Exercise 1 (Bound on C_{sol} for $F \in (H^1_{0,D}(\Omega_R))'$)**.** Let $F \in (H^1_{0,D}(\Omega_R))'$ and let $u \in H^1_{0,D}(\Omega_R)$ be the solution of the variational problem

$$a(u,v) = F(v)\,, \quad \forall v \in H^1_{0,D}(\Omega_R)\,.$$

Following the hint, we write

$$u = u^+ + w\,, \tag{1}$$

where $u^+ \in H^1_{0,D}(\Omega_R)$ is the solution of the variational problem

$$a^+(u^+, v) = F(v)\,. \tag{2}$$

Note that, using the sign property of the real part of the DtN map (1.11), we have the following inequality for all $v \in H^1_{0,D}(\Omega_R)$:

$$\begin{aligned} \Re(a^+(v,v)) &\geq A_{\min} k^{-2} \left\|\nabla u\right\|^2_{L^2(\Omega_R)} + n_{\min} \left\|v\right\|^2_{L^2(\Omega_R)} \\ &\geq \min\{A_{\min}, n_{\min}\} \left\|v\right\|^2_{H^1_k(\Omega_R)}\,. \end{aligned} \tag{3}$$

Therefore, the Lax–Milgram theorem ensures that eq. (2) is a valid definition of u^+. Taking $v = u^+$ in the inequality (3) and using the fact that $a^+(u^+, u^+) = F(u^+)$, we obtain

$$\left\|u^+\right\|^2_{H^1_k(\Omega_R)} \leq \frac{|a^+(u^+,u^+)|}{\min\{A_{\min}, n_{\min}\}} = \frac{|F(u^+)|}{\min\{A_{\min}, n_{\min}\}} \leq \frac{\left\|F\right\|_{H^1_k(\Omega_R)'} \left\|u^+\right\|_{H^1_k(\Omega_R)}}{\min\{A_{\min}, n_{\min}\}}\,,$$

i.e.

$$\left\|u^+\right\|_{H^1_k(\Omega_R)} \leq \frac{\left\|F\right\|_{H^1_k(\Omega_R)'}}{\min\{A_{\min}, n_{\min}\}}\,. \tag{4}$$

Next, w satisfies

$$\begin{aligned} a(w,v) &= a(u,v) - a(u^+,v) \\ &= a(u,v) - \left(a^+(u^+,v) - \int_{\Omega_R} 2nu^+\overline{v}\right) \\ &= F(v) - F(v) + \int_{\Omega_R} 2nu^+\overline{v} \\ &= \int_{\Omega_R} 2nu^+\overline{v}\,, \end{aligned}$$

for all $v \in H^1_{0,D}(\Omega_R)$. We remark that $2nu^+ \in L^2(\Omega_R)$: in fact

$$\left\|2nu^+\right\|_{L^2(\Omega_R)} \leq 2n_{\max} \left\|u^+\right\|_{H^1_k}\,.$$

Therefore, by definition of C_{sol}, one has

$$\left\|w\right\|_{H^1_k(\Omega_R)} \leq 2n_{\max} C_{\text{sol}} \left\|u^+\right\|_{H^1_k}\,. \tag{5}$$

We now get the desired inequality by injecting the estimates (4) and (5) in the decomposition (1), i.e.

$$\boxed{\|u\|_{H^1_k(\Omega_R)} \leq \frac{1+2n_{\max}C_{\mathrm{sol}}}{\min\{A_{\min}, n_{\min}\}} \|F\|_{H^1_k(\Omega_R)'} \,.}$$

Exercise 2 (Linear growth of C_{sol})**.** We show that for any k_0 and R_0, there exists a constant $C > 0$ such that for all k, R such that $kR \geq k_0 R_0$, there exists $u \in C^\infty_{\mathrm{comp}}(B_R)$ satisfying

$$\frac{\|u\|_{H^1_k(B_R)}}{\|k^-2\Delta u + u\|_{L^2(B_R)}} \geq CkR\,. \tag{6}$$

In other words, when $\Omega^- = \emptyset$, $A = I$, $n = 1$, C_{sol} grows at least linearly in kR.

Let us fix k_0 and R_0 and pick some arbitrary $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}, \mathbb{R})$ with support in $(0, 1)$; let $v(\boldsymbol{x}) := e^{ikx_1}$, $w(\boldsymbol{x}) := \chi\left(\frac{|\boldsymbol{x}|}{R}\right)$ and let $u(\boldsymbol{x}) := v(\boldsymbol{x})w(\boldsymbol{x})$, which is indeed in $C^\infty_{\mathrm{comp}}(B_R)$.

Estimate of $\|u\|_{H^1_k(B_R)}$: Since $|v(\boldsymbol{x})| = 1$ for all $\boldsymbol{x} \in B_R$, one has

$$\|u\|^2_{L^2(B_R)} = \|w\|_{L^2(B_R)} = \int_0^R \left|\chi\left(\frac{r}{R}\right)\right|^2 c_d r^{d-1} dr\,,$$

where c_d is the $(d-1)$-dimensional volume of the sphere

$$\mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\| = 1\}\,.$$

This reduces to

$$\|u\|_{L^2(B_R)} = KR^{d/2}\,, \quad K := \sqrt{c_d \int_0^1 |\chi(u)|^2 u^{d-1}\, du}\,.$$

Since $\|u\|_{H^1_k(B_R)} \geq \|u\|_{L^2(B_R)}$, we conclude that

$$\|u\|_{H^1_k(B_R)} \geq KR^{d/2}\,. \tag{7}$$

Estimate of $\left\|k^{-2}\Delta u + u\right\|_{L^2(B_R)}$: We write

$$\Delta u = w\Delta v + 2\nabla v \cdot \nabla w + v\Delta w$$

and observe that $\Delta v = -k^2 v$, so that

$$\begin{aligned} k^{-2}\Delta u + u &= 2k^{-2}\left(\nabla v \cdot \nabla w + v\Delta w\right) \\ &= 2i\left(k^{-1}\frac{\partial w}{\partial x_1}\right)v + (k^{-2}\Delta w)v\,. \end{aligned}$$

We estimate the first term by writing

$$\left\| \left(k^{-1} \frac{\partial w}{\partial x_1} \right) v \right\| \leq k^{-1} \left\| \nabla w \right\|_{L^2(B_R)}$$

and

$$k^{-1} \left\| \nabla w \right\|_{L^2(B_R)}^2 = c_d k^{-1} \sqrt{\int_0^R \frac{1}{R^2} \left| \chi' \left(\frac{r}{R} \right) \right|^2 r^{d-1} dr} = \frac{K' R^{d/2}}{kR} \,,$$

where $K' = \sqrt{c_d \int_0^1 \left| \chi'(u) \right|^2 u^{d-1} du}$. For the second term, we start from the following expression of the Laplacian of w:

$$\Delta w(\boldsymbol{x}) = \frac{1}{R^2} \chi'' \left(\frac{|\boldsymbol{x}|}{R} \right) + \frac{2}{R \, |\boldsymbol{x}|} \chi' \left(\frac{|\boldsymbol{x}|}{R} \right) \,.$$

It follows that

$$\begin{aligned} \left\| v \Delta w \right\|_{L^2(B_R)}^2 &\leq 2 c_d^2 \left(\int_0^R R^{-4} \left| \chi'' \left(\frac{r}{R} \right) \right|^2 r^{d-1} dr + \int_0^R 4 R^{-2} r^{-2} \left| \chi' \left(\frac{r}{R} \right) \right| r^{d-1} dr \right) \\ &= 2 c_d^2 R^{d-4} \left(\int_0^1 \left[\left| \chi''(u) \right|^2 + 4 u^2 \left| \chi'(u) \right|^2 \right] u^{d-1} \, du \right) \,, \end{aligned}$$

i.e.

$$\left\| (k^{-2} \Delta w) v \right\|_{L^2(B_R)} \leq \frac{K'' R^{d/2}}{(kR)^2} \,, \quad K'' := \sqrt{c_d \int_0^1 \left[\left| \chi''(u) \right|^2 + 4 u^2 \left| \chi'(u) \right|^2 \right] u^{d-1} \, du} \,.$$

We conclude that

$$\left\| k^{-2} \Delta u + u \right\|_{L^2(B_R)} \leq \frac{K' R^{d/2}}{kR} \left(1 + \frac{1}{(kR)} \frac{K''}{K'} \right) \,. \tag{8}$$

Combining the estimates (7) and (8), we have proved that if $kR \geq k_0 R_0$, then

$$\frac{\left\| u \right\|_{H_k^1(B_R)}}{\left\| k^{-2} \Delta u + u \right\|_{L^2(B_R)}} \geq C k R$$

with

$$C := \frac{K k_0 R_0}{K' k_0 R_0 + K''} \,.$$

Exercise 3 (Morawetz bound for a star-shaped domain.)**.**

a) Proof of identity (2.7)

(i) Some calculus identities. Let us prove the following identity:

$$2\Re\big\{\nabla v \cdot (\boldsymbol{x} \cdot \nabla)\overline{\nabla v}\big\} = \nabla \cdot \Big[|\nabla v|^2 \, \boldsymbol{x}\Big] - d\,|\nabla v|^2 \, . \tag{9}$$

Proof.

$$\begin{aligned}
2\Re\big\{\nabla v \cdot (\boldsymbol{x} \cdot \nabla)\overline{\nabla v}\big\} &= \nabla v \cdot (\boldsymbol{x} \cdot \nabla)\overline{\nabla v} + \overline{\nabla v} \cdot (\boldsymbol{x} \cdot \nabla)\nabla v \\
&= \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial v}{\partial x_i} x_j \frac{\partial}{\partial x_j}\frac{\partial \overline{v}}{\partial x_i} + \frac{\partial \overline{v}}{\partial i} x_j \frac{\partial}{\partial x_j}\frac{\partial v}{\partial x_i} \\
&= \sum_{i,j=1}^{d} x_j \frac{\partial}{\partial x_j}\left(\frac{\partial v}{\partial x_i}\frac{\partial \overline{v}}{\partial x_i}\right) \\
&= \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left[x_j \left|\frac{\partial v}{\partial x_i}\right|^2\right] - \left|\frac{\partial v}{\partial x_i}\right|^2 \\
&= \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(x_j \sum_{i=1}^{d}\left|\frac{\partial v}{\partial x_i}\right|^2\right) - \sum_{j=1}^{d}\left(\sum_{i=1}^{d}\left|\frac{\partial v}{\partial x_i}\right|^2\right) \\
&= \nabla \cdot \Big[|\nabla v|^2 \, \boldsymbol{x}\Big] - d\,|\nabla v|^2 \, . \qquad \square
\end{aligned}$$

A similar proof also gives

$$2\Re\big([\boldsymbol{x} \cdot \overline{\nabla v}]v\big) = \nabla \cdot \Big(|\nabla v|^2 \, \boldsymbol{x}\Big) - d\,|v|^2 \, , \tag{10}$$

and additionally, let us state without proof the last needed identity:

$$2\Re\big[(\boldsymbol{x} \cdot \overline{\nabla v})\Delta v\big] = \nabla \cdot \big(\Re\big[(\boldsymbol{x} \cdot \overline{\nabla v})\nabla v\big]\big) - 2\Re\big\{\nabla v \cdot (\boldsymbol{x} \cdot \nabla)\overline{\nabla v}\big\} - 2\,|\nabla v|^2 \, . \tag{11}$$

With those identities at hand, we now work out the different terms of identity (2.7) separately.

(ii) First term. We start by focusing on the first term of $\mathcal{M}_{\beta,\alpha}v$, i.e. the term $\boldsymbol{x} \cdot \nabla v$. Combining eqs. (9) and (11), we get, on the one hand,

$$2\Re\big[(\boldsymbol{x} \cdot \overline{\nabla v})\Delta v\big] = \nabla \cdot \Big[2\Re\big([\boldsymbol{x} \cdot \overline{\nabla v}]\nabla v\big) - |\nabla v|^2 \, \boldsymbol{x}\Big] - (2-d)\,|\nabla v|^2 \, ,$$

and using identity (10), on the other hand,

$$2\Re\big[(\boldsymbol{x} \cdot \overline{\nabla v})v\big] = \nabla \cdot \Big[|v|^2 \, \boldsymbol{x}\Big] - d\,|v|^2 \, .$$

Hence,

$$\begin{aligned}
2\Re\big[(\boldsymbol{x} \cdot \overline{\nabla v})\mathcal{L}v\big] = \nabla \cdot \Big[&2k^{-1}\Re\big([\boldsymbol{x} \cdot \nabla\overline{v}]k^{-1}\nabla v\big) + \Big(|v|^2 - k^{-2}\,|\nabla v|^2\Big)\boldsymbol{x}\Big] \\
&- d\,|v|^2 - (2-d)\,|\nabla v|^2 \, .
\end{aligned} \tag{12}$$

(iii) Second term. It is straightforward to check that

$$\nabla \cdot \big(\overline{i\beta v}\nabla v\big) = \overline{i\beta v}\Delta v - i\beta\,|\nabla v|^2 - \overline{v}(i\nabla\beta\cdot\nabla v)$$

hence

$$2\Re\big(\overline{i\beta v}\Delta v\big) = \nabla\cdot\big[2\Re\big(\overline{i\beta v}\nabla v\big)\big] + 2\Re\big(\overline{v}i\nabla\beta\cdot\nabla v\big)\,.$$

Since $\Re\big(\overline{i\beta v}v\big) = 0$, we thus obtain

$$2\Re\Big(\overline{(-ik\beta v)}\mathcal{L}v\Big) = \nabla\cdot\Big[2k^{-1}\Re\Big(\overline{(-ik\beta v)}k^{-1}\nabla v\Big)\Big] - 2\Re\big(\overline{v}i\nabla\beta\cdot k^{-1}\nabla v\big)\,. \quad (13)$$

(iv) Third term The same arguments lead to

$$2\Re(\overline{\alpha v}\mathcal{L}v) = \nabla\cdot\big[2k^{-1}\Re\big(\overline{\alpha v}k^{-1}\nabla v\big)\big] - 2\Re\big(\overline{v}(k^{-1}\nabla\alpha)\cdot(k^{-1}\nabla v)\big) + 2\alpha\,|v|^2\,. \quad (14)$$

Now, identity (2.7) is obtained by summing together eqs. (12), (13) and (14).

b) Special case where $\beta = r$

For the choice $\beta(\boldsymbol{x}) = r$, and $\alpha\in\mathbb{R}$, the expression of $\mathcal{M}_{\alpha,r}u$ now reads

$$\mathcal{M}_{\alpha,\beta}u = r\left(u_r - iku + \frac{\alpha u}{r}\right)\,.$$

Noting that $\nabla\beta\cdot\nabla v = \frac{\boldsymbol{x}\cdot\nabla v}{r} = v_r$, identity (2.7) becomes

$$\begin{aligned}2\Re\big(\overline{\mathcal{M}_{r,\alpha}v}\mathcal{L}v\big) = \nabla\cdot\Big[2k^{-1}\Re\Big(\overline{\mathcal{M}_{\beta,\alpha}k^{-1}\nabla v}\Big) + (|v|^2 - k^{-2}\,|\nabla v|^2)\boldsymbol{x}\Big]\\ - 2\Re\big(i\overline{v}(k^{-1}v_r)\big) - (d-2\alpha)|v|^2 - (2\alpha - d + 2)k^{-2}\,|\nabla v|^2\,.\end{aligned}$$

To obtain the desired result, it suffices to replace the first term of the second line according to the expression

$$-\Re\big(i\overline{v}k^{-1}v_r\big) = \Re\Big(\overline{(iv)}k^{-1}v_r\Big) = |v|^2 + k^{-2}\,|v_r|^2 - \big|k^{-1}v_r - iv\big|^2\,,$$

and rearrange the terms.

c) Decay of the boundary term

One can write, for any u

$$\boldsymbol{Q}_{R_1,\alpha}(u)\cdot\hat{\boldsymbol{x}} = 2k^{-2}\Re\left(u_r\overline{\mathcal{M}_{\alpha,r}u}\right) + r(|u|^2 - k^{-2}\,|\nabla u|^2)\,. \quad (15)$$

Atkinson-Wilcox expansions: We are interested in eq. (15) in the case where u is an outgoing solution of $\mathcal{L}u = 0$. We wish to estimate each term using Atkinson-Wilcox expansion (Lemma 1.4). First, we write

$$u(\boldsymbol{x}) = \frac{e^{ikr}}{r^{(d-1)/2}} g(\boldsymbol{x}) + O\left(\frac{1}{r^{(d+1)/2}}\right) . \tag{16}$$

where $g(\boldsymbol{x}) = f_0(\hat{\boldsymbol{x}})$. Note that since $g(t\boldsymbol{x}) = g(\boldsymbol{x})$ for all $t > 0$, it follows that $\boldsymbol{x} \cdot \nabla g(\boldsymbol{x}) = 0$. Furthermore,

$$\nabla g(\boldsymbol{x}) = \nabla f_0(\hat{\boldsymbol{x}}) \frac{r^2 - \boldsymbol{Z}}{r^3} ,$$

where $\boldsymbol{Z}$ is the vector field defined by $\boldsymbol{Z}_i = x_i^2$ Importantly, the above expression and Lemma 1.4 imply that

$$\nabla g(\boldsymbol{x}) = O\left(\frac{1}{r}\right) .$$

It follows from this remark and Lemma 1.4 that

$$\nabla u(x) = ik\hat{\boldsymbol{x}} f_0(\hat{\boldsymbol{x}}) \frac{e^{ikr}}{r^{(d-1)/2}} + O\left(\frac{1}{r^{(d+1)/2}}\right) , \tag{17}$$

and in particular that the radial derivative dominates the gradient of u at infinity:

$$|u_r|^2 = |\nabla u(x)|^2 + O\left(\frac{1}{r^d}\right) . \tag{18}$$

Taking the scalar product of (17) with $\hat{\boldsymbol{x}}$ and adding $-ikr$ times eq. (16) provides us with the last expansion that we need:

$$\mathcal{M}_{\alpha,r} u(\boldsymbol{x}) = O\left(\frac{1}{r^{(d-1)/2}}\right) . \tag{19}$$

Estimate of $\boldsymbol{Q}_{R_1,\alpha}(u) \cdot \hat{\boldsymbol{x}}$: We now go on to prove that $\boldsymbol{Q}_{r,\alpha}(u) \cdot \hat{\boldsymbol{x}} = O(r^{-d})$, which implies that

$$\int_{\Gamma_{R_1}} \boldsymbol{Q}_{R_1,\alpha}(u) \cdot \hat{\boldsymbol{x}} = O\left(\frac{1}{R_1}\right) .$$

We have

$$\boldsymbol{Q}_{r,\alpha}(u) \cdot \hat{\boldsymbol{x}} = 2k^{-2} r \Re\left(u_r \overline{\left[u_r - iku + \frac{\alpha}{r} u\right]} \right) + r(|v|^2 - k^{-2} |\nabla v|^2)$$

which can be conveniently rewritten as

$$\boldsymbol{Q}_{r,\alpha}(u) \cdot \hat{\boldsymbol{x}} = k^{-2} r \left(|u_r|^2 - |\nabla u|^2 \right) + 2k^{-2} \frac{|\mathcal{M}_{\alpha,R_1} u|^2}{r} - 2k^{-2} \frac{\alpha^2 |u|^2}{r} , \tag{20}$$

using the identity $2\Re(z_1\overline{z_2}) = |z_1+z_2|^2 - |z_1|^2 - |z_2|^2$. Using the previous asymptotics expansions, namely eqs. (16), (18) and (19), we deduce easily that

$$\boldsymbol{Q}_{r,\alpha}(u)\cdot\hat{\boldsymbol{x}} = O\left(\frac{1}{r^d}\right)$$

as announced.

d) A boundary integral on Γ_R

Let us assume that u is an outgoing solution of $\mathcal{L}u = 0$ in $\mathbb{R}^d \setminus \overline{B_R}$ for some $R > 0$. By integrating identity (2.11) over $\Omega := B_{R_1} \setminus \overline{B_R}$, where $R_1 > R$, with $v = u$, we find

$$\begin{aligned}\int_{\Gamma_{R_1}} &\boldsymbol{Q}_{R_1,\alpha}(u)\cdot\hat{\boldsymbol{x}} - \int_{\Gamma_R} \boldsymbol{Q}_{R,\alpha}(u)\cdot\hat{\boldsymbol{x}}\\ &= -\int_\Omega (2\alpha-(d-1))(|v|^2 - k^{-2}|\nabla v|^2) + k^{-2}(|\nabla v|^2 - |v_r|^2) + \left|k^{-1}v_r - iv\right|^2\end{aligned}$$

Setting $\alpha = \frac{d-1}{2}$ eliminates the first term in the rhs, so that

$$\int_{\Gamma_{R_1}} \boldsymbol{Q}_{R_1,\alpha}(u)\cdot\hat{\boldsymbol{x}} - \int_{\Gamma_R} \boldsymbol{Q}_{R,\alpha}(u)\cdot\hat{\boldsymbol{x}} \geq 0$$

(indeed, $|v_r| = \left|\frac{\boldsymbol{x}}{r}\cdot\nabla v\right| \leq |\nabla v|$). Sending $R_1 \to \infty$, we conclude that

$$\int_{\Gamma_R} \boldsymbol{Q}_{R,\alpha}(u)\cdot\hat{\boldsymbol{x}} \leq 0\,.$$

e) A boundary integral on Γ_D

We remark that if $u \in H^2(\Omega_R)$ vanishes identically on the $C^{1,1}$ boundary Γ_D, then

$$\gamma(\nabla u) = \frac{\partial u}{\partial n}\boldsymbol{n}\,,$$

where $\boldsymbol{n}$ is a C^0 unit normal vector on Γ_D. Indeed, one has in general

$$\gamma(\nabla u) = \frac{\partial u}{\partial n}\boldsymbol{n} + \nabla_{\Gamma_D} u$$

where ∇_{Γ_D} denotes the tangential gradient on Γ_D but $\nabla_{\Gamma_D} u = 0$ whenever $\gamma u = 0$ on Γ_D. Hence,

$$\begin{aligned}\boldsymbol{Q}_{\beta,\alpha}(u)\cdot\boldsymbol{n} &= 2k^{-2}\Re\left(\frac{\partial u}{\partial n}\overline{\left[\frac{\partial u}{\partial n}\boldsymbol{x}\cdot\boldsymbol{n} - ik\beta\gamma u + \alpha\gamma u\right]}\right) + \boldsymbol{x}\cdot\boldsymbol{n}\left(|\gamma u|^2 - k^{-2}\left|\frac{\partial u}{\partial n}\right|^2\right)\\ &= k^{-2}\boldsymbol{x}\cdot\boldsymbol{n}\left|\frac{\partial u}{\partial n}\right|^2.\end{aligned}$$

which immediately implies the desired result.

f) Proof of the Morawetz bound

We now gather the previous facts to derive a bound for C_{sol} in the case where Ω_- is a Lipschitz star-shaped domain with respect to the origin. We start from identity (2.7) where $\alpha = \frac{d-1}{2}$ and $\beta = r$. It reads

$$2\Re\Big(\overline{[ru_r - ikru + \alpha u]\, f}\Big) = \nabla \cdot Q_{\beta,\alpha}(u) - |u|^2 - k^{-2} |\nabla u|^2 \,.$$

We integrate on Ω_R for and apply Green's theorem for the divergence term:

$$\int_{\Omega_R} 2\Re\big(\overline{\mathcal{M}_{\beta,\alpha} u} f\big) + \|u\|^2_{H^1_k(\Omega_R)} = \\ - \int_{\Gamma_D} Q_{\beta,\alpha}(u) \cdot \boldsymbol{\nu} + \int_{\Gamma_R} Q_{\beta,\alpha}(u) \cdot \hat{\boldsymbol{x}}$$

where $\boldsymbol{\nu}$ is the unit normal vector on Γ_D pointing outwards of Ω_-. Using part d), part e) and the fact that Ω_- is star-shaped with respect to the origin (so that, by Lemma 2.13, $\boldsymbol{x} \cdot \boldsymbol{\nu}(\boldsymbol{x}) \geq 0$) for all $x \in \Gamma_D$), we are led to

$$\|u\|^2_{H^1_k(\Omega_R)} \leq \int_{\Omega_R} 2\Re\big(\overline{\mathcal{M}_{\beta,\alpha} u} f\big) \leq 2 \left\|\mathcal{M}_{\beta,\alpha} u\right\|_{L^2(\Omega_R)} \|f\|_{L^2(\Omega_R)} \,.$$

Using the inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, the fact that $|ikr + \alpha|^2 = k^2 r^2 + \alpha^2$, and the bound $r \leq R$ on Ω_R, we have

$$\begin{aligned} \|\mathcal{M}_{\beta,\alpha} u\|^2_{L^2(\Omega_R)} &\leq 2R^2 \|\nabla u\|^2_{L^2(\Omega_R)} + 2[(kR)^2 + \alpha^2] \|u\|^2_{L^2(\Omega_R)} \\ &\leq 2k^2 R^2 \left(1 + \frac{\alpha^2}{k^2 R^2}\right) \|u\|^2_{H^1_k(\Omega_R)} \,. \end{aligned}$$

Hence, since $\alpha = \frac{d-1}{2}$, we have obtained

$$\|u\|_{H^1_k(\Omega_R)} \leq 2kR \sqrt{1 + \left(\frac{d-1}{2kR}\right)^2} \, \|f\|_{L^2(\Omega_R)} \,,$$

i.e.

$$\boxed{C_{\mathrm{sol}} \leq 2kR \sqrt{1 + \left(\frac{d-1}{2kR}\right)^2}}\,.$$

Exercise 4 (Proof of Lemma 2.16 in a special case)**.** We prove lemma 2.16 in the case where $\Omega_- = \emptyset$ and $A = I$. We apply the inequality (2.14) to the vector field ∇u, where u is the H^2 solution of problem (2.8). It gives

$$\int_{B_R} |\Delta u|^2 - \sum_{i,j=1}^{d} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \geq -2k\Re\langle (\nabla_T \gamma u), \nabla_T (\mathrm{DtN}_k \gamma u) \rangle_{\Gamma_R} \,,$$

where we have used the property $(\gamma \nabla u)_T = \nabla_T(\gamma u)$ on Γ_R. One can show that ∇_T and DtN_k commute on Γ_R, either by exploiting rotational invariance, or by using the definition of DtN_k and ∇_T in terms of Fourier series on Γ_R. Hence,

$$|u|^2_{H^2(B_R)} \leq \|f\|^2_{L^2(B_R)} + 2k\Re\langle(\nabla_T \gamma u), \mathrm{DtN}_k(\nabla_T \gamma u)\rangle \, .$$

By Lemma 1.7, the second term of the rhs is negative. Hence, in this particular case, we have established the inequality

$$\boxed{|u|^2_{H^2(B_R)} \leq \|f\|^2_{L^2(B_R)}}$$

which is stronger than (2.9).