mathcal{R}}\right) = \mathcal{O}\left(\frac{1}{\mathcal{R}^2}\right) \quad \text{as } \mathcal{R} \to \infty, \text{ uniformly for } \mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\},$$

we obtain the following bound on the third term in (4.19),

$$\left|D^2\chi_\mathcal{R}(n(\mathbf{x}))(\phi_\Gamma^+(\mathbf{x}) - \phi_{\mathrm{ML}}(\mathbf{x}))\right| = \mathcal{O}\left(\frac{1}{\mathcal{R}}\right) \text{ as } \mathcal{R} \to \infty, \text{ uniformly for } \mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\}. \quad (4.22)$$

Using (4.21) and the fact that

$$|\nabla\chi_\mathcal{R}(n(\mathbf{x}))| = \mathcal{O}\left(\frac{1}{\mathcal{R}}\right) \text{ as } \mathcal{R} \to \infty, \text{ uniformly for } \mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\},$$

we obtain the following bound on the fourth term in (4.19),

$$\left|\nabla\chi_\mathcal{R}(n(\mathbf{x})) \otimes_s (\nabla\phi_\Gamma^+(\mathbf{x}) - \nabla\phi_{\mathrm{ML}}(\mathbf{x}))\right| = \mathcal{O}\left(\frac{1}{\mathcal{R}}\right) \text{ as } \mathcal{R} \to \infty, \text{ uniformly for } \mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\}. \quad (4.23)$$

Using (4.22) and (4.23) in (4.19) then proves that $D^2\phi^+(\mathbf{x}) \geq (1 - o(1))$ as $\mathcal{R} \to \infty$, uniformly for $\mathbf{x} \in \Omega_+ \cap \{n \leq \mathcal{R}\}$. ∎

Checking A2–A4 for ϕ^-. By the discussion about the (n,s)-coordinate system in §4.1 and Parts (ii) and (iii) of Theorem 4.2, given any $\mathbf{x} \in \Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$ there exist (n,s) such that $\mathbf{x} = (n,s)$ in the orthogonal coordinate system defined by Γ, and $n \in (0, 1/\kappa(s))$. The definition of ϕ_Γ^-, (4.12), and the analogues of (4.4) and (4.6) for Ω_- then imply that

$$\nabla \phi_\Gamma^-(n,s) = (-C_\Gamma + n)\widehat{\mathbf{e}}_n^-(s), \tag{4.24}$$

and

$$D^2\phi_\Gamma^-(n,s) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{(C_\Gamma - n)\kappa(s)}{1 - n\,\kappa(s)} \end{pmatrix}. \tag{4.25}$$

Therefore, on Γ (i.e. $n=0$) $\nabla\phi_\Gamma = -C_\Gamma \widehat{\mathbf{e}}_n^- = C_\Gamma \mathbf{n}$, which fulfils part of A2. To check the final requirement of A2, namely that $\nabla \cdot \mathbf{Z} = \Delta\phi$ is continuous across Γ, note that equations (4.16) and (4.25) imply that

$$D^2\phi_\Gamma^-(0,s) = \begin{pmatrix} 1 & 0 \\ 0 & C_\Gamma \end{pmatrix} = D^2\phi_\Gamma^+(0,s),$$

and so $\Delta\phi$ is continuous (being a particular linear combination of elements of the matrix).

For the uniform convexity condition, A4, we need to show that there exists a $\theta > 0$ such that $D^2\phi^-(\mathbf{x}) \geq \theta$ for all $\mathbf{x} \in \Omega_-$. Our first step is to show that $D^2\phi_\Gamma^-(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$. Writing $(C_\Gamma - n)\kappa(s)$ as $C_\Gamma\kappa(s) - 1 + (1 - n\kappa(s))$, we see that if $C_\Gamma\kappa(s) \geq 1$ then the smallest value of the (2,2)-element of $D^2\phi_\Gamma^-$ as a function of $n \in [0, 1/\kappa(s))$ is one, occurring when $n = 0$. If $C_\Gamma\kappa(s) < 1$ then the smallest value is zero, occuring when $n = C_\Gamma$; this corresponds to the quadratic term in ϕ_Γ^- "kicking in too soon" and making the derivative of ϕ_Γ^- in the $\widehat{\mathbf{e}}_n^-$ direction positive. The choice $C_\Gamma = 1/\kappa_*$ therefore ensures that $D^2\phi_\Gamma^-(n,s) \geq 1$ for all s and for all $n \in (0, 1/\kappa(s))$.

The next step is to prove that ϕ_ε is uniformly convex.

Lemma 4.4 **(ϕ_ε is uniformly convex)** *With ϕ_ε defined by (4.14), $D^2\phi_\varepsilon(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in D$ (where D is the set defined by (4.13)).*

We assume this result for the moment and use it to prove uniform convexity of ϕ^-.

Lemma 4.5 **(ϕ^- is uniformly convex if ε is small enough)** *Given $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, for all $\varepsilon \leq \varepsilon_0$, $D^2\phi^-(\mathbf{x}) \geq (1-\delta)$ for all $\mathbf{x} \in \Omega_-$.*

Proof of Lemma 4.5. Differentiating twice the definition of ϕ^-, equation (4.11), yields that

$$D^2\phi^- = \chi_- D^2\phi_\Gamma^- + \big(1 - \chi_-\big)D^2\phi_\varepsilon + D^2\chi_-\big(\phi_\Gamma^- - \phi_\varepsilon\big) + 2\nabla\chi_- \otimes_s \big(\nabla\phi_\Gamma^- - \nabla\phi_\varepsilon\big). \tag{4.26}$$

Now $D^2\phi_\Gamma^-(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$, and so certainly for all $\mathbf{x} \in \operatorname{supp}\chi_- = \{n : 0 \leq n \leq 2n_0/3\}$. Furthermore, Lemma 4.4 implies that $D^2\phi_\varepsilon(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in D = \operatorname{supp}(1-\chi_-)$. These two facts imply that the first two terms in (4.26) are ≥ 1 for all $\mathbf{x} \in \Omega_-$.

We now prove that the third and fourth terms of (4.26) are $o(1)$ as $\varepsilon \to 0$. Since $D^2\chi_-$ and $\nabla\chi_-$ have support only in $\{n : n_0/3 \leq n \leq 2n_0/3\}$, it is sufficient to prove that $\phi_\varepsilon \to \phi_\Gamma^-$ and $\nabla\phi_\varepsilon \to \nabla\phi_\Gamma^-$ as $\varepsilon \to 0$ on this set. These limits follow from the facts that $\phi_\Gamma^- \in C(\Omega_-)$ and $\nabla\phi_\Gamma^- \in (C(\Omega_- \setminus \{n : n \geq n_0\}))^d$ using a standard property of mollifiers, namely that if U is open and $f \in C(U)$ then $f_\varepsilon \to f$ uniformly on compact subsets of U (e.g. [23, §C.4 Theorem 6]). ■

All that remains is to prove Lemma 4.4, i.e. that ϕ_ε is uniformly convex.

Proof of Lemma 4.4. We split the proof up into two cases: (i) $B_\varepsilon(\mathbf{x}) \cap \overline{\mathcal{M}_{\Omega_-}} = \emptyset$, and (ii) $B_\varepsilon(\mathbf{x}) \cap \overline{\mathcal{M}_{\Omega_-}} \neq \emptyset$.

In Case (i), we differentiate under the integral sign in the expression for ϕ_ε in (4.14) in which the $\mathbf{x}$-dependence under the integral sign is in ϕ_Γ^-; this is allowed since, from above, $\phi_\Gamma^- \in C^2(\Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}})$. Using the facts that (with $C_\Gamma = 1/\kappa_*$) $D^2\phi_\Gamma^-(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$ and $\int_{B_\varepsilon(\mathbf{x})} \eta_\varepsilon(\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y} = 1$ in the resulting expression shows that $D^2\phi_\varepsilon \geq 1$.

In Case (ii), we begin by recalling from Theorem 4.2 that $\mathcal{M}_{\Omega_-}$ is a tree with finitely many vertices and edges. Following [14], we introduce the terminology that a vertex with degree ≥ 3 is a *bifurcation point*, and a vertex with degree equal to one is a *terminal point*.

If $B_\varepsilon(\mathbf{x}) \cap \overline{\mathcal{M}_{\Omega_-}} \neq \emptyset$ then there are now three different cases:

1. there are no bifurcation points or terminal points of $\mathcal{M}_{\Omega_-}$ in $B_\varepsilon(\mathbf{x})$,
2. there are no bifurcation points of $\mathcal{M}_{\Omega_-}$ in $B_\varepsilon(\mathbf{x})$, but at least one terminal point,
3. there is at least one bifurcation point of $\mathcal{M}_{\Omega_-}$ in $B_\varepsilon(\mathbf{x})$ (and possibly also terminal points).

We first consider Case 1 and then show afterwards how Cases 2 and 3 can be reduced to the first case. We let $\Sigma := \overline{\mathcal{M}_{\Omega_-}} \cap B_\varepsilon(\mathbf{x})$ and differentiate under the integral sign in the expression for ϕ_ε in (4.14) in which the $\mathbf{x}$-dependence under the integral sign is in η_ε. Since $\partial_{x_i}\eta_\varepsilon(\mathbf{x}-\mathbf{y}) = -\partial_{y_i}\eta_\varepsilon(\mathbf{x}-\mathbf{y})$, we find that

$$\partial_i\partial_j\phi_\varepsilon(\mathbf{x}) = \int_{B_\varepsilon(\mathbf{x})} \phi_\Gamma^-(\mathbf{y})\partial_i\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{4.27}$$

where, to avoid an excess of notation, we have omitted the $\mathbf{x}$- or $\mathbf{y}$-dependence from the derivatives, but highlight that on the left-hand side they are in $\mathbf{x}$, and under the integral on the right-hand side they are in $\mathbf{y}$.

Our plan is to integrate the right-hand side of (4.27) by parts to move the differentiation from η_ε to ϕ_Γ^-, and then use the fact that $D^2\phi_\Gamma^-(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \Sigma$. Let Σ divide $B_\varepsilon(\mathbf{x})$ into B^+ and B^-, and let $\boldsymbol{\nu}$ be the unit normal to Σ pointing into B^+. In order to apply the divergence theorem in B^+ and B^- we need some information about the smoothness of Σ. Theorem 4.2 and the fact that we are in Case 1 above imply that Σ is analytic; thus $\partial B^\pm$ are Lipschitz and applying the divergence theorem in $B^\pm$ is allowed by, e.g., [40, Theorem 3.34]. Integrating by parts (and recalling that $\boldsymbol{\nu}$ points into B^+), we have that

$$\int_{B^+} \phi_\Gamma^-(\mathbf{y})\partial_i\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B^+} \partial_i\phi_\Gamma^-(\mathbf{y})\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_\Sigma \boldsymbol{\nu}_i(\mathbf{y})\phi_\Gamma^-(\mathbf{y})\,\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}s(\mathbf{y})$$

(the integral over $\partial B^+ \cap \partial B_\varepsilon(\mathbf{x})$ equals zero as η_ε is zero here). A similar result holds for the integral over B^- (with the sign of the integral over Σ reversed), and thus, since ϕ_Γ^- is continuous across Σ,

$$\int_{B_\varepsilon(\mathbf{x})} \phi_\Gamma^-(\mathbf{y})\partial_i\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B_\varepsilon(\mathbf{x})} \partial_i\phi_\Gamma^-(\mathbf{y})\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{4.28}$$

where $\partial_i\phi_\Gamma^-(\mathbf{y})$ in the integral on the right-hand side is understood piecewise.

Integrating by parts again we have that

$$\int_{B_\varepsilon(\mathbf{x})} \partial_i\phi_\Gamma^-(\mathbf{y})\partial_j\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = -\int_{B_\varepsilon(\mathbf{x})} \partial_j\partial_i\phi_\Gamma^-(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_\Sigma \big[\partial_i\phi_\Gamma^-(\mathbf{y})\big]_-^+\boldsymbol{\nu}_j(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{4.29}$$

and then putting (4.27), (4.28), and (4.29) together we obtain that

$$\partial_i\partial_j\phi_\varepsilon(\mathbf{x}) = \int_{B_\varepsilon(\mathbf{x})} \partial_j\partial_i\phi_\Gamma^-(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} + \int_\Sigma \big[\partial_i\phi_\Gamma^-(\mathbf{y})\big]_-^+\boldsymbol{\nu}_j(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{4.30}$$

where $\partial_j\partial_i\phi_\Gamma^-(\mathbf{y})$ in the first integral on the right-hand side is understood piecewise.

If we can show that

$$\big[\partial_i\phi_\Gamma^-(\mathbf{y})\big]_-^+\,\boldsymbol{\nu}_j(\mathbf{y})\,\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geq 0 \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^d \text{ and } \mathbf{y} \in \Sigma, \tag{4.31}$$

then, using this in (4.30) along with the facts that $D^2_{\mathbf{y}}\phi(\mathbf{y}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \Sigma$, $\eta_\varepsilon \geq 0$, and $\int_{B_\varepsilon(\mathbf{x})} \eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = 1$, we find that

$$D^2_{\mathbf{x}}\phi_\varepsilon(\mathbf{x}) \geq \int_{B_\varepsilon(\mathbf{x})} D^2_{\mathbf{y}}\phi(\mathbf{y})\eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}(\mathbf{y}) \geq \int_{B_\varepsilon(\mathbf{x})} \eta_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} = 1,$$

which is the result.

We now prove that the inequality (4.31) holds. Since $\mathbf{y}$ is in Σ and is not a bifurcation point then there exist $(n_1, s_1), (n_2, s_2)$ such that, in (n, s)-coordinates, $\mathbf{y} = (n_j, s_j)$, $j = 1, 2$, with $n_1 = n_2$

but $s_1 \neq s_2$. Let $(0, s_1)$ be the closest point on Γ to $\mathbf{y}$ on the $+$ side of Σ, and $(0, s_2)$ be the closest point on Γ to $\mathbf{y}$ on the $-$ side of Σ. The expression for $\nabla\phi_\Gamma^-$, (4.24), implies that

$$\left[\partial_i \phi_\Gamma^-(\mathbf{y})\right]_-^+ \boldsymbol{\nu}_j(\mathbf{y}) = -(C_\Gamma - n_1)\big(\widehat{\mathbf{e}}_n^-(s_1) - \widehat{\mathbf{e}}_n^-(s_2)\big)_i \boldsymbol{\nu}_j(\mathbf{y}).$$

Since $n_1 \leq C_\Gamma$ (as $n_1 \leq 1/\kappa(s_1)$ and $C_\Gamma = 1/\kappa_*$) it is sufficient to prove that

$$\big(\widehat{\mathbf{e}}_n^-(s_1) - \widehat{\mathbf{e}}_n^-(s_2)\big) \otimes \boldsymbol{\nu}(\mathbf{y}) \leq 0 \quad \text{for all } \mathbf{y} \in \Sigma, \tag{4.32}$$

in the sense of quadratic forms. Recall that $\boldsymbol{\nu}(\mathbf{y})$ is the unit normal vector to Σ at $\mathbf{y}$ that points into B^+, and let $\boldsymbol{\tau}(\mathbf{y})$ be a unit tangent vector to Σ at $\mathbf{y}$ (there are two possible choices for $\boldsymbol{\tau}$, but which one we choose will not matter in what follows). Recall that if $\mathbf{a} \otimes \mathbf{b} \leq 0$ and $\mathbf{B}$ is an orthogonal matrix then $\mathbf{Ba} \otimes \mathbf{Bb} \leq 0$. Therefore, since $\boldsymbol{\nu}(\mathbf{y})$ and $\boldsymbol{\tau}(\mathbf{y})$ are orthonormal for every $\mathbf{y} \in \Sigma$, we can verify that (4.32) holds for a given $\mathbf{y} \in \Sigma$ by working in the $\{\boldsymbol{\nu}(\mathbf{y}), \boldsymbol{\tau}(\mathbf{y})\}$ basis. We find that the inequality (4.32) will hold if

(a) the component of $\big(\widehat{\mathbf{e}}_n^-(s_1) - \widehat{\mathbf{e}}_n^-(s_2)\big)$ in the $\boldsymbol{\nu}(\mathbf{y})$ direction is ≤ 0, and

(b) the component of $\big(\widehat{\mathbf{e}}_n^-(s_1) - \widehat{\mathbf{e}}_n^-(s_2)\big)$ in the $\boldsymbol{\tau}(\mathbf{y})$ direction equals zero.

Since $\boldsymbol{\nu}$ points into B^+, (a) holds. Furthermore, since $(0, s_1)$ and $(0, s_2)$ lie on the circle with centre $\mathbf{y}$, the definition of $\mathcal{M}_{\Omega_-}$ and elementary geometry imply that the tangent line to Γ at $(0, s_1)$ is the reflection of the tangent line to Γ at $(0, s_2)$ in the tangent line of Σ at $\mathbf{y}$; this implies that (b) holds.

We have now proved the result for the first of the three cases outlined above. Case 2 can be reduced to Case 1 by extending Σ continuously so that the extended curve divides $B_\varepsilon(\mathbf{x})$ into two parts. Since ϕ_Γ^- and $\nabla\phi_\Gamma^-$ are continuous across the extension, the argument proceeds as before. For Case 3, first extend Σ at all terminal points as in Case 2. This extended curve now divides $B_\varepsilon(\mathbf{x})$ into a finite number of pieces (≥ 3), and the argument in Case 1 for two pieces generalises in an obvious way. ∎

4.4 Modifications needed to the above arguments in 3-d

The definition of ϕ in 3-d is exactly the same as the definition in 2-d given in §4.2 (i.e. equations (4.8)–(4.14)). Indeed, $\phi_\Gamma^\pm$ are defined only in terms of the distance function, ϕ_{ML} only in terms of r, and ϕ_ε only in terms of ϕ_Γ^- and η_ε, and thus all these quantities are well-defined when $d = 3$. The only difference is that we now define κ^* and κ_* to be the maximum and minimum of the principal curvatures respectively. (Recall that, given $\mathbf{x} \in \Gamma$, the two principal curvatures at $\mathbf{x}$ are such that the curvature of any 1-d curve on Γ passing though $\mathbf{x}$ lies between the principal curvatures.) As in the 2-d case we choose $C_\Gamma = 1/\kappa_*$.

In the proof of Lemma 4.1 for $d = 2$ in §4.3 we used the (n, s)-coordinate system defined by Γ to verify that

(i) $\nabla\phi_\Gamma^+ = \nabla\phi_\Gamma^- = C_\Gamma \mathbf{n}$ and $\Delta\phi_\Gamma^+ = \Delta\phi_\Gamma^-$ on Γ (this gave Condition A2), and

(ii) $D^2\phi_\Gamma^+(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_+$ and $D^2\phi_\Gamma^-(\mathbf{x}) \geq 1$ for all $\mathbf{x} \in \Omega_- \setminus \Sigma$ (this was needed for Condition A4).

The rest of the argument in §4.3 that ϕ satisfies Condition A is valid both in 2-d and in 3-d. Indeed, the only other part of the argument that depended on the dimension was the proof of Lemma 4.4 (the uniform convexity of ϕ_ε). This proof relied on the results about the geometry of the medial axis in 2-d given in Theorem 4.2. An appropriate analogue of Theorem 4.2 holds in the 3-d case. Indeed, the analogue of Part (i) of Theorem 4.2 in 3-d is that, roughly speaking, if Γ is piecewise analytic (i.e. is the finite union of analytic surfaces) then $\mathcal{M}_{\Omega_-}$ is also piecewise analytic; see [13, Theorem 2.1 and Corollary 2.1] for a more precise statement of this result and its proof. The analogue of Part (ii) is that $\mathcal{M}_{\Omega_-}$ has the same homotopy type as Ω_-, and thus if Ω_- is simply connected then so is $\mathcal{M}_{\Omega_-}$ (i.e. every closed curve on $\mathcal{M}_{\Omega_-}$ can be continuously shrunk down to a point); see [36]. Finally, Part (iii) of Theorem 4.2 holds in 3-d as well as in 2-d. Using this information about the geometry of $\mathcal{M}_{\Omega_-}$, we can generalise the proof of Lemma 4.4 from 2-d to 3-d in a straightforward manner.

Therefore, to prove that ϕ defined in §4.2 satisfies Condition A when $d = 3$, we only need to shows that (i) and (ii) above hold. As in the 2-d case, we do this in coordinate systems defined by Γ, but now these will only be local to each $\mathbf{x}$, instead of well-defined in all of Ω_+ or $\Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$. Indeed, whereas in 2-d it is straightforward to construct orthogonal coordinate systems for all of Ω_+ and $\Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$, in 3-d it is not. However, for (i), given an $\mathbf{x} \in \Gamma$ we can construct an orthogonal coordinate system defined by Γ in a neighbourhood of that $\mathbf{x}$ and calculate $\nabla\phi^{\pm}_{\Gamma}(\mathbf{x})$ and $D^2\phi^{\pm}_{\Gamma}(\mathbf{x})$ in this coordinate system; for (ii), given an $\mathbf{x} \in \Omega_+$ or $\Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$ we can construct an orthogonal coordinate system defined by Γ in a neighbourhood of that $\mathbf{x}$ and calculate $D^2\phi^{\pm}(\mathbf{x})$ in this coordinate system.

We now give the details of the coordinate systems that we use in Ω_+. Given a point P in Ω_+, let $\mathbf{r}_0$ be the position vector of the closest point on Γ to P (this is unique since Ω_- is convex). Introduce a coordinate system on Γ in a neighbourhood of $\mathbf{r}_0$ with coordinates (s,t) such that $\partial\mathbf{r}_0/\partial s$ and $\partial\mathbf{r}_0/\partial t$ are unit vectors in the principal directions at $\mathbf{r}_0$ (and are hence orthonormal). (If $\mathbf{r}_0$ is an umbilical point, i.e. Γ is locally spherical at $\mathbf{r}_0$, then just chose (s,t) such that $\partial\mathbf{r}_0/\partial s$ and $\partial\mathbf{r}_0/\partial t$ are orthonormal tangent vectors.) Let $\mathbf{n}(s,t)$ be the outward-pointing unit normal vector, and defined $\kappa_1(s)$ and $\kappa_2(t)$ by

$$\frac{\partial^2\mathbf{r}_0}{\partial s^2}(s,t) = -\kappa_1(s)\,\mathbf{n}(s,t) \quad \text{and} \quad \frac{\partial^2\mathbf{r}_0}{\partial t^2}(s,t) = -\kappa_2(t)\,\mathbf{n}(s,t)$$

respectively. By the definition of the principal directions, $\kappa_1(s)$ and $\kappa_2(t)$ are the principal curvatures. Our definitions of κ_* and κ^* imply that $\kappa_* \le \kappa_1(s), \kappa_2(t) \le \kappa^*$, and the fact that Ω_- is convex implies that $\kappa_* \ge 0$. We then have that

$$\frac{\partial\mathbf{n}}{\partial s}(s,t) = \kappa_1(s)\frac{\partial\mathbf{r}_0}{\partial s}(s,t) \quad \text{and} \quad \frac{\partial\mathbf{n}}{\partial t}(s,t) = \kappa_2(t)\frac{\partial\mathbf{r}_0}{\partial t}(s,t) \tag{4.33}$$

(compare to (4.3)).

The position vector, $\mathbf{r}$, of P can then be expressed as

$$\mathbf{r}(n,s,t) = \mathbf{r}_0(s,t) + n\,\mathbf{n}(s,t),$$

where, as before, $n = \operatorname{dist}(\mathbf{r},\Gamma)$. The definition of the basis vectors and the relations in (4.33) imply that

$$\begin{aligned}
\mathbf{e}^+_n(n,s,t) &:= \frac{\partial\mathbf{r}}{\partial n}(n,s,t) = \mathbf{n}(s,t),\\
\mathbf{e}_s(n,s,t) &:= \frac{\partial\mathbf{r}}{\partial s}(n,s,t) = \big(1 + n\,\kappa_1(s)\big)\frac{\partial\mathbf{r}_0}{\partial s}(s,t),\\
\mathbf{e}_t(n,s,t) &:= \frac{\partial\mathbf{r}}{\partial t}(n,s,t) = \big(1 + n\,\kappa_2(t)\big)\frac{\partial\mathbf{r}_0}{\partial t}(s,t),
\end{aligned}$$

and thus

$$h_n := \left|\frac{\partial\mathbf{r}}{\partial n}\right| = 1, \quad h_s := \left|\frac{\partial\mathbf{r}}{\partial s}\right| = 1 + n\,\kappa_1(s), \quad \text{and} \quad h_t := \left|\frac{\partial\mathbf{r}}{\partial s}\right| = 1 + n\,\kappa_2(t).$$

Since the coordinate system is orthogonal, everything goes through as in the 2-d case, with

$$\nabla\psi = \frac{1}{h_n}\frac{\partial\psi}{\partial n}\widehat{\mathbf{e}}^+_n + \frac{1}{h_s}\frac{\partial\psi}{\partial s}\widehat{\mathbf{e}}_s + \frac{1}{h_t}\frac{\partial\psi}{\partial t}\widehat{\mathbf{e}}_t$$

for scalar functions $\psi : \Omega_+ \to \mathbb{R}$, and

$$D\mathbf{v} = \begin{pmatrix} \frac{\partial v^n}{\partial n} & 0 & 0 \\ 0 & \frac{v^n}{h_s}\frac{\partial h_s}{\partial n} & 0 \\ 0 & 0 & \frac{v^n}{h_t}\frac{\partial h_t}{\partial n} \end{pmatrix}$$

for vector fields $\mathbf{v} : \Omega_+ \to \mathbb{R}^d$ such that $v^s = v^t = 0$ and v^n is a function of n only. The coordinate system in Ω_- is analogous, except that now $h_s = 1 - n\,\kappa_1(s)$ and $h_t = 1 - n\,\kappa_2(t)$. Therefore, for a given (n, s, t), the coordinate system breaks down when $n = 1/\max(\kappa_1(s), \kappa_2(t))$, and so the earliest breakdown is at $n = 1/\kappa^*$.

Performing the 3-d analogues of the 2-d calculations in §4.3, we see that (as in the 2-d case) (i) $\nabla\phi_\Gamma^+ = \nabla\phi_\Gamma^- = C_\Gamma\mathbf{n}$ and $\Delta\phi_\Gamma^+ = \Delta\phi_\Gamma^-$ on Γ, and (ii) the choice $C_\Gamma = 1/\kappa_*$ ensures that $D^2\phi_\Gamma^\pm \geq 1$ in Ω_+ and $\Omega_- \setminus \overline{\mathcal{M}_{\Omega_-}}$.

5 Nonexistence of a Z satisfying either Condition A or Condition B for nonconvex Ω_-

In this section, we show that if Ω_- is nonconvex, then the condition that $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ (Condition A2 or B2) and the nonnegativity condition on $\partial_i\mathbf{Z}_j$ (Condition A4 or B4) cannot be satisfied simultaneously. We restrict our attention to C^2 domains since both Conditions A and B assume this smoothness of Γ.

Lemma 5.1 *If Ω_- is a bounded C^2 domain that is nonconvex then there does not exist a real-valued $\mathbf{Z} \in (C^1(\overline{\Omega_+ \cap B_R}))^d$, for any R such that $\overline{\Omega_-} \subset B_R$, satisfying both*

- *$\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ for some constant $C_\Gamma > 0$, and*
- *$\partial_i\mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\mathbf{x} \in \overline{\Omega_+ \cap B_R}$.*

Proof. Since Ω_- is nonconvex and C^2, there exists a one-dimensional curve $\Gamma^* \subset \Gamma$ that has negative curvature. That is, if $\Gamma^* := \{\mathbf{r}_0(s) : a \leq s \leq b\}$ and $\kappa(s)$ is defined by (4.1) then there exists a constant $\kappa_0 < 0$ such that $\kappa(s) \leq \kappa_0$ for all $s \in (a, b)$.

The idea of the proof is to lift Γ^* off Γ in the normal direction, calculate the derivative of the length of the lifted curve with respect to the distance from Γ in two different ways (one using the curvature, the other using the fact that $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ) and reach a contradiction.

Let $\mathbf{r}(s;\varepsilon) := \mathbf{r}_0(s) + \varepsilon\mathbf{n}(s)$, and thus $\{\mathbf{r}(s;\varepsilon) : a \leq s \leq b\}$ is the curve Γ^* lifted outwards in the normal direction by ε. This definition and the expression (4.3) for $\mathrm{d}\mathbf{n}/\mathrm{d}s$ imply that

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s;\varepsilon) = \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s) + \varepsilon\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}(s) = \big(1 + \varepsilon\kappa(s)\big)\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s}(s),$$

and so

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s;\varepsilon)\right|^2 = \big(1 + \varepsilon\kappa(s)\big)^2. \tag{5.1}$$

Let $I(\varepsilon)$ denote the length of $\{\mathbf{r}(s;\varepsilon) : a \leq s \leq b\}$, i.e.

$$I(\varepsilon) := \int_a^b \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s;\varepsilon)\right| \mathrm{d}s.$$

The equation (5.1) implies that, for sufficiently small ε,

$$I(\varepsilon) = \int_a^b \big(1 + \varepsilon\kappa(s)\big)\mathrm{d}s,$$

and thus

$$\frac{\mathrm{d}I}{\mathrm{d}\varepsilon}(0) = \int_a^b \kappa(s)\,\mathrm{d}s \leq \kappa_0(b-a) < 0. \tag{5.2}$$

On the other hand, since $\mathbf{Z} = C_\Gamma\mathbf{n}$ on Γ,

$$\mathbf{r}(s;\varepsilon) = \mathbf{r}_0(s) + \varepsilon\,C_\Gamma\mathbf{Z}\big(\mathbf{r}_0(s)\big). \tag{5.3}$$

To condense notation, let x_j denote the j-th component of $\mathbf{r}_0$. Differentiating (5.3) to obtain $\mathrm{d}\mathbf{r}/\mathrm{d}s$, we find that

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s;\varepsilon)\right|^2 = 1 + 2\,\varepsilon\,C_\Gamma\partial_i\mathbf{Z}_j\big(\mathbf{r}_0(s)\big)\frac{\mathrm{d}x_i}{\mathrm{d}s}(s)\frac{\mathrm{d}x_j}{\mathrm{d}s}(s) + \mathcal{O}(\varepsilon^2) \qquad \text{as } \varepsilon \to 0.$$

Using this in the definition of $I(\varepsilon)$ yields

$$I(\varepsilon) = \int_a^b \left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s}(s;\varepsilon)\right| \mathrm{d}s = \int_a^b \mathrm{d}s + \varepsilon\, C_\Gamma \int_a^b \partial_i \mathbf{Z}_j\big(\mathbf{r}_0(s)\big)\frac{\mathrm{d}x_i}{\mathrm{d}s}(s)\frac{\mathrm{d}x_j}{\mathrm{d}s}(s)\,\mathrm{d}s + \mathcal{O}(\varepsilon^2),$$

and thus

$$\frac{\mathrm{d}I}{\mathrm{d}\varepsilon}(0) = C_\Gamma \int_a^b \partial_i \mathbf{Z}_j\big(\mathbf{r}_0(s)\big)\frac{\mathrm{d}x_i}{\mathrm{d}s}(s)\frac{\mathrm{d}x_j}{\mathrm{d}s}(s)\,\mathrm{d}s.$$

The facts that (i) $\partial_i \mathbf{Z}_j(\mathbf{x})\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and $\mathbf{x} \in \overline{\Omega_+ \cap B_R}$ and (ii) $C_\Gamma > 0$ then imply that $(\mathrm{d}I/\mathrm{d}\varepsilon)(0) \geq 0$, contradicting (5.2). ■

Remark 5.2 (Can one of Conditions A and B be satisfied when Ω_- is only convex (as opposed to uniformly convex)?) *In Lemma 4.1 we constructed a $\mathbf{Z}$ (equal to the gradient of a scalar function ϕ) satisfying Condition A when Ω_- is a uniformly convex domain, and we just showed in Lemma 5.1 above that there does not exist a $\mathbf{Z}$ satisfying either Condition A or Condition B when Ω_- is nonconvex.*

The task remains to construct a $\mathbf{Z}$ (either the gradient of some function ϕ or otherwise) satisfying either Condition A or Condition B when Ω_- is a smooth convex domain (i.e. with Γ allowed to contain straight line segments). In 2-d such a $\mathbf{Z}$ was essentially constructed in Ω_+ by [48, §4]; indeed, (in the notation of that paper) the extension of the vector field l to B^c satisfies the parts of Condition B that concern Ω_+. Given this fact, one might ask why we did not use the construction of [48, §4] in §4. The reason is that the construction of $\mathbf{Z}$ for uniformly convex domains in §4 is such that the 2-d version generalises almost immediately to 3-d, but this is not the case for the construction in [48, §4].

6 Conclusion: identities for the Helmholtz equation

In this conclusion we attempt to place this paper's use of Morawetz's identities into a wider context. We do this with the following two diagrams, Figures 1 and 2, which contrast the properties and uses of Green's identity with those of Morawetz's identities.

We make the following two remarks regarding Figure 2.

(i) The k-explicit bounds in A2 are for the interior impedance problem, i.e. the problem of finding u such that $\Delta u + k^2 u = -f$ in Ω_- and $\partial u/\partial n - \mathrm{i}\eta u = g$ on Γ for given f, g, and η (with $\eta \in \mathbb{R} \setminus \{0\}$). For this problem, one can use the multiplier $\mathbf{Z} \cdot \nabla u + \alpha u$ (i.e. $\beta = 0$ in $\mathcal{Z}u$) and, furthermore, in all the references in the figure ([41], [18], and [29]) $\mathbf{Z}$ is chosen to be $\mathbf{x}$. The resulting identity is then equivalent to adding the Rellich identity with multiplier $\mathbf{x} \cdot \nabla u$ (introduced by Rellich in [52]) to Green's identity multiplied by α, and this is how this method of obtaining bounds was understood in [41], [18], and [29]. Note that the analogue of these bounds for the time-harmonic Maxwell equations was obtained in [30, Theorem 4.6], [43, Theorem 5.4.5] using the Maxwell analogue of the $\mathbf{x} \cdot \nabla u + \alpha u$ multiplier; see [43, §5.3]. (The review in [10, §5.3.2] contains more discussion of these results and these multipliers.)

(ii) To obtain the results in A1, A2, and B, one needs a vector field $\mathbf{Z}$ in the domain where the PDE is posed (since the identity is applied in this domain). In contrast, to obtain the integral equation results, C1 and C2, one needs a vector field in *both* the interior *and* exterior domains (i.e. Ω_- and Ω_+). This is because (as we discussed in §1.5–1.6 and saw in §3) the identity is applied in both Ω_- and Ω_+.

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$$\text{Green's identity:} \quad \overline{v}\big(\Delta u + k^2 u\big) = \nabla \cdot \big[\overline{v}\nabla u\big] - \nabla u \cdot \overline{\nabla v} + k^2 u \overline{v}$$

When $v = u$, non-divergence terms on the right-hand side are not single-signed.

A. Difficult to prove k-explicit bounds for Helmholtz problems.

B. Variational (i.e. weak) formulations of Helmholtz problems are not coercive for large k.

C. Only prove boundary integral equations are coercive up to a compact perturbation (see §1.4).

Figure 1: The consequences of Green's identity for the analysis and numerical analysis of the Helmholtz equation.

$$\text{Morawetz's identities:} \quad \overline{\mathcal{Z}v}\big(\Delta u + k^2 u\big) = \nabla \cdot [\ \ldots\ ] + \ \ldots\ ,$$
$$\text{where } \mathcal{Z}v = \mathbf{Z} \cdot \nabla v - \mathrm{i}k\beta v + \alpha v$$

(when $v = u$ these are equations (2.1), (2.3)).

When $v = u$, non-divergence terms on the right-hand side are single-signed.
(under non-negativity condition on $\partial_i \mathbf{Z}_j$).

A1. Can prove k-explicit bounds for exterior Helmholtz problems: [47], [46].

A2. Can prove k-explicit bounds for interior impedance problem: [41], [18], [29].

B. New coercive variational formulations of exterior Dirichlet and interior impedance problems: [44].

C1. New coercive boundary integral equation for exterior Dirichlet problem (equation (1.34)): [58].

C2. Coercivity of standard boundary integral equation for exterior Dirichlet problem: this paper.

Figure 2: The consequences of Morawetz's identities for the analysis and numerical analysis of the Helmholtz equation.

References

[1] I. M. Babuška and S. A. Sauter. Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM Review*, pages 451–484, 2000.

[2] B. Beckermann, S. A. Goreinov, and E. E. Tyrtyshnikov. Some remarks on the Elman estimate for GMRES. *SIAM Journal on Matrix Analysis and Applications*, 27(3):772–778, 2006.

[3] T. Betcke, S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and M. Lindner. Condition number estimates for combined potential boundary integral operators in acoustics and their bound-

ary element discretisation. *Numerical Methods for Partial Differential Equations*, 27(1):31–69, 2011.

[4] T. Betcke, J. Phillips, and E. A. Spence. Spectral decompositions and non-normality of boundary integral operators in acoustic scattering. *IMA J. Num. Anal. to appear*, 2014.

[5] T. Betcke and E. A. Spence. Numerical estimation of coercivity constants for boundary integral operators in acoustic scattering. *SIAM Journal on Numerical Analysis*, 49(4):1572–1601, 2011.

[6] Y. Boubendir and C. Turc. Wave-number estimates for regularized combined field boundary integral operators in acoustic scattering problems with Neumann boundary conditions. *IMA Journal of Numerical Analysis*, 33(4):1176–1225, 2013.

[7] O. Bruno, T. Elling, and C. Turc. Regularized integral equations and fast high-order solvers for sound-hard acoustic scattering problems. *International Journal for Numerical Methods in Engineering*, 91(10):1045–1072, 2012.

[8] L. A. Caffarelli and A. Friedman. The free boundary for elastic-plastic torsion problems. *Transactions of the American Mathematical Society*, 252:65–97, 1979.

[9] S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and M. Lindner. Condition number estimates for combined potential boundary integral operators in acoustic scattering. *Journal of Integral Equations and Applications*, 21(2):229–279, 2009.

[10] S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and E. A. Spence. Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering. *Acta Numerica*, 21(1):89–305, 2012.

[11] S. N. Chandler-Wilde and P. Monk. Wave-number-explicit bounds in time-harmonic scattering. *SIAM Journal on Mathematical Analysis*, 39(5):1428–1455, 2008.

[12] S. N. Chandler-Wilde and E. A. Spence. Coercive modifications of the double-layer potential operator on Lipschitz domains. *In preparation*, 2014.

[13] F. Chazal and R. Soufflet. Stability and finiteness properties of medial axis and skeleton. *Journal of Dynamical and Control Systems*, 10(2):149–170, 2004.

[14] H. I. Choi, S. W. Choi, and H. P. Moon. Mathematical theory of medial axis transform. *pacific journal of mathematics*, 181(1):57–88, 1997.

[15] D. Colton and R. Kress. *Integral Equation Methods in Scattering Theory.* John Wiley & Sons Inc., New York, 1983.

[16] M. Costabel. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.*, 19:613–626, 1988.

[17] M. Costabel. Some historical remarks on the positivity of boundary integral operators. In O. Steinbach M. Schanz, editor, *Boundary element analysis: mathematical aspects and applications*, Lecture notes in Applied and Computational Mechanics, pages 1–27. Springer, 2007.

[18] P. Cummings and X. Feng. Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations. *Mathematical Models and Methods in Applied Sciences*, 16(1):139–160, 2006.

[19] V. Domínguez, I. G. Graham, and V. P. Smyshlyaev. A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering. *Numerische Mathematik*, 106(3):471–510, 2007.

[20] S. C. Eisenstat, H. C. Elman, and M. H. Schultz. Variational iterative methods for nonsymmetric systems of linear equations. *SIAM Journal on Numerical Analysis*, pages 345–357, 1983.

[21] H. C. Elman. *Iterative Methods for Sparse Nonsymmetric Systems of Linear Equations.* PhD thesis, Yale University, 1982.

[22] S. Esterhazy and J. M. Melenk. On stability of discretizations of the Helmholtz equation. In I. G. Graham, T. Y. Hou, O. Lakkis, and R. Scheichl, editors, *Numerical Analysis of Multiscale Problems*, volume 83 of *Lecture Notes in Computational Science and Engineering*, pages 285–324. Springer, 2012.

[23] L. C. Evans. *Partial differential equations.* American Mathematical Society Providence, RI, 1998.

[24] W. D. Evans and D. J. Harris. Sobolev embeddings for generalized ridged domains. *Proceedings of the London Mathematical Society*, 3(1):141–175, 1987.

[25] R. L. Foote. Regularity of the distance function. *American Mathematical Society*, 92(1), 1984.

[26] M. Ganesh and S. Hawkins. A fully discrete Galerkin method for high frequency exterior acoustic scattering in three dimensions. *Journal of Computational Physics*, 230:104–125, 2011.

[27] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer Verlag, 2001. Reprint of the 1998 edition.

[28] I. G. Graham, M. Löhndorf, J. M. Melenk, and E. A. Spence. When is the error in the h-BEM for solving the Helmholtz equation bounded independently of k? *preprint*, 2013.

[29] U. Hetmaniuk. Stability estimates for a class of Helmholtz problems. *Commun. Math. Sci*, 5(3):665–678, 2007.

[30] R. Hiptmair, A. Moiola, and I. Perugia. Stability results for the time-harmonic Maxwell equations with impedance boundary conditions. *Mathematical Models and Methods in Applied Sciences*, 21(11):2263–2287, 2011.

[31] G. C. Hsiao and W. L. Wendland. A finite element method for some integral equations of the first kind. *Journal of Mathematical Analysis and Applications*, 58(3):449–481, 1977.

[32] G. C. Hsiao and W. L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences.* Springer, 2008.

[33] N. Kuznetsov, V. G. Maz'ya, and B. R. Vainberg. *Linear water waves: a mathematical approach.* Cambridge University Press, 2002.

[34] M. N. Le Roux. Equations intégrales pour le probleme du potentiel électrique dans le plan. *C. R. Acad. Sci. Paris Sér. A*, 278:541–544, 1974.

[35] Y. Y. Li and L. Nirenberg. The distance function to the boundary, Finsler geometry, and the singular set of viscosity solutions of some Hamilton-Jacobi equations. *Communications on Pure and Applied Mathematics*, 58(1):85–146, 2005.

[36] A. Lieutier. Any open bounded subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis. *Computer-Aided Design*, 36(11):1029–1046, 2004.

[37] C. Mantegazza and A. C. Mennucci. Hamilton-Jacobi equations and distance functions on Riemannian manifolds. *Applied Mathematics and Optimization*, 47(1):1–26, 2003.

[38] S. Marburg. Six boundary elements per wavelength: is that enough? *Journal of Computational Acoustics*, 10(01):25–51, 2002.

[39] V. G. Maz'ya. On the steady problem of small oscillations of a fluid in the presence of a submerged body. *Proceedings of the Sobolev's Seminar No. 2, Novosibirsk: Institute of Mathematics, Sibirian Branch, Acad. Sci. USSR. (In Russian)*, pages 57–79, 1977.

[40] W. McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.

[41] J. M. Melenk. *On generalized finite element methods.* PhD thesis, The University of Maryland, 1995.

[42] R. B. Melrose and M. E. Taylor. Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Advances in Mathematics*, 55(3):242 – 315, 1985.

[43] A. Moiola. *Trefftz-discontinuous Galerkin methods for time-harmonic wave problems.* PhD thesis, Seminar for applied mathematics, ETH Zürich, 2011.

[44] A. Moiola and E. A. Spence. Is the Helmholtz equation really sign-indefinite? *SIAM Review*, page to appear, 2014.

[45] C. S. Morawetz. The decay of solutions of the exterior initial-boundary value problem for the wave equation. *Communications on Pure and Applied Mathematics*, 14(3):561–568, 1961.

[46] C. S. Morawetz. Decay for solutions of the exterior problem for the wave equation. *Communications on Pure and Applied Mathematics*, 28(2):229–264, 1975.

[47] C. S. Morawetz and D. Ludwig. An inequality for the reduced wave operator and the justification of geometrical optics. *Communications on Pure and Applied Mathematics*, 21:187–203, 1968.

[48] C. S. Morawetz, J. V. Ralston, and W. A. Strauss. Decay of solutions of the wave equation outside nontrapping obstacles. *Communications on Pure and Applied Mathematics*, 30(4):447–508, 1977.

[49] J. Nečas. *Les méthodes directes en théorie des équations elliptiques.* Masson, 1967.

[50] J. C. Nédélec. *Acoustic and electromagnetic equations: integral representations for harmonic problems.* Springer Verlag, 2001.

[51] J. C. Nédélec and J. Planchard. Une méthode variationnelle d'éléments finis pour la résolution numérique d'un probleme extérieur dans $\mathbb{R}^3$. *Revue française d'automatique, informatique, recherche opérationnelle*, 7(R3):105–129, 1973.

[52] F. Rellich. Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. *Mathematische Zeitschrift*, 46(1):635–636, 1940.

[53] K. F. Riley, M. P. Hobson, and S. J. Bence. *Mathematical methods for physics and engineering: a comprehensive guide.* Cambridge University Press, 2nd edition, 2002.

[54] R. T. Rockafellar. *Convex analysis.* Princeton Univ Press, 1970.

[55] S. A. Sauter and C. Schwab. *Boundary Element Methods.* Springer-Verlag, Berlin, 2011.

[56] E. A. Spence. Bounds on acoustic layer potentials via oscillatory integral techniques. *preprint*, 2013.

[57] E. A. Spence. Wavenumber-explicit bounds in time-harmonic acoustic scattering. *preprint*, 2013.

[58] E. A. Spence, S. N. Chandler-Wilde, I. G. Graham, and V. P. Smyshlyaev. A new frequency-uniform coercive boundary integral equation for acoustic scattering. *Communications on Pure and Applied Mathematics*, 64(10):1384–1415, 2011.

[59] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements.* Springer, New York, 2008.

[60] O. Steinbach and W. L. Wendland. On C. Neumann's method for second-order elliptic systems in domains with non-smooth boundaries. *J. Math. Anal. Appl.*, 262:733–748, 2001.

[61] W. A. Strauss. Dispersal of waves vanishing on the boundary of an exterior domain. *Communications on Pure and Applied Mathematics*, 28(2):265–278, 1975.

[62] L. N. Trefethen and M. Embree. *Spectra and pseudospectra.* Princeton University Press, Princeton, NJ, 2005.