Exercises of lecture 1

Exercise 1 (Sign of imaginary part of DtN_k). Let $\phi \in H^{1/2}(\Gamma_R) \setminus \{0\}$ and let u be the outgoing solution of

$$(-k^{-2}\Delta - 1)u = 0 \quad \text{in } B \setminus \overline{B_R} \quad \gamma u = \phi \text{ on } \Gamma_R\,.$$

We consider a large ball B_ρ where $\rho > R$ will be sent to infinity in what follows, and denote

$$\Omega := B_\rho \setminus \overline{B}_R\,.$$

On the boundary $\partial\Omega$, which has the two connected components Γ_R and Γ_ρ, we denote by $\boldsymbol{\nu}_{\partial\Omega}$ the outward pointing normal vector. Notice that

$$\forall x \in \Gamma_R\,,\ \boldsymbol{\nu}_{\partial\Omega}(x) = -\boldsymbol{\nu}(x) \quad \text{and} \quad \forall x \in \Gamma_\rho\,,\ \boldsymbol{\nu}_{\partial\Omega}(x) = \boldsymbol{\nu}(x) = \hat{\boldsymbol{x}}\,.$$

One has $\Delta u \in L^2(\Omega)$ since $u \in H^1(\Omega)$ and $\Delta u = -k^2 u$. Therefore, we may apply Green's formula in Ω:

$$\int_\Omega \overline{u}\Delta u + \int_\Omega |\nabla u|^2 = \langle \partial_\nu u, \gamma u\rangle_{\partial\Omega}\,.$$

In the previous equation, we take the imaginary part and take into account the following facts:

- $\Delta u = -k^2 u$ so that $\overline{u}\,\Delta u$ is real,
- $\gamma u = \phi$ and $\partial_\nu u = k\,\mathrm{DtN}_k\,\phi$ on Γ_R,
- u is in $H^2(\mathcal{O})$ for some open neighborhood of Γ_ρ, so that $\partial_\nu u = \frac{\partial u}{\partial \rho}$ on Γ_ρ.

This leads to

$$\Im\langle \mathrm{DtN}_k\,\phi, \phi\rangle = k^{-1}\Im\left(\int_{\Gamma_\rho} \overline{u}\frac{\partial u}{\partial\rho}\right),$$

and this holds for all $\rho > R$. It follows from Lemma 1.4 that

$$|u(x)|^2 = \frac{|f_0(\hat{\boldsymbol{x}})|}{\rho^{d-1}} + O(\rho^{-d})\,,$$

$$\overline{u(x)}\frac{\partial u}{\partial\rho}(x) = ik\frac{|f_0(\hat{\boldsymbol{x}})|^2}{\rho^{d-1}} + O\left(\rho^{-d}\right)\,,$$

which implies

$$\int_{\Gamma_\rho} |u(x)|^2 = \int_{|\omega|=1} |f_0(\omega)|^2 \, d\sigma(\omega) + O(\rho^{-1})\,,$$

$$\int_{\Gamma_\rho} \overline{u}\frac{\partial u}{\partial \rho} = ik \int_{|\omega|=1} |f_0(\omega)|^2 \, d\sigma(\omega) + O(\rho^{-1})\,.$$

One must have

$$\int_{|\omega|=1} |f_0(\omega)|^2 \, d\sigma(\omega) > 0,$$

otherwise we would have $f_0 = 0$ so $u = 0$ by Lemma 1.4 (but we assumed $\gamma u = \phi \neq 0$). Hence, for ρ sufficiently large,

$$\Im\left(\int_{\Gamma_\rho} \overline{u}\frac{\partial u}{\partial \rho}\right) > 0\,,$$

concluding the proof.

Exercise 2 (Equivalence between **EDP** and variational problem)**.**

First, let u be a solution of EDP with data $g_D = 0$ and let $\tilde{u} = u_{|\Omega_R}$. We first remark that $\nabla \cdot (A\nabla\tilde{u}) \in L^2(\Omega_R)$ in the weak sense and it is equal to $-k^{-2}f - n\tilde{u}$. Now, fix $\phi \in C^\infty_{\mathrm{comp}}(\Omega_R)$. Applying Lemma 1.8, we find

$$\int_{\Omega_R} \overline{\phi}\nabla\cdot(A\nabla u) + \int_{\Omega}(A\nabla u)\cdot\overline{\nabla\phi} = \left\langle \frac{\partial u}{\partial \nu_A}, \gamma\phi \right\rangle_{\partial\Omega_R}\,,$$

Since $\gamma\phi = 0$ on Γ_D, we have

$$\langle \partial_{\nu_A} u, \gamma\phi\rangle_{\partial\Omega_R} = \langle \partial_\nu u, \gamma\phi\rangle_{\Gamma_R} = k\,\langle \mathrm{DtN}_k\gamma\tilde{u}, \gamma\phi\rangle_{\Gamma_R}$$

hence

$$a(\tilde{u},\phi) = -k^{-2}\int_{\Omega_R}\overline{\phi}\nabla\cdot(A\nabla u) + \int_{\Omega_R} nu\overline{\phi} = -\int_{\Omega_R} f\overline{\phi}\,.$$

By the density of $C^\infty_{\mathrm{comp}}(\Omega_R)$ in $H^1_{0,D}(\Omega_R)$ and the continuity of the bilinear form a and the linear form F, it follows that

$$\forall v \in H^1_{0,D}(\Omega_R)\,,\quad a(\tilde{u},v) = F(v)\,.$$

Reciprocally, let us now fix a solution $\tilde{u}$ of the variational problem (1.21) with a rhs $F(v) = \int_{\Omega_R} f\overline{v}$. We introduce U_R the outgoing solution of

$$(-k^{-2}\Delta - 1)U_R = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_R}\,, \quad \text{and} \quad \gamma U_R = \gamma\tilde{u} \text{ on } \Gamma_R\,,$$

and consider

$$u(x) = \begin{cases} \tilde{u}(\boldsymbol{x}) & |\boldsymbol{x}| < R\,, \\ U_R(\boldsymbol{x}) & |\boldsymbol{x}| > R\,. \end{cases}$$

This piecewise $H^1_{\mathrm{loc}}(\Omega_R \cup (\mathbb{R}^d \setminus \overline{B_R}))$ function has matching traces, so it is in $H^1_{\mathrm{loc}}(\Omega^+)$, it is obviously outgoing, and satisfies $\gamma u = 0$. It remains to check that

$$k^{-2}\nabla \cdot (A\nabla u) + nu = -f \quad \text{ in } \Omega^+ ,$$

in the weak sense (where f also denotes the extension of f by 0 on $\Omega^+ \setminus \Omega_R$). In other words, for every $\phi \in C^\infty_{\mathrm{comp}}(\Omega^+)$ we must show that

$$I := \int_{\Omega^+} k^{-2} u \nabla \cdot (A\overline{\nabla\phi}) + nu\overline{\phi} + f\overline{\phi} = 0 .$$

Letting $\rho > \mathrm{diam}(\mathrm{supp}\,\phi)$, we split the integral as

$$\begin{aligned} I = &\int_{\Omega_R} k^{-2}\tilde{u}\nabla \cdot (A\overline{\nabla\phi}) + n\tilde{u}\overline{\phi} + f\overline{\phi} \\ &+ \int_{B_\rho \setminus \overline{B_R}} k^{-2} U_R \overline{\Delta\phi} + U_R\overline{\phi} \end{aligned}$$

We next apply Green's theorem for the first integral, which is possible since $\phi \in H^2(\Omega_R)$. This leads to

$$\begin{aligned} I = &\int_{\Omega_R} -k^{-2}(A\nabla\tilde{u}) \cdot \overline{\nabla\phi} + n\tilde{u}\overline{\phi} + f\overline{\phi} + k^{-2}\left\langle \gamma\tilde{u}, \frac{\partial\phi}{\partial\nu}\right\rangle_{\Gamma_R} \\ &+ \int_{B_\rho \setminus \overline{B_R}} k^{-2} U_R \overline{\Delta\phi} + U_R\overline{\phi} . \end{aligned}$$

For the second integral, we apply the formula

$$\int_{B_\rho \setminus \overline{B}_R} U_R \Delta\overline{\phi} - \Delta U_R \overline{\phi} = \left\langle \gamma U_R, \frac{\partial\phi}{\partial\nu_{\partial\Omega}}\right\rangle_{\Gamma_R \cup \Gamma_\rho} - \left\langle \frac{\partial U_R}{\partial\nu_{\partial\Omega}}, \gamma\phi\right\rangle_{\Gamma_R \cup \Gamma_\rho}$$

where $\nu_{\partial\Omega}$ is the normal vector pointing out of $\Omega := B_\rho \setminus \overline{B_R}$. This can be proved applying Green's theorem twice in $B_\rho \setminus \overline{B_R}$ and noticing that $\Delta U_R = -k^2 U_R \in L^2(B_\rho \setminus \overline{B_R})$. Furthermore, we have

- $\nu_{\partial\Omega} = -\nu$ on Γ_R,
- $\frac{\partial U_R}{\partial\nu_{\partial\Omega}} = -k\, \mathrm{DtN}_k \gamma U_R$ on Γ_R,
- $\gamma U_R = \gamma\tilde{u}$ on Γ_R,
- $\phi = 0$ in a neighborhood of Γ_ρ so that $\gamma\phi = 0$ and $\frac{\partial\phi}{\partial\nu} = 0$ on Γ_ρ.

From those remarks, we conclude that

$$\int_{B_\rho \setminus \overline{B_R}} k^{-2} U_R \Delta\overline{\phi} + U_R\overline{\phi} = -k^{-2}\left\langle \gamma\tilde{u}, \frac{\partial\phi}{\partial\nu}\right\rangle_{\Gamma_R} + k^{-1}\left\langle \mathrm{DtN}_k\gamma\tilde{u}, \gamma\phi\right\rangle_{\Gamma_R} .$$

We inject this expression in I to find:

$$I = -a(\tilde{u}, \phi) + F(\phi) = 0 ,$$

concluding the proof.

Exercise 3 (Well posedness of the variational formulation)**.** Let us assume that Ω_{-}, A and n satisfy Assumption 1.1, and let $\tilde{u}$ and $\tilde{v}$ be two solutions of the variational problem (1.21). Let $\tilde{w} = \tilde{u} - \tilde{v}$, then $\tilde{w} \in H^1_{0,D}(\Omega_R)$ and satisfies

$$a(\tilde{w}, \phi) = a(\tilde{u}, \phi) - a(\tilde{u}, \phi) = 0\,, \quad \forall \phi \in H^1_{0,D}(\Omega_R)\,.$$

Applying this to $\phi = \tilde{w}$ and taking the imaginary part, it follows that

$$\Im \langle \mathrm{DtN}_k \gamma \tilde{w}, \gamma \tilde{w} \rangle_{\Gamma_R} = 0\,.$$

From Lemma 1.7, we conclude that $\gamma \tilde{w} = 0$ on Γ_R. We may thus consider

$$\tilde{w}'(x) := \begin{cases} \tilde{w}(x) & x \in \Omega_R \\ 0 & x \in B_\rho \setminus \overline{\Omega_R} \end{cases}$$

which defines a function in $H^1_{0,D}(\Omega_\rho)$, where $\Omega_\rho = \Omega^+ \cap B_\rho$. One can check that $\tilde{w}'$ satisfies the variational problem (1.21) with $F = 0$. Thus, by Lemma 1.11, there exists $w \in H^1_{\mathrm{loc}}(\Omega^+)$ such that $\tilde{w}' = w_{|\Omega_\rho}$, where w is a solution of the EDP

$$k^{-2} \nabla \cdot (A \nabla w) + nu = 0\,,$$

on Ω^+. Since w is identically 0 in some ball far from B_R, it must be 0 everywhere by the unique continuation principle. It follows that $\tilde{w} = 0$, and thus, $\tilde{u} = \tilde{v}$, establishing the uniqueness of the solution to the variational problem (1.21).