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When is the error in the h–BEM for solving the Helmholtz equation bounded independently of k?

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Abstract We consider solving the sound-soft scattering problem for the Helmholtz equation with the h–version of the boundary element method using the standard second-kind combined-field integral equations. We obtain sufficient conditions for the relative best approximation error to be bounded independently of k. For certain geometries, these rigorously justify the commonly-held belief that a fixed number of degrees of freedom per wavelength is sufficient to keep the relative best approximation error bounded independently of k. We then obtain sufficient conditions for the Galerkin method to be quasi-optimal, with the constant of quasi-optimality independent of k. Numerical experiments indicate that, while these conditions for quasi-optimality are sufficient, they are not necessary for many geometries.

Keywords Helmholtz equation · high frequency · boundary integral equation · boundary element method · pollution effect

Mathematics Subject Classification (2000) 35J05 · 65N38 · 65R20

1 Introduction

Integral equations are often used to solve acoustic, electromagnetic, and elastic scattering problems in homogeneous media. In this paper, we consider solving the sound-soft scattering problem for the Helmholtz equation in two or three dimensions using the standard second-kind combined-field integral equations. We write these integral equations as

$$A'_{k,\eta} v = f \tag{1.1}$$

and

$$A_{k,\eta} \phi = g. \tag{1.2}$$

The operators $A'_{k,\eta}$ and $A_{k,\eta}$ are defined by

$$A'_{k,\eta} := \frac{1}{2} I + D'_k - \mathrm{i}\eta S_k, \qquad A_{k,\eta} := \frac{1}{2} I + D_k - \mathrm{i}\eta S_k, \tag{1.3}$$

where $\eta \in \mathbb{R}\setminus\{0\}$ is the coupling parameter (which is usually taken to be proportional to k), S_k is the single-layer operator, D_k is the double-layer operator, and D'_k is the adjoint double-layer operator (these three

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integral operators are defined by equations (1.10) and (1.14) below). The unknowns f and g are defined in terms of the incident field by (1.11) and (1.13) respectively.

The equation (1.1) is the *direct* formulation, with the unknown v equal to the normal derivative on Γ of the total field, where Γ denotes the boundary of the obstacle. The equation (1.2) is the *indirect* formulation, and the physical meaning of ϕ is less clear than it is for v; it turns out that ϕ is the difference of traces of certain exterior and interior Helmholtz boundary value problems (BVPs); see [12, p.132].

We consider the equations (1.1) and (1.2) as equations in $L^2(\Gamma)$. Although there are several ways to solve integral equations such as these, we restrict attention to the the Galerkin method, i.e. approximations v_N and ϕ_N are sought in a finite dimensional approximation space $\mathscr{V}_N$ (where N is the dimension, i.e. the total number of degrees of freedom). In this paper we consider the h–version of the Galerkin method, i.e. $\mathscr{V}_N$ consists of piecewise polynomials of degree p for some fixed $p \geq 0$. In the majority of the paper Γ is C^2, in which case $\mathscr{V}_N$ will be the space of piecewise polynomials of degree p for some fixed $p \geq 0$ on shape regular meshes of diameter h, with h decreasing to zero (see, e.g., [40, Chapter 4] for specific realisations); in this case we denote $\mathscr{V}_N, v_N$, and ϕ_N by $\mathscr{V}_h$, v_h, and ϕ_h respectively, and note that $N \sim h^{-(d-1)}$, where d is the dimension. We also consider the case when Γ is the boundary of a 2–d polygon, and in this case $\mathscr{V}_N$ will consist of piecewise polynomials on a mesh appropriately graded towards the corners (we give more details below).

In this paper we investigate the following two questions.

Question 1: What are sufficient conditions on N for

$$\frac{\inf_{w_N \in \mathscr{V}_N} \|v - w_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} \quad \text{and} \quad \frac{\inf_{w_N \in \mathscr{V}_N} \|\phi - w_N\|_{L^2(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}} \tag{1.4}$$

to be bounded independently of k as $k \to \infty$? (In other words, what are sufficient conditions for the relative best approximation error to be bounded independently of k?)

Question 2: What are sufficient conditions on N for

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{\inf_{w_N \in \mathscr{V}_N} \|v - w_N\|_{L^2(\Gamma)}} \quad \text{and} \quad \frac{\|\phi - \phi_N\|_{L^2(\Gamma)}}{\inf_{w_N \in \mathscr{V}_N} \|\phi - w_N\|_{L^2(\Gamma)}} \tag{1.5}$$

to be bounded independently of k as $k \to \infty$? (In other words, what are sufficient conditions for the Galerkin method to be quasi-optimal, with the constant of quasi-optimality independent of k?)

Answering both Questions 1 and 2 then gives us sufficient conditions on N for the relative errors

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} \quad \text{and} \quad \frac{\|\phi - \phi_N\|_{L^2(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}} \tag{1.6}$$

to be bounded independently of k as $k \to \infty$.

Regarding Question 1: It is generally believed that, for both discretisations in the domain and on the boundary, employing a fixed number of degrees of freedom per wavelength is sufficient to keep the relative best approximation error bounded independently of k. However, to the authors' knowledge, this has only ever been rigorously proved for the Helmholtz equation posed in a 1-d interval, where an explicit expression for the solution is available.

Using results about the large–k asymptotics of the solution to the scattering problem, we prove that a fixed number of degrees of freedom per wavelength is sufficient to keep the quantity in (1.4) involving v bounded independently of k when *either* the obstacle is a 2-d smooth, convex domain with strictly positive curvature *or* the obstacle is a convex polygon. (Since we use results about the asymptotics of the solution of the scattering problem, these results only apply to the direct formulation (1.1), where the unknown in the integral equation is related in a simple way to the solution.) In 2-d the function v is a function of one spatial dimension, and thus these results are also, in some sense, one dimensional. However, the behaviour of v for these two geometries incorporates complicated features of the solution (rays hitting a point of tangency, rays hitting corners) that are not found when the Helmholtz equation is posed in a 1-d interval.

We also prove that, under the sole geometric restriction that Γ is C^2, the condition $hk^{(d+1)/2} \lesssim 1$ is sufficient to keep both the quantities in (1.4) bounded independently of k. Although this is a more restrictive condition than having a fixed number of degrees of freedom per wavelength (i.e. $hk \sim 1$), especially when $d = 3$, the novelty is that this result holds for general domains (even those trapping domains where the inverse of the Helmholtz operator blows up as $k \to \infty$), and for both the direct *and* indirect formulations (despite ϕ's lack of immediate physical relevance). These results are obtained by using the fact that the integral operators $A'_{k,\eta}$ and $A_{k,\eta}$ in (1.3) are compact perturbations of the identity when Γ is C^2, and then proving new k–explicit bounds on the operators S_k, D'_k, and D_k as mappings from $L^2(\Gamma) \to H^1(\Gamma)$.

Regarding Question 2: We prove that for C^2 star-shaped domains in 2- or 3-d, the quantities in (1.5) are bounded independently of k if $hk^{(d+1)/2} \lesssim 1$. (We expect that this result holds for general nontrapping domains, but the currently-available bounds on the inverses of $A'_{k,\eta}$ and $A_{k,\eta}$ for these domains are not sharp enough in their k–dependence to prove this.) Combining this result with the results addressing Question 1, we have that for C^2 star-shaped domains in 2- or 3-d, the quantities in (1.6) are bounded independently of k if $hk^{(d+1)/2} \lesssim 1$.

We discuss the relation of these results to other existing results in detail in §1.2.2, but we note here that the only other available bounds on (1.5) in the literature are valid when the obstacle is C^3 and piecewise analytic with strictly positive curvature [45] or is a ball [4], [18]. (An error analysis of the hp–BEM on analytic domains has recently been conducted in [28], [31]. However, since these techniques are geared towards a p–BEM, they yield a more restrictive condition than $hk^{(d+1)/2} \lesssim 1$ for k–independent quasi-optimality of the h–BEM; for more discussion see §4.2.)

We obtain the bound on the quantities in (1.5) using the classic abstract projection-method argument going back to Anselone [2] and Atkinson [3]. This argument treats the operators $A'_{k,\eta}$ and $A_{k,\eta}$ as compact perturbations of the identity, as is standard. The novelty, however, is that everything can be made k–explicit by using (i) recently-proved k–explicit bounds on the norms of $(A'_{k,\eta})^{-1}$ and $A^{-1}_{k,\eta}$ as operators from $L^2(\Gamma) \to L^2(\Gamma)$, and (ii) new k–explicit bounds on the operators S_k, D'_k, and D_k as operators from $L^2(\Gamma) \to H^1(\Gamma)$.

1.1 Formulation of the problem

Let $\Omega_- \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded Lipschitz open set with boundary $\Gamma := \partial\Omega_-$, such that the open complement $\Omega_+ := \mathbb{R}^d \setminus \overline{\Omega_-}$ is connected. Let $H^1_{\rm loc}(\Omega_+)$ denote the set of functions v such that v is locally integrable on Ω_+ and $\psi v \in H^1(\Omega_+)$ for every compactly supported $\psi \in C^\infty(\overline{\Omega_+}) := \{\psi|_{\Omega_+} : \psi \in C^\infty(\mathbb{R}^d)\}$. Let γ_+ denote the trace operator from Ω_+ to Γ. Let $\mathbf{n}$ be the outward-pointing unit normal vector to Ω_-, and let ∂_n^+ denote the normal derivative trace operator from Ω_+ to Γ that satisfies $\partial_n^+ u = \mathbf{n} \cdot \gamma_+(\nabla u)$ when $u \in H^2_{\rm loc}(\Omega_+)$. (We also call $\gamma_+ u$ the Dirichlet trace of u and $\partial_n^+ u$ the Neumann trace.)

Definition 1.1 (Sound-soft scattering problem) Given $k > 0$ and an incident plane wave $u^I(\mathbf{x}) = \exp({\rm i}k\mathbf{x} \cdot \widehat{\mathbf{a}})$ for some $\widehat{\mathbf{a}} \in \mathbb{R}^d$ with $|\widehat{\mathbf{a}}| = 1$, find $u^S \in C^2(\Omega_+) \cap H^1_{loc}(\Omega_+)$ such that the total field $u := u^I + u^S$ satisfies

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega_+, \qquad \gamma_+ u = 0 \quad \text{on } \Gamma,$$

and u^S satisfies the Sommerfeld radiation condition,

$$\frac{\partial u^S}{\partial r}(\mathbf{x}) - {\rm i}k u^S(\mathbf{x}) = o\left(\frac{1}{r^{(d-1)/2}}\right)$$

as $r := |\mathbf{x}| \to \infty$, uniformly in $\widehat{\mathbf{x}} := \mathbf{x}/r$.

It is well known that the solution to this problem exists and is unique; see, e.g., [12, Theorem 2.12].

The BVP in Definition 1.1 can be reformulated as an integral equation on Γ in two different ways. The first, the so-called *direct method*, uses Green's integral representation for the solution u, i.e.

$$u(\mathbf{x}) = u^I(\mathbf{x}) - \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y}) \partial_n^+ u(\mathbf{y}) \,{\rm d}s(\mathbf{y}), \quad \mathbf{x} \in \Omega_+, \tag{1.7}$$

where $\Phi_k(\mathbf{x},\mathbf{y})$ is the fundamental solution of the Helmholtz equation given by

$$\Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{i}}{4}H_0^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big),\ d=2, \qquad \Phi_k(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},\ d=3 \tag{1.8}$$

(note that to obtain (1.7) from the usual form of Green's integral representation one must use the fact that u^I is a solution of the Helmholtz equation in Ω_-; see, e.g., [12, Theorem 2.43]).

Taking the Dirichlet and Neumann traces of (1.7) on Γ, one obtains two integral equations for the unknown Neumann boundary value $\partial_n^+ u$:

$$S_k\partial_n^+ u = \gamma_+ u^I, \qquad \left(\frac{1}{2}I + D_k'\right)\partial_n^+ u = \partial_n^+ u^I, \tag{1.9}$$

where the integral operators S_k and D_k', the single-layer operator and the adjoint-double-layer operator respectively, are defined for $\psi \in L^2(\Gamma)$ by

$$S_k\psi(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad D_k'\psi(\mathbf{x}) := \int_\Gamma \frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma \tag{1.10}$$

(when Γ is Lipschitz, the integral defining D_k' is understood as a Cauchy principal value integral; see, e.g., [12, §2.3]).

Both integral equations in (1.9) fail to be uniquely solvable for certain values of k (for the first equation in (1.9) these are the k such that k^2 is a Dirichlet eigenvalue of the Laplacian in Ω_-, and for the second equation in (1.9) these are the k such that k^2 is a Neumann eigenvalue). The standard way to resolve this difficulty is to take a linear combination of the two equations, which yields the integral equation (1.1), where $v := \partial_n^+ u$ and

$$f(\mathbf{x}) = \partial_n^+ u^I(\mathbf{x}) - \mathrm{i}\eta\gamma_+ u^I(\mathbf{x}), \quad \mathbf{x}\in\Gamma. \tag{1.11}$$

Since Ω_+ is Lipschitz, standard trace results imply that the unknown Neumann boundary value $\partial_n^+ u$ is in $H^{-1/2}(\Gamma)$. When Ω_+ is C^2, elliptic regularity implies that $\partial_n^+ u \in L^2(\Gamma)$ (since $u\in H^2_{\mathrm{loc}}(\Omega_+)$), but $\partial_n^+ u \in L^2(\Gamma)$ even when Ω_+ is Lipschitz via a regularity result of Nečas [36, §5.1.2], [29, Theorem 4.24 (ii)]. Therefore, even for Lipschitz Ω_+ we can consider the integral equation (1.1) as an operator equation in $L^2(\Gamma)$, which is a natural space for the practical solution of second-kind integral equations since it is self-dual. It is well known that, for $\eta\neq 0$, $A_{k,\eta}'$ is a bounded and invertible operator on $L^2(\Gamma)$ (see [12, Theorem 2.27]).

Instead of using Green's integral representation to formulate the BVP as an integral equation, one can pose the ansatz

$$u^S(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \mathrm{i}\eta\int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \tag{1.12}$$

for $\phi\in L^2(\Gamma)$ and $\eta\in\mathbb{R}\setminus\{0\}$; this is the so-called *indirect method*. Imposing the boundary condition $\gamma_+ u^S = -\gamma_+ u^I$ on Γ leads to the integral equation (1.2) with

$$g := -\gamma_+ u^I \tag{1.13}$$

and where D_k is the double-layer operator, which is defined for $\psi\in L^2(\Gamma)$ by

$$D_k\psi(\mathbf{x}) = \int_\Gamma \frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma \tag{1.14}$$

(as with D_k', the integral defining D_k is understood as a Cauchy principal value integral when Γ is Lipschitz).

Although the unknowns in the integral equations (1.1) and (1.2) are different, the identities

$$\int_\Gamma \phi\, S_k\psi\,\mathrm{d}s = \int_\Gamma \psi\, S_k\phi\,\mathrm{d}s, \text{ and } \int_\Gamma \phi\, D_k\psi\,\mathrm{d}s = \int_\Gamma \psi\, D_k'\phi\,\mathrm{d}s, \tag{1.15}$$

for $\phi,\psi\in L^2(\Gamma)$ (see [12, Equation 2.37]), mean that $A_{k,\eta}$ and $A_{k,\eta}'$ are adjoint with respect to the real-valued $L^2(\Gamma)$ inner product, and so in particular satisfy

$$\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A_{k,\eta}'\|_{L^2(\Gamma)\to L^2(\Gamma)} \quad \text{and} \quad \|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|(A_{k,\eta}')^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}.$$

In this paper we consider solving the integral equations (1.1) and (1.2) using the Galerkin method. Given a finite-dimensional approximation space $\mathscr{V}_N \subset L^2(\Gamma)$, the Galerkin method for the direct integral equation (1.1) is

$$\text{find } v_N \in \mathscr{V}_N \text{ such that } \left(A'_{k,\eta} v_N, w_N\right)_{L^2(\Gamma)} = \left(f, w_N\right)_{L^2(\Gamma)} \text{ for all } w_N \in \mathscr{V}_N. \tag{1.16}$$

For the indirect integral equation (1.2), the Galerkin method is

$$\text{find } \phi_N \in \mathscr{V}_N \text{ such that } \left(A_{k,\eta} \phi_N, w_N\right)_{L^2(\Gamma)} = \left(g, w_N\right)_{L^2(\Gamma)} \text{ for all } w_N \in \mathscr{V}_N. \tag{1.17}$$

1.2 Statement of the main results and discussion

This paper contains six main theorems (Theorems 1.1, 1.2, 1.3, 1.4, 1.5, 1.6). The first three (1.1, 1.2, and 1.3) give sufficient conditions for the relative best approximation error to be bounded independently of k (and thus provide an answer to Question 1). The next two (1.4 and 1.5) give sufficient conditions for the h–version of the BEM to be quasi-optimal, with the constant of quasi-optimality independent of k (and thus provide an answer Question 2). The last one (1.6) gives bounds on the norms of S_k, D_k, and D'_k as operators from $L^2(\Gamma) \to H^1(\Gamma)$; these bounds are the main new ingredients used to prove Theorems 1.3, 1.4, and 1.5.

In what follows, $\mathscr{V}_h$ is the space of piecewise polynomials of degree p for some fixed $p \geq 0$ on shape regular meshes of diameter h, with h decreasing to zero. As above, u is the solution of the sound-soft scattering problem of Definition 1.1, and $v := \partial_n^+ u$.

We use the notation $a \lesssim b$ to mean $a \leq Cb$ for some constant C that is independent of k, η, and h. $a \gtrsim b$ means $b \lesssim a$. If $a \lesssim b$ and $b \lesssim a$ we write $a \sim b$.

1.2.1 Results concerning Question 1

Theorem 1.1 (Bound on the best approximation error for smooth convex domains) *If Ω_- is a 2-d, C^∞, convex domain with strictly positive curvature then, given $k_0 > 0$,*

$$\inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)} \lesssim hk \|v\|_{L^2(\Gamma)} \tag{1.18}$$

for all $k \geq k_0$. Thus, choosing $hk \lesssim 1$ keeps the relative best approximation error bounded independently of k.

The right-hand side of the bound (1.18) is not explicit in p. Nevertheless, using the same ideas used to prove (1.18), one can show that $\inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)} \lesssim (hk)^{p+1} \|v\|_{L^2(\Gamma)}$. Therefore, for fixed p, one still requires $hk \lesssim 1$ for this bound to prove that the relative best approximation error is bounded independently of k.

Theorem 1.2 (Bound on the best approximation error for convex polygons) *Let Ω_- be a convex polygon, and let*

$$M(u) := \sup_{\mathbf{x} \in \Omega_+} |u(\mathbf{x})|.$$

If $M(u) \lesssim 1$ then there exists a mesh on Γ with $\mathscr{O}(N)$ points such that, with $\mathscr{V}_N$ the corresponding space of piecewise polynomials, given $k_0 > 0$,

$$\inf_{w_N \in \mathscr{V}_N} \|v - w_N\|_{L^2(\Gamma)} \lesssim \frac{k}{N} \|v\|_{L^2(\Gamma)} \tag{1.19}$$

for all $k \geq k_0$. Thus, choosing $N \gtrsim k$ keeps the relative best approximation error bounded independently of k. (We give the details of the mesh in the proof of the theorem.)

Regarding the assumption $M(u) \lesssim 1$, the best currently-available bound on $M(u)$ is

$$M(u) \lesssim k^{1/2} \left(\log k\right)^{1/2}, \tag{1.20}$$

[24, Theorem 4.3], but numerical experiments in [14] and [13] indicate that $M(u) \lesssim 1$. Furthermore, if Ω_- is a star-shaped Lipschitz domain then the arguments in [35] can be used to show that, for any $R > 0$, $\|u\|_{L^2(\Omega_R)} \lesssim 1$, where $\Omega_R := \Omega_+ \cap \{\mathbf{x} : |\mathbf{x}| < R\}$; this is consistent with $M(u) \lesssim 1$, but does not imply it.

Theorem 1.3 (Bound on the best approximation error for general C^2 domains) *If $\Omega_- \subset \mathbb{R}^d$, with $d = 2$ or 3, and Γ is C^2, then, given $k_0 > 0$,*

$$\inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)} \lesssim hk^{(d+1)/2} \|v\|_{L^2(\Gamma)} \tag{1.21}$$

and

$$\inf_{w_h \in \mathscr{V}_h} \|\phi - w_h\|_{L^2(\Gamma)} \lesssim hk^{(d+1)/2} \|\phi\|_{L^2(\Gamma)} \tag{1.22}$$

for all $k \geq k_0$. Thus, choosing $hk^{(d+1)/2} \lesssim 1$ keeps the relative best approximation error bounded independently of k.

At first sight, it may seem surprising that Theorem 1.3 bounds the relative best approximation error for both direct and indirect formulations with no restriction on the geometry (apart from Γ being C^2), since $\|v\|_{L^2(\Gamma)}$ is strongly influenced by the geometry. Indeed, if Ω_- is a star-shaped Lipschitz domain then $k^{1/2} \lesssim \|v\|_{L^2(\Gamma)} \lesssim k$ (see §3.2), but if Ω_+ contains an elliptical cavity then there exist a sequence of wavenumbers $0 < k_1 < k_2 < \ldots$ with $k_m \to \infty$ as $m \to \infty$ and a constant $\gamma > 0$ such that $\|v\|_{L^2(\Gamma)} \gtrsim \exp(\gamma k_m)$ for all $m \geq 1$ (see [6, Equation 2.33], [12, §5.6.2], [43, §1.2]).

The results of Theorem 1.3 become more understandable when we note that the results of both Theorems 1.1 and 1.3 use the standard approximation theory result that, for $w \in H^1(\Gamma)$,

$$\inf_{w_h \in \mathscr{V}_h} \|w - w_h\|_{L^2(\Gamma)} \lesssim h \|w\|_{H^1(\Gamma)} \tag{1.23}$$

[40, Theorem 4.3.22(b)]. The bound (1.18) is then obtained from the bound

$$\|v\|_{H^1(\Gamma)} \lesssim k \|v\|_{L^2(\Gamma)} \tag{1.24}$$

for C^∞, convex domains with strictly positive curvature, and the bounds (1.21) and (1.22) are obtained from the bounds

$$\|v\|_{H^1(\Gamma)} \lesssim k^{(d+1)/2} \|v\|_{L^2(\Gamma)} \quad \text{and} \quad \|\phi\|_{H^1(\Gamma)} \lesssim k^{(d+1)/2} \|\phi\|_{L^2(\Gamma)} \tag{1.25}$$

for C^2 domains. The bound (1.24) is obtained using results about the large–k asymptotics of v from [32], converted into a format suitable for numerical analysis in [18]. The bounds in (1.25) are obtained by using the fact that v and ϕ satisfy the integral equations (1.1) and (1.2) respectively. Indeed, when Γ is C^2 the operators $A'_{k,\eta}$ and $A_{k,\eta}$ are compact perturbations of the identity, with S_k, D_k, and D'_k all mapping $L^2(\Gamma)$ to $H^1(\Gamma)$. In Theorem 1.6 below we prove k–explicit bounds on the norms of S_k, D_k, and D'_k from $L^2(\Gamma)$ to $H^1(\Gamma)$, and then taking the $H^1(\Gamma)$ norms of the integral equations (1.1) and (1.2) and using these bounds essentially yields (1.25).

When Ω_- is a convex polygon, $v \notin H^1(\Gamma)$ and thus we cannot use (1.23). Nevertheless, the bound (1.19) is obtained from a result analogous to (1.23) where the H^1–norm is replaced by a weighted H^1–norm, $\mathscr{V}_h$ is replaced by $\mathscr{V}_N$, and h is replaced by $1/N$. The analogue of the bound (1.24) is then obtained using results about the large–k asymptotics of v from [14], in a similar way to how (1.24) is obtained in the smooth convex case.

To the authors' knowledge, the only other Helmholtz BVP where rigorous results about the relative best approximation error are available is the Helmholtz equation posed on a 1-d interval (with an impedance boundary condition imposed at one end to ensure that the problem has a unique solution for all k). In this case, the bound $\|u\|_{H^1(\Omega)} \lesssim k\|u\|_{L^2(\Omega)}$ can be verified using the explicit expression for the solution.

1.2.2 Results concerning Question 2

Before stating the two theorems concerning Question 2, we need to make the following definition.

Definition 1.2 (Star-shaped) If $\Omega_- \subset \mathbb{R}^d$, $d = 2$ or 3, is a Lipschitz domain we say that it is *star-shaped* if there exists a constant $c > 0$ such that $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq c$ for every $\mathbf{x} \in \Gamma$ for which $\mathbf{n}(\mathbf{x})$ is defined. (This condition is sometimes known as being *star-shaped with respect to a ball*; see, e.g., [34, Remark 3.5].)

Theorem 1.4 (Sufficient conditions for the Galerkin method to be quasi-optimal) *If Ω_- is C^2 and star-shaped (in the sense of Definition 1.2) and $|\eta| \sim k$ then, given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if*

$$hk^{(d+1)/2} \leq C, \tag{1.26}$$

then both sets of Galerkin equations (1.16) and (1.17) have unique solutions which satisfy

$$\|v - v_h\|_{L^2(\Gamma)} \lesssim \inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)} \tag{1.27}$$

and

$$\|\phi - \phi_h\|_{L^2(\Gamma)} \lesssim \inf_{w_h \in \mathscr{V}_h} \|\phi - w_h\|_{L^2(\Gamma)} \tag{1.28}$$

respectively, for all $k \geq k_0$.

The assumption in Theorem 1.4 that Ω_- is star-shaped is there to ensure that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ is bounded independently of k when $|\eta| \sim k$. The numerical experiments in [7, §5] indicate that this property holds whenever Ω_- is nontrapping, however the best currently-available bound on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ for nontrapping domains has a positive power of k on the right-hand side; see (3.3) below. (The argument leading to (1.28) can be repeated with this worse bound on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$, however this yields quasi-optimality under a more restrictive mesh threshold than (1.26).)

Combining Theorems 1.3 and 1.4 we obtain the following result.

Corollary 1.1 (Bound on the relative errors in the Galerkin method) *If Ω_- is C^2 and star-shaped (in the sense of Definition 1.2) and $|\eta| \sim k$ then, given $k_0 > 0$, there exists a $C > 0$ (independent of k and h) such that if*

$$hk^{(d+1)/2} \leq C,$$

then both sets of Galerkin equations (1.16) and (1.17) have unique solutions which satisfy

$$\frac{\|v - v_h\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}} \lesssim 1 \quad \text{and} \quad \frac{\|\phi - \phi_h\|_{L^2(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}} \lesssim 1 \tag{1.29}$$

respectively, for all $k \geq k_0$.

Under a more restrictive mesh threshold we can sharpen the quasi-optimality estimates (1.27) and (1.28) to show that the Galerkin solutions v_h and ϕ_h are asymptotically just as good as the best possible approximations to v and ϕ from $\mathscr{V}_h$.

Theorem 1.5 (Sufficient conditions for the Galerkin method to be asymptotically optimal) *If Ω_- is C^2 and star-shaped (in the sense of Definition 1.2), $|\eta| \sim k$, and h is a function of k such that*

$$hk^{(3d-1)/2} \to 0 \quad \text{as} \quad k \to \infty,$$

then both sets of Galerkin equations (1.16) and (1.17) have unique solutions which satisfy

$$\frac{\|v - v_h\|_{L^2(\Gamma)}}{\inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)}} \quad \text{and} \quad \frac{\|\phi - \phi_h\|_{L^2(\Gamma)}}{\inf_{w_h \in \mathscr{V}_h} \|\phi - w_h\|_{L^2(\Gamma)}} \to 1 \quad \text{as} \quad h \to 0.$$

How sharp are these results? The numerical experiments discussed in §5 show that for a wide variety of 2–d domains, the quasi-optimality (1.27) holds even when $hk \lesssim 1$ These results suggest that the h–BEM does not suffer from the pollution effect, but we do not see this from Theorem 1.4. (For more discussion, see §5.)

How do these results compare with other results about the h–BEM in the literature? For second-kind integral equations such as (1.1) and (1.2), there are several classical approaches to error analysis, all based, in some sense, on the fact that each of $A'_{k,\eta}$ and $A_{k,\eta}$ is a k–dependent compact perturbation of a k–independent invertible operator. (This can easily be seen when Γ is C^1, since in this case D_k, D'_k, and S_k are compact [21, Theorem 1.2], but the result is true even when Γ is Lipschitz [14, Theorem 2.7], [12, Theorem 2.25].) Although these classical approaches establish quasi-optimality for the h–BEM applied to the integral equations (1.1) and (1.2), the k–dependence of both the constant of quasi-optimality and the threshold after which quasi-optimality holds has not been determined until now.

To the authors' knowledge, there exist in the literature two sets of results that give k–explicit quasi-optimality of the h–BEM applied to (1.1) and (1.2) (or any other integral equation used to solve the Helmholtz equation). These are

(a) results that use coercivity [18], [44], [45], and
(b) results that give sufficient conditions for quasi-optimality to hold in terms of how well the spaces $\mathscr{V}_h$ approximate the solution of certain adjoint problems [4], [28], [31].

Regarding (a): in [18], $A'_{k,\eta}$ and $A_{k,\eta}$ are proved to be coercive on $L^2(\Gamma)$ when Γ is the circle or sphere, $\eta = k$, and k is sufficiently large; i.e. it is shown that (for these domains) there exists a $k_0 > 0$ such that, with $\eta = k$,

$$\left|(A'_{k,\eta}\psi,\psi)_{L^2(\Gamma)}\right| \gtrsim \|\psi\|^2_{L^2(\Gamma)} \quad \text{for all } \psi \in L^2(\Gamma) \tag{1.30}$$

and for all $k \geq k_0$ (and similarly for $A_{k,\eta}$). In [45] it is proved that (1.30) holds when Ω_- is a C^3, piecewise analytic, 2- or 3-d domain with strictly positive curvature, $\eta \gtrsim k$, and k is sufficiently large [45, Theorem 1.2]. By Céa's Lemma, these coercivity results imply that, for these domains, the Galerkin solutions v_h and ϕ_h exist for any $h > 0$, and the error esimate

$$\|v - v_h\|_{L^2(\Gamma)} \lesssim \left(\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\right) \inf_{w_h\in\mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)} \tag{1.31}$$

and an analogous one for $\|\phi - \phi_h\|_{L^2(\Gamma)}$ hold for all sufficiently large k. To make the error estimate (1.31) fully k–explicit, we need a k–explicit bound on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ when $\eta \sim k$. When Γ is a circle or sphere and $\eta \sim k$, $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim k^{1/3}$ [12, Theorem 5.12]. The best currently-available bound on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ (when $\eta \sim k$) for smooth convex domains is $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \sim k^{1/2}$ [12, Theorem 5.14] (although this is unlikely to be sharp).

The quasi-optimality result (1.31) is quite different from (1.27); although quasi-optimality is established in (1.31) without any mesh threshold, the factor in front of the best approximation error grows with k. Nevertheless, the results (1.27) and (1.31) can be compared if we use the bound (1.18) on the best approximation error for 2-d smooth, convex domains with strictly positive curvature. Combining (1.31) with (1.18) and using the bounds on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ discussed above, we see that when Γ is the circle the Galerkin error is bounded independently of k (i.e. (1.29) holds) when k is sufficiently large and $hk^{4/3} \lesssim 1$. We also have that when Ω_- is a C^∞, piecewise analytic, 2–d domain with strictly positive curvature, the Galerkin error is bounded independently of k when k is sufficiently large and $hk^{3/2} \lesssim 1$. This result for C^∞, piecewise analytic, 2–d domains with strictly positive curvature is the same as that in Corollary 1.1, but the result for the circle and sphere is slightly sharper than that in Corollary 1.1 (although Corollary 1.1 holds for a much wider class of domains).

The final quasi-optimality result obtained via coercivity concerns a modification of the integral operator $A'_{k,\eta}$, denoted by $\mathscr{A}_k$, that can also be used to solve the sound-soft scattering problem. This operator was introduced in [44], and was proved to be coercive for all $k > 0$ when Ω_- is a star-shaped Lipschitz domain. Since $\|\mathscr{A}_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/2}$, the error estimate

$$\|v - v_N\|_{L^2(\Gamma)} \lesssim k^{1/2} \inf_{w_N\in\mathscr{V}_N} \|v - w_N\|_{L^2(\Gamma)} \tag{1.32}$$

holds for the Galerkin method applied to this integral equation on these domains. The error estimate (1.32) is similar to (1.31) (since they both come from Céa's lemma), although (1.32) is valid for a wider class of domains.

Regarding (b): this method (which is often attributed to Schatz [41]) was applied to the h–BEM in [4] and to the hp–BEM in [28] (using results in [31]). We discuss these results in more detail in §4.2, but note here that, when applied to the h–BEM, these techniques yield a more restrictive condition than $hk^{(d+1)/2} \lesssim 1$ for (1.27) and (1.28) to hold.

Finally, it is instructive to compare the results of Theorems 1.4 and 1.5 to results about the quasi-optimality and relative error of the h–FEM. The relevant results for the h–FEM in 1–d were obtained in [26] (see also [25]). These authors considered the Helmholtz equation posed in a 1–d finite interval (with an impedance boundary condition at one end of the interval to ensure that the solution exists for all k) and proved that the h–FEM is quasi-optimal in the H^1-semi-norm, with the constant of quasi-optimality independent of k, if $hk^2 \lesssim 1$ (this is shown in [25, Theorem 4.13] using [26, Lemma 3]), and numerical experiments indicate that this result is sharp [26, Figures 7-9], [25, §4.5.4 and Figure 4.12]. Furthermore, these authors showed that the relative error in both the H^1-semi-norm and the L^2-norm is bounded independently of k if $hk^{3/2} \lesssim 1$ [26, Equation 3.25], [25, Equation 4.5.15], with numerical experiments indicating that this is sharp [26, Figure 11], [25, Figure 4.13].

In [30, Proposition 8.2.7] it was proved that the h–FEM is quasi-optimal in 2– and 3–d (with the constant of quasi-optimality independent of k) when $hk^2 \lesssim 1$ and the domain is such that the solution is in H^2 and satisfies a certain stability estimate (when Ω_- is star-shaped this stability estimate holds for the interior impedance problem [30, Proposition 8.1.4] and the exterior Dirichlet problem [23, Proposition 3.3]). The numerical experiments in [5, §3] indicate that, at least for certain 2-d problems, the relative error in the L^2-norm is bounded independently of k if $hk^{3/2} \lesssim 1$, although this has yet to be proven. (We note, however, that [46, Theorem 6.1] proves that, under the stability estimate mentioned above, the weighted H^1-norms of $u-u_h$ and u_h are bounded by norms of the data in both 2- and 3-d if $hk^{3/2} \lesssim 1$.)

Comparing the best currently-available results about the quasi-optimality of the h–FEM and the h–BEM for star-shaped domains in 2-d (given by [30, Proposition 8.2.7] and Theorem 1.4 respectively), we see that quasi-optimality holds (with the constant independent of k) for the h–FEM if $hk^2 \lesssim 1$ and for the h–BEM if $hk^{3/2} \lesssim 1$, rigorously confirming the observation that the pollution effect is less pronounced for the h–BEM than for the h–FEM. [1]

1.2.3 Bounds on $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$

The main new ingredients in the proofs of Theorems 1.3, 1.4, and 1.5 are the following bounds on the norms of S_k, D_k, and D'_k as mappings from $L^2(\Gamma)\to H^1(\Gamma)$.

Theorem 1.6 (Bounds on $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$)

(i) If Γ is Lipschitz then S_k is a bounded operator from $L^2(\Gamma)$ to $H^1(\Gamma)$ and, given $k_0>0$,

$$\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim 1+k^{(d-1)/2} \tag{1.33}$$

for all $k\geq k_0$.

(ii) If Γ is C^2 then D_k and D'_k are bounded operators from $L^2(\Gamma)$ to $H^1(\Gamma)$ with

$$\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim 1+k^{(d+1)/2} \tag{1.34}$$

and

$$\left\|D'_k\right\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim 1+k^{(d+1)/2} \tag{1.35}$$

for all $k>0$.

These bounds should be compared to the following bounds proved in [11, Theorems 3.3 and 3.5] on the norms of S_k, D_k, and D'_k as mappings from $L^2(\Gamma)\to L^2(\Gamma)$ for general Lipschitz Ω_-,

$$\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-3)/2}, \quad \|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1+k^{(d-1)/2}, \quad \left\|D'_k\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1+k^{(d-1)/2}. \tag{1.36}$$

We see that the powers of k on the right-hand sides of (1.33), (1.34), and (1.35) are exactly one more than the respective powers of k on the right-hand sides of the bounds in (1.36). The reason that the bound (1.33) is not valid uniformly for $k>0$ is that it uses the bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in (1.36), and when $d=2$ the power of k in the latter bound blows up as $k\to 0$ (but this does not happen to the bounds on $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$).

The relationships (1.15) allow us to convert the bounds on S_k, D_k, and D'_k as mappings from $L^2(\Gamma)\to H^1(\Gamma)$ into bounds on these operators from $H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)$ for $|s|\leq 1/2$.

Corollary 1.2 *(i) If Γ is Lipschitz then $S_k: H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)$ for $|s|\leq 1/2$ and, given $k_0>0$,*

$$\|S_k\|_{H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)} \lesssim 1+k^{(d-1)/2} \tag{1.37}$$

for all $k\geq k_0$.

[1] Of course, if the goal is to compute the solution in (a subset of) the domain, then after using the h–BEM one must evaluate the integrals in Green's integral representation (1.7) or the ansatz (1.12). This adds to the computational cost of the h–BEM, but the question "which of the h–BEM and h–FEM achieves the goal of computing the solution with the least cost?" is independent of the question "to what extent does each method suffers from the pollution effect?"

(ii) If Γ is C^2 then D_k and D'_k are bounded operators from $H^{s-1/2}(\Gamma)$ to $H^{s+1/2}(\Gamma)$ for $|s| \leq 1/2$ with

$$\|D_k\|_{H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)} \lesssim 1 + k^{(d+1)/2} \tag{1.38}$$

and

$$\|D'_k\|_{H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)} \lesssim 1 + k^{(d+1)/2} \tag{1.39}$$

for all $k > 0$.

1.2.4 Results concerning Questions 1 and 2 for other boundary conditions

This paper is concerned with sound-soft acoustic scattering (i.e. the BVP has a zero Dirichlet boundary condition), but here we briefly discuss whether results about Questions 1 and 2 exist (or can be obtained) for other boundary conditions.

Regarding Question 1: Theorem 1.1, which concerns smooth convex domains, is proved using the asymptotic results of [32] (adapted by [18] into a format suitable for our use). The techniques of [32] are also applicable to the analogous BVP with zero Neumann or Robin boundary conditions (see [32, Sections 8 and 9]), and thus we expect results similar to Theorem 1.1 to hold for these BVPs (although with some non-trivial technical work required to prove this). Theorem 1.2, which concerns convex polygons, is proved using the asymptotics results of [14]. The analogous results for the corresponding BVP with a zero impedance boundary condition were proved in [15, Theorem 2.1], and therefore these results could be used to obtain the analogue of Theorem 1.2 for this BVP.

Regarding Question 2: to the authors' knowledge, there do not yet exist any results concerning this question posed for the standard combined-field integral equations used to solve the Neumann or impedance problems (i.e. [12, Equations 2.73 and 2.77]). There do exist, however, results concerning this question posed for certain modifications of the standard integral equations for the Neumann problem. Indeed, the standard combined-field integral equations (both direct and indirect) for the Neumann problem contain the hypersingular operator, and it is common to "regularise" this operator by either pre- or post-multiplying by a smoothing operator (see, e.g., the literature review in [8, §1]). In [8], a k-explicit analysis of two such (indirect) modifications is performed. In particular [8, Theorems 3.2 and 3.6] prove that these modifications are continuous and coercive in $L^2(\Gamma)$ when Γ is the circle or sphere (with k-explicit bounds for the norm and coercivity constant); Céa's lemma can therefore be used to prove a result analogous to (1.31). As discussed in §1.2.2, such a result does not quite answer Question 2, but it does give a bound on the relative error of the Galerkin solution (i.e. the analogue of (1.29)).

Outline of paper. In §2 we prove Theorem 1.6 (we do this first as Theorems 1.3, 1.4, and 1.5 depend on this result). In §3 we prove Theorems 1.1, 1.2, and 1.3. In §4 we prove Theorems 1.4 and 1.5. In §5 we give the results of numerical experiments concerning Question 2.

2 Proof of Theorem 1.6 (k–explicit bounds on $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$)

We begin by recapping (i) some facts about the surface gradient and (ii) the method that was used to prove the bounds (1.36) on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

Recap of facts about the surface gradient. Recall that for Γ Lipschitz there exists a unique operator ∇_Γ, the surface (or tangential) gradient, such that the mapping $\nabla_\Gamma : H^1(\Gamma) \to (L^2(\Gamma))^d$ is bounded and if w is C^1 in a neighbourhood of Γ then

$$\nabla w(\mathbf{x}) = \nabla_\Gamma w(\mathbf{x}) + \mathbf{n}(\mathbf{x}) \frac{\partial w}{\partial n}(\mathbf{x}) \tag{2.1}$$

for almost every $\mathbf{x} \in \Gamma$. For an explicit definition of ∇_Γ in terms of a parametrisation of Γ see, e.g., [12, Equation A.14]. The definition of ∇_Γ implies that, for $v \in H^1(\Gamma)$,

$$\|v\|_{H^1(\Gamma)} \sim \|\nabla_\Gamma v\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)}. \tag{2.2}$$

Another property of ∇_Γ that we use below is that, if Γ is C^2 and $\boldsymbol{\tau}(\mathbf{x})$ is a unit tangent vector to Γ at a point $\mathbf{x} \in \Gamma$, then $\boldsymbol{\tau}(\mathbf{x}) \cdot \nabla_\Gamma v(\mathbf{x})$ is the directional derivative of v along a curve with tangent $\boldsymbol{\tau}(\mathbf{x})$. That is, given a point $\mathbf{x} \in \Gamma$ and tangent vector $\boldsymbol{\tau}(\mathbf{x})$, let C be the curve on Γ passing through $\mathbf{x}$ with tangent vector $\boldsymbol{\tau}(\mathbf{x})$. Let $\mathbf{x}_h$ be a point on C such that the arc between $\mathbf{x}_h$ and $\mathbf{x}$ has length h, so $\mathbf{x}_h = \mathbf{x} + h\boldsymbol{\tau}(\mathbf{x}) + \mathscr{O}(h^2)$. Then

$$\boldsymbol{\tau}(\mathbf{x}) \cdot \nabla_\Gamma v(\mathbf{x}) = \lim_{h\to 0} \frac{v(\mathbf{x}_h) - v(\mathbf{x})}{h}. \tag{2.3}$$

To obtain the bounds on D_k and D'_k, we need the following lemma about the surface gradient of integral operators.

Lemma 2.1 *If $\phi \in L^1(\Gamma)$ and*

(i) $\kappa(\mathbf{x},\mathbf{y}) \in C(\Gamma \times \Gamma)$,
(ii) for all $\mathbf{y} \in \Gamma$, the map $\mathbf{x} \mapsto \kappa(\mathbf{x},\mathbf{y})$ is in $C^1(\Gamma \setminus \{\mathbf{y}\})$, and
(iii) $\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y}) \in L^\infty(\Gamma \times \Gamma \setminus \{(\mathbf{x},\mathbf{y}) : \mathbf{x} = \mathbf{y}\})$,

then

$$\nabla_{\Gamma,\mathbf{x}} \left(\int_\Gamma \kappa(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}) \right) = \int_\Gamma \nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}).$$

Proof (Sketch proof) Fix $\mathbf{x} \in \Gamma$, and let $\boldsymbol{\tau}(\mathbf{x})$ be a unit tangent vector at $\mathbf{x}$. With $\mathbf{x}_h$ defined above, we need to show that

$$\lim_{h\to 0} \int_\Gamma \left(\frac{\kappa(\mathbf{x}_h,\mathbf{y}) - \kappa(\mathbf{x},\mathbf{y})}{h} \right) \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}) = \int_\Gamma \boldsymbol{\tau}(\mathbf{x}) \cdot \nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}). \tag{2.4}$$

For $\varepsilon > 0$ we split the integral on the left-hand side of (2.4) into the integral over $\Gamma \cap B_\varepsilon(\mathbf{x})$ and the integral over $\Gamma \setminus B_\varepsilon(\mathbf{x})$. For the integral over $\Gamma \cap B_\varepsilon(\mathbf{x})$, the assumptions (i)-(iii) imply that $\kappa(\mathbf{x},\mathbf{y})$ is Lipschitz as a function of $\mathbf{x}$, and thus the integrand is bounded independently of h. Therefore the integral over $\Gamma \cap B_\varepsilon(\mathbf{x})$ tends to zero as $\varepsilon \to 0$. By differentiation under the integral sign (using the dominated convergence theorem; see, e.g., [22, Theorem 2.27]) the integral over $\Gamma \setminus B_\varepsilon(\mathbf{x})$ equals

$$\int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} \boldsymbol{\tau}(\mathbf{x}) \cdot \nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}).$$

By the assumption (iii) and another application of the dominated convergence theorem, this last integral tends to the right-hand side of (2.4) as $\varepsilon \to 0$.

Overview of the Riesz–Thorin method. The k–explicit bounds (1.36) on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ were obtained in [11] using the following idea. If T is an integral operator on Γ with kernel $t(\mathbf{x},\mathbf{y})$, i.e.,

$$T\phi(\mathbf{x}) = \int_\Gamma t(\mathbf{x},\mathbf{y})\, \phi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}),$$

then, using the definitions of the L^1- and L^∞-operator norms, it is straightforward to show that

$$\|T\|_{L^1(\Gamma)\to L^1(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{y}\in\Gamma} \int_\Gamma \big|t(\mathbf{x},\mathbf{y})\big|\, \mathrm{d}s(\mathbf{x}), \tag{2.5a}$$

$$\|T\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} = \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma} \int_\Gamma \big|t(\mathbf{x},\mathbf{y})\big|\, \mathrm{d}s(\mathbf{y}) \tag{2.5b}$$

(provided these integrals exist). The Riesz-Thorin interpolation theorem implies that

$$\|T\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \left(\|T\|_{L^1(\Gamma)\to L^1(\Gamma)} \right)^{1/2} \left(\|T\|_{L^\infty(\Gamma)\to L^\infty(\Gamma)} \right)^{1/2}$$

(see, e.g., [22, Theorem 6.27]), and thus a bound on $\|T\|_{L^2(\Gamma)\to L^2(\Gamma)}$ can be obtained by bounding the integrals on the right-hand sides of (2.5). In particular, if $|t(\mathbf{x},\mathbf{y})| \le \widetilde{t}(\mathbf{x},\mathbf{y})$, where $\widetilde{t}$ is such that $\widetilde{t}(\mathbf{x},\mathbf{y}) = \widetilde{t}(\mathbf{y},\mathbf{x})$, then

$$\|T\|_{L^2(\Gamma)\to L^2(\Gamma)} \le \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma} \int_\Gamma \widetilde{t}(\mathbf{x},\mathbf{y})\, \mathrm{d}s(\mathbf{y}). \tag{2.6}$$

To obtain a bound on $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, we can apply the bound (2.6) with $T = S_k$ and $\widetilde{t}(\mathbf{x},\mathbf{y})$ chosen as $|\Phi_k(\mathbf{x},\mathbf{y})|$. On the other hand, to obtain a bound on $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ we write D_k as $D_0 + (D_k - D_0)$ and apply (2.6) with $T = D_k - D_0$; we do this because the singularity of D_k is too strong for the operator itself to be bounded on $L^1(\Gamma)$ and $L^\infty(\Gamma)$ for general Lipschitz Γ.

In this section, we obtain bounds on $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, and $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$ by using the method above to obtain bounds on $\|\nabla_\Gamma(S_k - S_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$, $\|\nabla_\Gamma(D_k - D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and $\|\nabla_\Gamma(D'_k - D'_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$. In [42, §1.2] it is shown that the bounds (1.36) can also be obtained using Young's inequality, and we note that the bounds on $\|\nabla_\Gamma(S_k - S_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$, $\|\nabla_\Gamma(D_k - D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and $\|\nabla_\Gamma(D'_k - D'_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$ that we obtain below can also be obtained using this alternative method.

Since we plan on bounding quantities involving S_0, D_0, and D'_0, before we begin we recall that $\Phi_0(\mathbf{x},\mathbf{y})$ is defined when $d = 3$ by the second equation in (1.8) with $k = 0$, and when $d = 2$ by $\Phi_0(\mathbf{x},\mathbf{y}) := -\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|$.

Proof (Proof of the bound (1.33) *on* $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$*)* The fact that $S_k : L^2(\Gamma) \to H^1(\Gamma)$ for $k \geq 0$ follows from the harmonic analysis results summarised in, e.g., [33, Chapter 15], [12, Theorems 2.15 and 2.16]. The bound on $\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$ (1.33) follows by using (2.2) and combining the estimates $\|S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-3)/2}$ (proved in [11, Theorem 3.3]) and

$$\|\nabla_\Gamma S_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + k^{(d-1)/2}, \tag{2.7}$$

using the fact that, given $k_0 > 0$, there exists a C (depending on k_0 when $d = 2$) such that $k^{(d-3)/2} + 1 + k^{(d-1)/2} \leq C(1 + k^{(d-1)2})$ for all $k \geq k_0$. To obtain (2.7), note that $\nabla_\Gamma S_k$ equals the vector-valued boundary integral operator defined by

$$\nabla_\Gamma S_k \phi(\mathbf{x}) = \int_\Gamma \left(\nabla_{\mathbf{x}}\Phi_k(\mathbf{x},\mathbf{y}) - \mathbf{n}(\mathbf{x})\frac{\partial \Phi_k}{\partial n(\mathbf{x})}(\mathbf{x},\mathbf{y})\right)\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{2.8}$$

where the integral is understood as a Cauchy Principal Value; see [33, Chapter 15, §4]. When Γ is Lipschitz, the singularity in the integral on the right-hand side of (2.8) has the same strength as the singularity in the integral defining D_k, and thus the bound $\|\nabla_\Gamma S_k - \nabla_\Gamma S_0\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-1)/2}$ follows in exactly the same way as the bound $\|D_k - D_0\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{(d-1)/2}$ was proved in [11, Theorem 3.5] (indeed, the same $\widetilde{t}(\mathbf{x},\mathbf{y})$ in (2.6) can be used for both $D_k - D_0$ and $\nabla_\Gamma S_k - \nabla_\Gamma S_0$).

To prove the bounds on D_k and D'_k when $d = 2$ we need the following bounds on $H_1^{(1)}(t)$.

Lemma 2.2 (Bounds on the Hankel function $H_1^{(1)}(t)$) *There exist constants $c_j > 0$, $j = 1,\ldots,5$, such that*

$$\left|\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) - 1\right| \leq c_1 t^{1/2}, \tag{2.9}$$

$$\left|\left(\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) - 1\right)'\right| \leq c_2 t^{1/2} + c_3 t^{-1/2}, \tag{2.10}$$

$$\left|\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) - 1\right| \leq c_4 t, \quad \textit{and} \tag{2.11}$$

$$\left|\left(\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) - 1\right)'\right| \leq c_5(1+t), \tag{2.12}$$

for all $t > 0$. Furthermore, there exists a function $h(t)$ that is continuous on $[0,\infty)$ such that

$$\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) - 1 = t\,h(t) \tag{2.13}$$

for all $t \geq 0$.

We postpone the proof of Lemma 2.2 until after the proofs of the bounds (1.34) and (1.35). To put the bounds in Lemma 2.2 into context, we note that

$$\left|\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t)-1\right| \sim t^{1/2} \quad \text{as } t\to\infty \quad \text{and } \sim t^2\log(1/t) \quad \text{as } t\to 0,$$

and

$$\left|\left(\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t)-1\right)'\right| \sim t^{1/2} \quad \text{as } t\to\infty \quad \text{and } \sim t\log(1/t) \quad \text{as } t\to 0,$$

with the asymptotics as $t\to\infty$ following from the asymptotics of $H_\nu^{(1)}(t)$ as $t\to\infty$ for ν fixed [1, Equation 9.2.3], and the asymptotics as $t\to 0$ following from the power series of $J_0(t)$ and $Y_0(t)$ about $t=0$ [1, Equations 9.1.12 and 9.1.13]. Therefore, the bounds (2.9) and (2.10) are sharp as $t\to\infty$, but not as $t\to 0$, and neither (2.11) nor (2.12) are sharp as either $t\to\infty$ or $t\to 0$.

The key point is that, although the bounds (2.9)–(2.12) are generally not sharp as $t\to\infty$ and $t\to 0$, they are valid for all $t>0$. We need this property for the proofs below since we let $t=k|\mathbf{x}-\mathbf{y}|$ and this quantity can be arbitrarily small (since $\mathbf{y}$ can be equal to $\mathbf{x}$) and arbitrarily large (since k can be arbitrarily large).

The reason we need two different bounds on each of $\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t)-1$ and $(\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t)-1)'$ is the following. We use these bounds to bound the kernels of $\nabla_\Gamma(D_k-D_0)$ and $\nabla_\Gamma(D'_k-D'_0)$, and when doing this we have two contradictory requirements. On the one hand, we would like large powers of t in the bounds, since, with $t=k|\mathbf{x}-\mathbf{y}|$, these would show that the kernel is well-behaved when $\mathbf{y}=\mathbf{x}$. On the other hand, we would like small powers of t, since these would lead to small powers of k in the resulting bounds on $\|\nabla_\Gamma(D_k-D_0)\|_{L^2(\Gamma)\to H^1(\Gamma)}$ and $\|\nabla_\Gamma(D'_k-D'_0)\|_{L^2(\Gamma)\to H^1(\Gamma)}$. The remedy is to use the bounds with large powers of t, (2.11) and (2.12), to show that the kernels of the integral operators are non-singular, and then use the bounds with small powers of t, (2.9) and (2.10), to obtain bounds with small powers of k on the norms.

Before proving the bounds (1.34) and (1.35), we state and prove one final lemma that we use in the proofs of (1.34) and (1.35).

Lemma 2.3 *We have that*

$$\left|\mathrm{e}^{\mathrm{i}t}(\mathrm{i}t-1)+1\right| \le 2t \quad \textit{for all } t\ge 0, \tag{2.14}$$

$$\left|\mathrm{e}^{\mathrm{i}t}(\mathrm{i}t-1)+1\right| \le \frac{t^2}{2} \quad \textit{for all } t\ge 0, \tag{2.15}$$

and there exists a function $g(t)$ that is continuous on $[0,\infty)$ such that

$$\mathrm{e}^{\mathrm{i}t}(\mathrm{i}t-1)+1 = t^2 g(t) \tag{2.16}$$

for all $t\ge 0$. Furthermore,

$$|\mathrm{e}^{\mathrm{i}t}-1|\le t \quad \textit{and} \quad |\mathrm{e}^{\mathrm{i}t}-1|\le\sqrt{2t} \quad \textit{for all } t\ge 0. \tag{2.17}$$

Proof The bounds (2.14) and (2.15) are proved in [11, Lemma 3.4], and (2.16) follows from Taylor's theorem. To obtain (2.17), we observe that

$$|\mathrm{e}^{\mathrm{i}t}-1| = 2|\sin(t/2)| = \min\Big(2, 2|\sin(t/2)|\Big) \le \min(2,t),$$

where we have used that $|\sin x|\le x$ for $x>0$. Since $\min(2,t)\le t$ and $\min(2,t)\le\sqrt{2t}$, the bounds (2.17) follow. (Note that the second bound in (2.17) is proved in this way in [11, Page 11].)

Proof (Proof of the bound (1.34) *on* $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$*)* When Γ is C^2, $D_k : L^2(\Gamma)\to H^1(\Gamma)$ for all $k\ge 0$; see [37, Theorem 4.4.1]. (Note that [37, Theorem 4.4.1] is proved using [37, Theorem 4.3.1], which is valid if the "surface Γ [is] regular enough", however one can check that Γ being C^2 is sufficient.)

We already have that $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1+k^{(d-1)/2}$ for general Lipschitz domains from [11, Theorem 3.5], and so, by (2.2), we only need to show that

$$\|\nabla_\Gamma D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1+k^{(d+1)/2}.$$

Since

$$\|\nabla_\Gamma D_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \|\nabla_\Gamma (D_k - D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)} + \|\nabla_\Gamma D_0\|_{L^2(\Gamma)\to L^2(\Gamma)},$$

we only need to show that

$$\|\nabla_\Gamma (D_k - D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1 + k^{(d+1)/2} \tag{2.18}$$

for all $k > 0$.

Following the Riesz–Thorin method outlined above, we aim to prove (2.18) by applying (2.6) with $T = \nabla_\Gamma (D_k - D_0)$. The definition of D_k (1.14) implies that

$$(D_k - D_0)\phi(\mathbf{x}) = \int_\Gamma \kappa(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

where

$$\kappa(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\left(\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}\big(\mathrm{i}k|\mathbf{x}-\mathbf{y}| - 1\big) + 1\right)\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} \tag{2.19}$$

for $d = 3$, and

$$\kappa(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\left(\frac{\mathrm{i}\pi}{2}k|\mathbf{x}-\mathbf{y}|H_1^{(1)}\big(k|\mathbf{x}-\mathbf{y}|\big) - 1\right)\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2} \tag{2.20}$$

for $d = 2$, where $\mathbf{n}(\mathbf{y})$ is the outward-pointing unit normal vector to Ω_- at $\mathbf{y} \in \Gamma$.

Our plan for the rest of the proof is as follows. We use Lemma 2.1 to show that

$$\nabla_{\Gamma,\mathbf{x}}(D_k - D_0)\phi(\mathbf{x}) = \int_\Gamma \nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\,\phi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}). \tag{2.21}$$

We then find a $\widetilde{\kappa}(\mathbf{x},\mathbf{y})$ such that $\widetilde{\kappa}(\mathbf{x},\mathbf{y})$ is in $L^1(\Gamma)$ as a function of $\mathbf{y}$, $\widetilde{\kappa}(\mathbf{x},\mathbf{y}) = \widetilde{\kappa}(\mathbf{y},\mathbf{x})$, and

$$|\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})| \lesssim \widetilde{\kappa}(\mathbf{x},\mathbf{y}). \tag{2.22}$$

The consequence of the Riesz–Thorin theorem (2.6) then implies that

$$\|\nabla_\Gamma (D_k - D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \operatorname*{ess\,sup}_{\mathbf{x}\in\Gamma}\int_\Gamma \widetilde{\kappa}(\mathbf{x},\mathbf{y})\mathrm{d}s(\mathbf{y}). \tag{2.23}$$

In rest of the proof, we use the notation $a \lesssim b$ to mean that $a \leq Cb$ where C is independent of k *and* independent of $\mathbf{x}$ and $\mathbf{y}$ (so, in particular, any factors of $|\mathbf{x}-\mathbf{y}|$ must be given explicitly in the bound).

We now need to verify that the assumptions (i)-(iii) of Lemma 2.1 hold. Since Γ is C^2, $\mathbf{n}$ is C^1, and thus the expressions (2.19) and (2.20) show that $\kappa(\mathbf{x},\mathbf{y})$ is continuous for $(\mathbf{x},\mathbf{y}) \in \Gamma\times\Gamma$, except possibly when $\mathbf{x} = \mathbf{y}$. Writing $(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})$ as $|\mathbf{x}-\mathbf{y}|\widehat{(\mathbf{x}-\mathbf{y})}\cdot\mathbf{n}(\mathbf{y})$ and using the properties (2.16) and (2.13), we see that $\kappa(\mathbf{x},\mathbf{y})$ is continuous for all $(\mathbf{x},\mathbf{y}) \in \Gamma\times\Gamma$, and thus the assumption (i) holds. The assumption (ii) follows immediately from the expressions (2.19) and (2.20).

To prove that (iii) holds, we use (2.3) to find an explicit expression for $\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})$ when $\boldsymbol{\tau}(\mathbf{x})$ is an arbitrary unit tangent vector to $\mathbf{x} \in \Gamma$. We make use of the fact that

$$\kappa(\mathbf{x},\mathbf{y}) = f\big(|\mathbf{x}-\mathbf{y}|\big)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y}), \tag{2.24}$$

where

$$f(s) := -\frac{1}{4\pi}\left(\mathrm{e}^{\mathrm{i}ks}\big(\mathrm{i}ks - 1\big) + 1\right)\frac{1}{s^3} \quad \text{for } d = 3, \tag{2.25}$$

and

$$f(s) := \frac{1}{2\pi}\left(\frac{\mathrm{i}\pi}{2}ksH_1^{(1)}(ks) - 1\right)\frac{1}{s^2} \quad \text{for } d = 2. \tag{2.26}$$

Given a point $\mathbf{x} \in \Gamma$ and unit tangent vector $\boldsymbol{\tau}(\mathbf{x})$, let C be the curve on Γ passing through $\mathbf{x}$ with tangent vector $\boldsymbol{\tau}(\mathbf{x})$. Let $\mathbf{x}_h$ be a point on C such that the arc between $\mathbf{x}_h$ and $\mathbf{x}$ has length h, so $\mathbf{x}_h = \mathbf{x} + h\boldsymbol{\tau}(\mathbf{x}) + \mathscr{O}(h^2)$. By expanding $|\mathbf{x}_h - \mathbf{y}|^2 = |(\mathbf{x}_h - \mathbf{x}) + (\mathbf{x}-\mathbf{y})|^2$ and using Taylor's theorem, we find that

$$|\mathbf{x}_h - \mathbf{y}| = |\mathbf{x}-\mathbf{y}| + h\,\boldsymbol{\tau}(\mathbf{x})\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} + \mathscr{O}(h^2) \quad \text{as } h \to 0. \tag{2.27}$$

With $f(s)$ any differentiable function of s, Taylor's theorem and the expressions (2.3) and (2.27) imply that

$$\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}f\big(|\mathbf{x}-\mathbf{y}|\big)=\boldsymbol{\tau}(\mathbf{x})\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}f'\big(|\mathbf{x}-\mathbf{y}|\big). \tag{2.28}$$

Furthermore,

$$\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma}\big((\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\big)=\boldsymbol{\tau}(\mathbf{x})\cdot\mathbf{n}(\mathbf{y})=\boldsymbol{\tau}(\mathbf{x})\cdot\big(\mathbf{n}(\mathbf{y})-\mathbf{n}(\mathbf{x})\big). \tag{2.29}$$

Using (2.24), (2.28), and (2.29), we then have that

$$\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})=\boldsymbol{\tau}(\mathbf{x})\cdot\frac{(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}f'\big(|\mathbf{x}-\mathbf{y}|\big)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})+f\big(|\mathbf{x}-\mathbf{y}|\big)\boldsymbol{\tau}(\mathbf{x})\cdot\big(\mathbf{n}(\mathbf{y})-\mathbf{n}(\mathbf{x})\big). \tag{2.30}$$

Recall that our goal is to show that $\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})$ is bounded on $\Gamma\times\Gamma\setminus\{(\mathbf{x},\mathbf{y}):\mathbf{x}=\mathbf{y}\}$, for any tangent vector $\boldsymbol{\tau}(\mathbf{x})$, and find a function $\widetilde{\kappa}(\mathbf{x},\mathbf{y})$ such that (2.22) holds.

The case $d=3$. Using the bounds (2.14) and (2.15) and the definition of f (2.25), we find that

$$|f(s)|\lesssim k^2s^{-1}\quad\text{and}\quad|f'(s)|\lesssim k^2s^{-2}\quad\text{for all } k,s>0.$$

Using these bounds in (2.30), along with the bounds

$$|\mathbf{n}(\mathbf{x})-\mathbf{n}(\mathbf{y})|\lesssim|\mathbf{x}-\mathbf{y}|\quad\text{and}\quad|(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})|\lesssim|\mathbf{x}-\mathbf{y}|^2 \tag{2.31}$$

(valid when Γ is C^2; see, e.g., [17, Theorem 2.2]), we find that

$$\big|\tau(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\big|\lesssim k^2\quad\text{for all }(\mathbf{x},\mathbf{y})\in\Gamma\times\Gamma\text{ with }\mathbf{x}\neq\mathbf{y}. \tag{2.32}$$

Since $\boldsymbol{\tau}(\mathbf{x})$ was an arbitrary unit tangent vector, the bound (2.32) implies that $\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})$ is bounded on $\Gamma\times\Gamma\setminus\{(\mathbf{x},\mathbf{y}):\mathbf{x}=\mathbf{y}\}$. Therefore, we can use Lemma 2.1 to obtain that (2.21) and (2.22) hold with $\widetilde{\kappa}(\mathbf{x},\mathbf{y})=k^2$. The bound (2.23) then yields the bound (2.18) on $\|\nabla_\Gamma(D_k-D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}$ when $d=3$.

The case $d=2$. Using the bounds (2.11) and (2.12) and the definition of f (2.26), we obtain that

$$|f(s)|\lesssim\frac{k}{s}\quad\text{and}\quad|f'(s)|\lesssim\frac{k(1+ks)}{s^2}+\frac{k}{s^2}\quad\text{for all }k,s>0.$$

Using these bounds in (2.30), along with the bounds in (2.31), we find that

$$\big|\tau(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\big|\lesssim k+k^2|\mathbf{x}-\mathbf{y}|\quad\text{for all }(\mathbf{x},\mathbf{y})\in\Gamma\times\Gamma\text{ with }\mathbf{x}\neq\mathbf{y}. \tag{2.33}$$

Since $\boldsymbol{\tau}(\mathbf{x})$ was an arbitrary unit tangent vector, the bound (2.33) shows that $\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})$ is bounded on $\Gamma\times\Gamma\setminus\{(\mathbf{x},\mathbf{y}):\mathbf{x}=\mathbf{y}\}$, and thus (2.21) and (2.22) hold with $\widetilde{\kappa}(\mathbf{x},\mathbf{y})$ equal to the right-hand side of (2.33). However, the consequence of the Riesz–Thorin theorem (2.23) then yields the bound

$$\|\nabla_\Gamma(D_k-D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim k+k^2,$$

which is weaker than (2.18) when k is large. Having established that $\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})$ is bounded and thus that (2.21) holds, we now seek a $\widetilde{\kappa}(\mathbf{x},\mathbf{y})$ with milder growth in k.

Using (2.9) and (2.10), we obtain that

$$|f(s)|\lesssim\frac{k^{1/2}}{s^{3/2}}\quad\text{and}\quad|f'(s)|\lesssim\frac{1}{s^2}\left[k^{3/2}s^{1/2}+\frac{k^{1/2}}{s^{1/2}}\right]\quad\text{for all }k,s>0.$$

Using these bounds in (2.30), we find that

$$\big|\tau(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\kappa(\mathbf{x},\mathbf{y})\big|\lesssim k^{3/2}|\mathbf{x}-\mathbf{y}|^{1/2}+\frac{k^{1/2}}{|\mathbf{x}-\mathbf{y}|^{1/2}} \tag{2.34}$$

for all $(\mathbf{x},\mathbf{y})\in\Gamma\times\Gamma$ with $\mathbf{x}\neq\mathbf{y}$ and for all $k>0$. Since

$$\sup_{\mathbf{x}\in\Gamma}\int_\Gamma\frac{1}{|\mathbf{x}-\mathbf{y}|^{1/2}}\mathrm{d}s(\mathbf{y})<\infty\quad\text{when } d=2,$$

using the bound (2.34) in (2.23) yields the bound $\|\nabla_\Gamma(D_k-D_0)\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim k^{3/2}+k^{1/2}$. This bound then implies the result (2.18) when $d=2$, because there exists a $C>0$ such that $k^{1/2}+k^{3/2}\leq C(1+k^{3/2})$ for all $k>0$.

Whereas a bound on $\|D_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$ immediately yields a bound on $\|D'_k\|_{L^2(\Gamma)\to L^2(\Gamma)}$, we need to do a little bit extra work to obtain the bound on $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$ from that on D_k.

Proof (Proof of the bound (1.35) *on* $\|D'_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$*)* The kernel of D'_k is identical to that of D_k except that it involves $\mathbf{n}(\mathbf{x})$ instead of $\mathbf{n}(\mathbf{y})$. Inspecting the proof of the bound on $\|D_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$, we see that this difference means that the proof of the bound for D'_k follows from the proof of the bound for D_k if we can show that

$$\left|\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\big((\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{x})\big)\right| \lesssim |\mathbf{x}-\mathbf{y}|. \tag{2.35}$$

Using (2.3), we have that

$$\boldsymbol{\tau}(\mathbf{x})\cdot\nabla_{\Gamma,\mathbf{x}}\big((\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{x})\big) = (\mathbf{x}-\mathbf{y})\cdot\left[\lim_{h\to 0}\frac{\mathbf{n}(\mathbf{x}_h)-\mathbf{n}(\mathbf{x})}{h}\right]. \tag{2.36}$$

Since Γ is C^2, $\mathbf{n}$ is C^1, and the quantity in square brackets is finite; the bound (2.35) then follows.

Proof (Proof of Corollary 1.2) Let $\mathscr{C}: H^s(\Gamma)\to H^s(\Gamma)$ denote the operation of complex conjugation, i.e.,

$$\mathscr{C}u(\mathbf{x}) := \overline{u(\mathbf{x})}, \quad \mathbf{x}\in\Gamma,$$

so that $\mathscr{C}$ is an anti-linear bounded operator on $H^s(\Gamma)$ for $|s|\le 1$. Then, if A^* denotes the adjoint of a bounded linear operator A on $L^2(\Gamma)$, the relations (1.15) imply that

$$S_k^* = \mathscr{C}S_k\mathscr{C} \quad \text{and} \quad D_k^* = \mathscr{C}D'_k\mathscr{C}.$$

The relations (1.15) can then be written in terms of the duality pairing on Γ as

$$\langle S_k\phi,\psi\rangle_\Gamma = \langle \phi, S_k^*\psi\rangle_\Gamma \quad \text{and} \quad \langle D_k\phi,\psi\rangle_\Gamma = \langle\phi, D_k^*\psi\rangle_\Gamma. \tag{2.37}$$

We concentrate on proving the bound on S_k (1.37); the bounds (1.38) and (1.39) on D_k and D'_k respectively follow in a similar manner.

We begin by proving that

$$\|S_k\|_{H^{-1}(\Gamma)\to L^2(\Gamma)} \le \|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}. \tag{2.38}$$

Indeed, using (2.37) we have that, for $\psi\in L^2(\Gamma)$,

$$\begin{aligned}\|S_k\psi\|_{L^2(\Gamma)} = \|S_k^*\psi\|_{L^2(\Gamma)} = \sup_{\phi\in L^2(\Gamma),\phi\neq 0}\frac{\left|\langle S_k^*\psi,\phi\rangle_\Gamma\right|}{\|\phi\|_{L^2(\Gamma)}} &= \sup_{\phi\in L^2(\Gamma),\phi\neq 0}\frac{|\langle\psi,S_k\phi\rangle_\Gamma|}{\|\phi\|_{L^2(\Gamma)}}\\ &\le \sup_{\phi\in L^2(\Gamma),\phi\neq 0}\frac{\|\psi\|_{H^{-1}(\Gamma)}\|S_k\phi\|_{H^1(\Gamma)}}{\|\phi\|_{L^2(\Gamma)}}\\ &\le \|\psi\|_{H^{-1}(\Gamma)}\|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}.\end{aligned}$$

Since $L^2(\Gamma)$ is dense in $H^{-1}(\Gamma)$ the last inequality shows that $S_k: H^{-1}(\Gamma)\to L^2(\Gamma)$ and that (2.38) holds. We then have that

$$\|S_k\|_{H^{s-1/2}(\Gamma)\to H^{s+1/2}(\Gamma)} \le \|S_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$$

by interpolation (see, e.g., [29, Theorems B.2 and B.11]), and the result (1.37) follows.

Proof (Proof of Lemma 2.2) We use the following integral representation of $H_0^{(1)}$,

$$H_0^{(1)}(t) = -\frac{2\mathrm{i}}{\pi}\mathrm{e}^{\mathrm{i}t}\int_0^\infty \frac{\mathrm{e}^{-rt}}{r^{1/2}(r-2\mathrm{i})^{1/2}}\mathrm{d}r, \quad t>0, \tag{2.39}$$

[39, Chapter 7, Equation 13.07], [38, §2.12, Equation 12.31], where the branch cut of $(r-2\mathrm{i})^{1/2}$ is taken so that $\Re(r-2\mathrm{i})^{1/2}\geq 0$ for $r\in[0,\infty)$ (note that for this branch, $\Im(r-2\mathrm{i})^{1/2}\leq 0$ for $r\in[0,\infty)$). Using (2.39) and the facts that $H_1^{(1)}(t) = -H_0^{(1)'}(t)$ and $1 = t\int_0^\infty \mathrm{e}^{-rt}\mathrm{d}r$, we obtain

$$\frac{\mathrm{i}\pi}{2}\mathrm{e}^{-\mathrm{i}t}tH_1^{(1)}(t) - 1 = -t\int_0^\infty \frac{\mathrm{e}^{-rt}}{r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})}\mathrm{d}r, \quad t>0. \tag{2.40}$$

Since $\Im(r-2\mathrm{i})^{1/2}\leq 0$ for $r\in[0,\infty)$, we have

$$\Im(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i}) \leq -1 \quad \text{for } r\in[0,\infty),$$

and then

$$|r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i}| \geq |\Im(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})| \geq 1 \quad \text{for } r\in[0,\infty). \tag{2.41}$$

Using (2.41) and the estimate $|(r-2\mathrm{i})^{1/2}|\geq\sqrt{2}$ for $r\in[0,\infty)$, we can estimate the modulus of the right hand side of (2.40) by

$$t\int_0^\infty \frac{\mathrm{e}^{-rt}}{(2r)^{1/2}}\mathrm{d}r$$

and calculating this integral leads to the bound

$$\left|\frac{\mathrm{i}\pi}{2}\mathrm{e}^{-\mathrm{i}t}tH_1^{(1)}(t) - 1\right| \leq \sqrt{\frac{\pi t}{2}}$$

(which is [11, Equation (1.24)]). Combining this bound with the triangle inequality and the second bound in (2.17) gives the bound (2.9).

To obtain (2.10), we first rewrite (2.40) as

$$\frac{\mathrm{i}\pi}{2}tH_1^{(1)}(t) = \mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}t}t\int_0^\infty \frac{\mathrm{e}^{-rt}}{r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})}\mathrm{d}r, \quad t>0. \tag{2.42}$$

Differentiating both sides of (2.42) and estimating the integrals exactly as before, we obtain (2.10).

To obtain (2.11), note that the integral representation (2.40) gives

$$\begin{aligned}\left|\frac{\mathrm{i}\pi}{2}\mathrm{e}^{-\mathrm{i}t}tH_1^{(1)}(t) - 1\right| &\leq t\int_0^\infty \frac{\mathrm{e}^{-rt}}{|r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})|}\mathrm{d}r\\ &\leq t\int_0^\infty \frac{1}{|r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})|}\mathrm{d}r,\end{aligned} \tag{2.43}$$

and the integral on the right-hand side of (2.43) is finite. Using this last bound, the first bound in (2.17), and the triangle inequality, we obtain (2.11).

Differentiating (2.42) and using (2.43) and (2.44), we see that to prove (2.12) we only need to show that

$$\left|\int_0^\infty \frac{t\,r\mathrm{e}^{-rt}\mathrm{d}r}{r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})}\right| \lesssim 1+t, \quad \text{for } t>0. \tag{2.44}$$

To prove (2.44) we split the integral over $(0,\infty)$ into integrals over $(0,2)$ and $(2,\infty)$ so that we can use the inequality

$$(r-2\mathrm{i})^{1/2} \geq \max(r^{1/2},\sqrt{2}). \tag{2.45}$$

Considering the integral over $(0,2)$ and using (2.45) and (2.41), we have

$$\left|\int_0^2 \frac{t\,r\mathrm{e}^{-rt}\mathrm{d}r}{r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})}\right| \leq \frac{1}{\sqrt{2}}\int_0^2 t\,r^{1/2}\mathrm{e}^{-rt}\mathrm{d}r = \frac{1}{\sqrt{2t}}\int_0^{2t} s^{1/2}\mathrm{e}^{-s}\mathrm{d}s \lesssim t, \tag{2.46}$$

where we have used the fact that $\exp(-s)\leq 1$ for $s\geq 0$ to estimate the last integral.

For the second integral, we use (2.45) and (2.41) to obtain

$$\left|\int_2^\infty \frac{t\,r\mathrm{e}^{-rt}\mathrm{d}r}{r^{1/2}(r-2\mathrm{i})^{1/2}(r^{1/2}(r-2\mathrm{i})^{1/2}+r-\mathrm{i})}\right| \leq t\int_2^\infty \mathrm{e}^{-rt}\mathrm{d}r = \mathrm{e}^{-2t} \lesssim 1. \tag{2.47}$$

Combining the bounds (2.46) and (2.47), we obtain (2.44), and thus (2.12).

Finally, the claim (2.13) follows from (2.42) and Taylor's theorem.

3 Proofs of Theorems 1.1, 1.2, and 1.3 (concerning the relative best approximation errors)

The proofs in this section use upper bounds on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$, and the proofs in §4 use upper bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|A^{-1}_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$. We therefore give a summary of these bounds here.

3.1 Recap of upper bounds on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$

Theorem 3.1 (Upper bounds on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ [11]) *If $\Omega_- \subset \mathbb{R}^d$, $d = 2$ or 3, and Γ is Lipschitz then*

$$\|A'_{k,\eta}\|_{L^2(\Gamma)} = \|A_{k,\eta}\|_{L^2(\Gamma)} \lesssim 1 + k^{(d-1)/2}\left(1+\frac{|\eta|}{k}\right), \tag{3.1}$$

for all $k > 0$ and $\eta \in \mathbb{R}$ [11, Theorem 3.6].

Note that (3.1) follows from the bounds (1.36) discussed in §2.

Theorem 3.2 (Upper bounds on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|A^{-1}_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ [16], [43])

(i) If $\Omega_- \subset \mathbb{R}^d$, $d = 2$ or 3, is a Lipschitz domain that is star-shaped (in the sense of Definition 1.2) then

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A^{-1}_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim \left(1+\frac{1+k}{|\eta|}\right), \tag{3.2}$$

for all $k > 0$ [16, Theorem 4.3].

(ii) If $\Omega_+ \subset \mathbb{R}^d$, $d = 2$ or 3, is nontrapping (in the sense of [43, Definition 1.1]) or *Ω_- is a nontrapping polygon (in the sense of [43, Definition 1.2]), then, given $k_0 > 0$,*

$$\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} = \|A^{-1}_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{3/2}\left(1+\frac{k}{|\eta|}\right) \tag{3.3}$$

for all $k \geq k_0$ and $\eta \in \mathbb{R}\setminus\{0\}$ [43, Theorem 1.10].

3.2 Proofs of Theorems 1.1, 1.2, and 1.3

We first prove lower bounds on v and ϕ.

Lemma 3.1 *If Γ is Lipschitz and v and ϕ are the solutions of (1.1) and (1.2) respectively, then, given $k_0 > 0$,*

$$\|v\|_{L^2(\Gamma)} \gtrsim k^{(3-d)/2} \quad \text{and} \quad \|\phi\|_{L^2(\Gamma)} \gtrsim k^{(1-d)/2} \tag{3.4}$$

for all $k \geq k_0$.

Proof From the integral equations (1.1) and (1.2) and the definitions of f and g, (1.11) and (1.13) respectively, we have that

$$\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\|v\|_{L^2(\Gamma)} \geq \|f\|_{L^2(\Gamma)} = k\|\mathbf{n}\cdot\widehat{\mathbf{a}} - \eta/k\|_{L^2(\Gamma)}$$

and

$$\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\|\phi\|_{L^2(\Gamma)} \geq \|\gamma_+ u^I\|_{L^2(\Gamma)} \sim 1.$$

Choosing $\eta = 0$ and using the bounds (3.1) on $\|A'_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$, we obtain the bounds (3.4).

Note that the upper bound on v for star-shaped domains, $\|v\|_{L^2(\Gamma)} \lesssim k$, mentioned in §1.2, follows from the bound (3.2) on $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and the fact that $\|f\|_{L^2(\Gamma)} \sim k$ (when $|\eta| \sim k$). (In a similar manner, we also have that $\|\phi\|_{L^2(\Gamma)} \lesssim 1$.)

Proof (Proof Theorem 1.1) As discussed in §1.2.1, if we can prove the bound (1.24) then the result (1.18) follows from (1.23). Note that we only need to show that there exists a $k_1 > 0$ and $C > 0$ (with C independent of k) such that

$$\|v\|_{H^1(\Gamma)} \leq Ck\|v\|_{L^2(\Gamma)} \qquad \text{for all } k \geq k_1, \tag{3.5}$$

since then, given $k_0 > 0$, we have that

$$\|v\|_{H^1(\Gamma)} \leq C'k\|v\|_{L^2(\Gamma)} \qquad \text{for all } k \geq k_0,$$

where

$$C' := \max\left\{C, \frac{\max_{k_0\leq k\leq k_1}\|v\|_{H^1(\Gamma)}}{\min_{k_0\leq k\leq k_1}\left(k\|v\|_{L^2(\Gamma)}\right)}\right\}.$$

In [18, Theorem 5.4, Corollary 5.5] it is proved that there exists $k_1 > 0$ such that, for all $k \geq k_1$,

$$v(\mathbf{x}) = kV(\mathbf{x},k)\exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{a}}), \tag{3.6}$$

and the estimates

$$|V(\mathbf{x},k)| \lesssim 1 \quad \text{and} \quad |D^n_{\Gamma,\mathbf{x}}V(\mathbf{x},k)| \lesssim 1 + k^{(n-1)/3}$$

hold uniformly for $\mathbf{x} \in \Gamma$, for all $n \geq 1$, where $D_{\Gamma,\mathbf{x}}$ is any first order differential operator on Γ. Thus

$$\|V(\,.\,,k)\|_{L^2(\Gamma)} \lesssim 1 \qquad \text{and} \qquad \|\nabla_\Gamma V(\,.\,,k)\|_{L^2(\Gamma)} \lesssim 1 \tag{3.7}$$

for all $k \geq k_1$. Now, by differentiating (3.6) we obtain

$$\nabla_{\Gamma,\mathbf{x}}v(\mathbf{x}) \;=\; k\Big(\mathrm{i}\,v(\mathbf{x})\big(\widehat{\mathbf{a}} - (\widehat{\mathbf{a}}\cdot\mathbf{n}(\mathbf{x}))\mathbf{n}(\mathbf{x})\big) \;+\; \nabla_{\Gamma,\mathbf{x}}V(\mathbf{x},k)\exp(\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{a}})\Big), \tag{3.8}$$

and thus

$$|v|_{H^1(\Gamma)} \;\leq\; k\big(\|v\|_{L^2(\Gamma)} + \|\nabla_\Gamma V(\cdot,k)\|_{L^2(\Gamma)}\big)$$

Using the lower bound on v in (3.4) and the bound on $\nabla_\Gamma V$ (3.7), we obtain that $|v|_{H^1(\Gamma)} \lesssim k\|v\|_{L^2(\Gamma)}$ for all $k \geq k_1$. This implies (3.5), and so we are done.

Proof (Proof Theorem 1.2) We first introduce some notation. Let Ω_- be a convex polygon with n_s sides and let the vertices of the polygon be numbered P_1 to P_{n_s}. Let $\omega_m \in (\pi, 2\pi)$ be the exterior angle at P_m. We use the convention that $P_{n_s+1} = P_1$ and $\omega_{n_s+1} = \omega_1$. Let Γ_m denote the side of the polygon connecting the vertices P_m and P_{m+1}, let L_m denote its length.

We now recall some results about the behaviour of v on Γ that were originally proved in [14] and recapped in [12, §3.3.1]. Let $v(s)$ equal $v(\mathbf{x},k)$ restricted Γ_m, where s denotes the distance of $\mathbf{x}$ from P_m. We use the following decomposition of $v(s)$,

$$v(s) = V_0(s) + k\Big[V_m^+(s)\mathrm{e}^{\mathrm{i}ks} + V_m^-(L_m - s)\mathrm{e}^{-\mathrm{i}ks}\Big], \quad s \in [0, L_m] \tag{3.9}$$

[12, Equation 3.36]. The function $V_0(s)$ equals $V_0(\mathbf{x},k)$ restricted to Γ_m, where

$$V_0(\mathbf{x},k) = \begin{cases} 2\dfrac{\partial u^I}{\partial n}(\mathbf{x}) & \text{on illuminated sides} \\[2ex] 0 & \text{on shadow sides,} \end{cases}$$

where the shadow is defined to be such that $\mathbf{n}(\mathbf{x})\cdot\widehat{\mathbf{a}} \geq 0$ (i.e. sides with grazing incidence are also in the shadow). Given $k_0 > 0$, the functions $V_m^+(t)$, $m = 1,\ldots,n_s$, satisfy the bounds

$$\left|\frac{\partial^n}{\partial t^n}V_m^+(t)\right| \lesssim M(u)\frac{k^n}{(kt)^{\alpha_m+n}} \qquad \text{if } t \leq 1/k \tag{3.10}$$

for all $k \geq k_0$ and $n \geq 0$, where $\alpha_m := 1 - \pi/\omega_m \in (0, 1/2)$ [12, Theorem 3.9], [14, Corollary 3.4]. Similar bounds hold for $V_m^-(t)$ with α_m replaced by α_{m+1}. Furthermore, given $k_0 > 0$,

$$\left|\frac{\partial^n}{\partial t^n}V_m^\pm(t)\right| \lesssim M(u)\frac{k^n}{(kt)^{1/2+n}} \qquad \text{if } t \geq 1/k \tag{3.11}$$

for all $k \geq k_0$ and $n \geq 0$ [12, Theorem 3.10], [14, Theorem 3.2], and similarly for $V_m^-(t)$ after replacing α_m by α_{m+1}.

These results mean that, in a k–dependent neighbourhood of P_m,

$$V_m^+(s) \sim s^{-\alpha_m} \quad \text{and} \quad (V_m^+)'(s) \sim s^{-\alpha_m - 1} \quad \text{as } s \to 0,$$

(and similarly for V_m^-) where the omitted constant depends on k. Therefore,

$$s^{\beta_m}|V_m^+(s)| \text{ and } s^{\beta_m+1}|(V_m^+)'(s)| \in L^2(\Gamma_m) \text{ if } \beta_m > \alpha_m - 1/2.$$

This behaviour motivates the definition of the following weighted norm. Given β_m, $m = 1, \ldots, n_s$, with $\alpha_m - 1/2 < \beta_m < 0$, let

$$\|v\|^2_{L^2_{\rm w}(\Gamma)} := \sum_{m=1}^{n_s} \|v\|^2_{L^2_{\rm w}(\Gamma_m)},$$

where

$$\begin{aligned}\|v\|^2_{L^2_{\rm w}(\Gamma_m)} :=& \underbrace{\int_0^{L_m/2} s^{2\beta_m} |v(s)|^2 \, {\rm d}s}_{=:I_1} + \underbrace{\int_{L_m/2}^{L_m} (L_m - s)^{2\beta_{m+1}} |v(s)|^2 \, {\rm d}s}_{=:I_2} \\ &+ \underbrace{\int_0^{L_m/2} s^{2\beta_m+2} |v'(s)|^2 \, {\rm d}s}_{=:I_3} + \underbrace{\int_{L_m/2}^{L_m} (L_m - s)^{2\beta_{m+1}+2} |v'(s)|^2 \, {\rm d}s}_{=:I_4}. \end{aligned} \qquad (3.12)$$

The decomposition (3.9), the bounds (3.10) and (3.11), and the fact that $\beta_m > \alpha_m - 1/2$ then imply that $\|v\|_{L^2_{\rm w}(\Gamma)} < \infty$.

The approximation of such functions v by piecewise polynomials of fixed degree on graded meshes is classical, with sample references being [10], [19]. The result [19, Lemma 2.10] implies that if the mesh on Γ_m is given by $(s_i)_{i=1}^{2N}$, with

$$s_i := \frac{L_m}{2}\left(\frac{i}{N}\right)^{q_m} \quad \text{and} \quad s_{N+i} := L_m - \frac{L_m}{2}\left(\frac{N-i}{N}\right)^{q_m} \quad \text{for } i = 0, \ldots, N,$$

and $q_m > -1/\beta_m$, then

$$\inf_{w_N \in \mathscr{V}_N} \|v - w_N\|_{L^2(\Gamma)} \lesssim \frac{1}{N} \|v\|_{L^2_{\rm w}(\Gamma)}, \qquad (3.13)$$

where $\mathscr{V}_N$ is the corresponding space of piecewise polynomials of fixed degree. Recalling the beginning of the proof of Theorem 1.1, we see that the result (1.19) follows from (3.13) if we can show that there exists a $k_1 > 0$ such that

$$\|v\|_{L^2_{\rm w}(\Gamma)} \lesssim k \|v\|_{L^2(\Gamma)} \text{ for all } k \geq k_1. \qquad (3.14)$$

That is, with $I_j, j = 1, \ldots, 4$, defined as in (3.12), we need to show that, for every m,

$$I_1 + I_2 + I_3 + I_4 \lesssim k^2 \|v\|^2_{L^2(\Gamma)}. \qquad (3.15)$$

We now bound each of the I_j separately. Before we begin, we note that $2\beta_m + 1 > 0$ since $2\alpha_m \geq 0$. Using (3.9), we have that

$$I_1 \lesssim \int_0^{L_m/2} s^{2\beta_m} |V_0(s)|^2 \, {\rm d}s + k^2 \int_0^{L_m/2} s^{2\beta_m} |V_m^+(s)|^2 \, {\rm d}s + k^2 \int_0^{L_m/2} s^{2\beta_m} |V_m^-(L_m - s)|^2 \, {\rm d}s. \qquad (3.16)$$

Recall that $V_0(s)$ is $\partial u^I/\partial n$ restricted to Γ_m, and

$$\frac{\partial u^I}{\partial n}(\mathbf{x}) = {\rm i}k \, {\rm e}^{{\rm i}k\mathbf{x}\cdot\widehat{\mathbf{a}}} \widehat{\mathbf{a}} \cdot \mathbf{n}(\mathbf{x}), \qquad (3.17)$$

and thus the first term on the right-hand side of (3.16) is $\lesssim k^2$. For the second term on the right-hand side of (3.16), we assume that $kL_m \geq 2$, split the integral into integrals over $(0, 1/k)$ and $(1/k, L_m/2)$, and use the bounds (3.10) and (3.11) to find that

$$k^2 \int_0^{L_m/2} s^{2\beta_m} |V_m^+(s)|^2 \, {\rm d}s \lesssim k^2 \big(M(u)\big)^2 \left[\int_0^{1/k} \frac{s^{2\beta_m}}{(ks)^{2\alpha_m}} {\rm d}s + \int_{1/k}^{L_m/2} \frac{s^{2\beta_m}}{ks} {\rm d}s\right]$$

$$\lesssim \frac{k^2\big(M(u)\big)^2}{k^{2\beta_m+1}}\left[\int_0^1 \frac{t^{2\beta_m}}{t^{2\alpha_m}}\mathrm{d}t + \int_1^{kL_m/2} t^{2\beta_m-1}\mathrm{d}t\right]$$
$$\lesssim k^2\big(M(u)\big)^2\left[\frac{1}{k^{2\beta_m+1}}+\frac{1}{k}\right],$$
$$\lesssim k^2\big(M(u)\big)^2 \quad \text{using the fact that } 2\beta_m+1>0.$$

For the third term on the right-hand side of (3.16), we use the fact that $kL_m \geq 2$ and the bound (3.11) to obtain

$$k^2\int_0^{L_m/2} s^{2\beta_m}|V_m^-(L_m-s)|^2\,\mathrm{d}s \lesssim k^2\big(M(u)\big)^2\int_0^{L_m/2}\frac{s^{2\beta_m}}{k(L_m-s)}\mathrm{d}s,$$
$$\lesssim k\big(M(u)\big)^2.$$

Therefore, putting the bound on the terms on the right-hand side of (3.16) together we have that

$$I_1 \lesssim k^2 + k^2\big(M(u)\big)^2. \tag{3.18}$$

In a similar way, we find that an identical bound holds for I_2.

To determine the k–dependence of I_3 and I_4, we need to estimate $v'(s)$. Differentiating (3.9), we have that

$$v'(s) = V_0'(s) + \mathrm{i}k^2\left[\mathrm{e}^{\mathrm{i}ks}V_m^+(s) - \mathrm{e}^{-\mathrm{i}ks}V_m^-(L-s)\right] + k\left[\mathrm{e}^{\mathrm{i}ks}(V_m^+)'(s) - \mathrm{e}^{-\mathrm{i}ks}(V_m^-)'(L-s)\right]. \tag{3.19}$$

The function $V_0'(s)$ is the surface gradient on Γ_m of (3.17), and thus $\sim k^2$. Since our only lower bound on $\|v\|_{L^2(\Gamma)}$ is $\|v\|_{L^2(\Gamma)} \gtrsim k^{1/2}$ (3.4), we need to estimate the term in (3.19) involving $V_0'(s)$ in way other than $\|V_0'\|_{L^2(\Gamma_m)} \lesssim k^2 \lesssim k^{3/2}\|v\|_{L^2(\Gamma)}$ (as this last inequality is too weak to give us (3.14)). Our plan is to express $V_0'(s)$ in terms of $V_0(s)$, and thus in terms of $v(s)$, $V_m^+(s)$, and $V_m^-(s)$. Taking the surface gradient of (3.17), and recalling that $\mathbf{n}(\mathbf{x})$ is constant on Γ_m, we have that

$$\nabla_{\Gamma,\mathbf{x}}\left(\frac{\partial u^I}{\partial n}(\mathbf{x})\right) = -k^2\,\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\widehat{\mathbf{a}}}\,(\widehat{\mathbf{a}}\cdot\mathbf{n}(\mathbf{x}))\,(\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x})) = \mathrm{i}k(\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x}))\frac{\partial u^I}{\partial n}(\mathbf{x}),$$

where $\boldsymbol{\tau}(\mathbf{x})$ is a unit tangent vector on Γ_m. If $\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x}) = 0$ (i.e. the incident wave is perpendicular Γ_m) then $V_0'(s) = 0$ and (3.19) implies that

$$|v'(s)| \lesssim k^2\left[|V_m^+(s)| + |V_m^-(L_m-s)|\right] + k\left[|(V_m^+)'(L-s)| + |(V_m^-)'(s)|\right]. \tag{3.20}$$

If $\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x}) \neq 0$ then $V_0'(s) = \mathrm{i}kAV_0(s)$ with $A = \widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x})$ (which is constant on each side). Therefore,

$$v'(s) = \mathrm{i}kA\left[v(s) - k\big(V_m^+(s)\mathrm{e}^{\mathrm{i}ks} + V_m^-(L_m-s)\mathrm{e}^{-\mathrm{i}ks}\big)\right] + \mathrm{i}k^2\left[\mathrm{e}^{\mathrm{i}ks}V_m^+(s) - \mathrm{e}^{-\mathrm{i}ks}V_m^-(L_m-s)\right]$$
$$+ k\left[\mathrm{e}^{\mathrm{i}ks}(V_m^+)'(s) - \mathrm{e}^{-\mathrm{i}ks}(V_m^-)'(L_m-s)\right],$$

and

$$|v'(s)| \lesssim k|v(s)| + k^2\left[|V_m^+(s)| + |V_m^-(L_m-s)|\right] + k\left[|(V_m^+)'(s)| + |(V_m^-)'(L_m-s)|\right] \tag{3.21}$$

We proceed assuming that $\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x}) \neq 0$ (and thus (3.21) holds); the argument in the case when $\widehat{\mathbf{a}}\cdot\boldsymbol{\tau}(\mathbf{x}) = 0$ (and (3.20) holds) is almost identical.

Using (3.21) in the definition of I_3, we have that

$$I_3 \lesssim k^2\int_0^{L_m/2} s^{2\beta_m+2}|v(s)|^2\,\mathrm{d}s + k^4\int_0^{L_m/2} s^{2\beta_m+2}\Big(|V_m^+(s)|^2 + |V_m^-(L_m-s)|^2\Big)\,\mathrm{d}s$$
$$+ k^2\int_0^{L_m/2} s^{2\beta_m+2}\Big(|(V_m^+)'(s)|^2 + |(V_m^-)'(L_m-s)|^2\Big)\,\mathrm{d}s. \tag{3.22}$$

Now, the first term on the right-hand side of (3.22) is $\lesssim k^2\|v\|^2_{L^2(\Gamma)}$. Using the bounds (3.10) and (3.11), the second term on the right-hand side of (3.22) is

$$\begin{aligned}
&\lesssim k^4\big(M(u)\big)^2\left[\int_0^{1/k}\frac{s^{2\beta_m+2}}{(ks)^{2\alpha_m}}\mathrm{d}s+\int_{1/k}^{L_m/2}\frac{s^{2\beta_m+2}}{ks}\mathrm{d}s+\int_0^{L_m/2}\frac{s^{2\beta_m+2}}{k(L_m-s)}\mathrm{d}s\right]\\
&\lesssim k^4\big(M(u)\big)^2\left[\frac{1}{k^{2\beta_m+3}}+\frac{1}{k}+\frac{1}{k}\right]\\
&\lesssim k^3\big(M(u)\big)^2 \qquad \text{since } 2\beta_m+3>2.
\end{aligned}$$

Using the bounds (3.10) and (3.11) again, the third term on the right-hand side of (3.22) is

$$\begin{aligned}
&\lesssim k^2\big(M(u)\big)^2\left[\int_0^{1/k}s^{2\beta_m+2}\frac{k^2}{(ks)^{2\alpha_m+2}}\mathrm{d}s+\int_{1/k}^{L_m/2}s^{2\beta_m+2}\frac{k^2}{(ks)^3}\mathrm{d}s+\int_0^{L_m/2}s^{2\beta_m+2}\frac{k^2}{(k(L_m-s))^3}\mathrm{d}s\right]\\
&\lesssim k^2\big(M(u)\big)^2\left[\frac{1}{k^{2\beta_m+1}}+\frac{1}{k}+\frac{1}{k}\right]\\
&\lesssim k^2\big(M(u)\big)^2 \qquad \text{since } 2\beta_m+1>0.
\end{aligned}$$

Therefore,

$$I_3\lesssim k^2\|v\|^2_{L^2(\Gamma)}+k^3\big(M(u)\big)^2+k^2\big(M(u)\big)^2; \tag{3.23}$$

in a similar way, we find that an identical bound holds for I_4. Using the bounds (3.18), (3.23) and their counterparts for I_2 and I_4, we have that

$$I_1+I_2+I_3+I_4\lesssim k^2+k^2\big(M(u)\big)^2+k^2\|v\|^2_{L^2(\Gamma)}+k^3\big(M(u)\big)^2, \tag{3.24}$$

If $M(u)\lesssim 1$, then the right-hand side of (3.24) is $\lesssim k^2\|v\|^2_{L^2(\Gamma)}+k^3$. Since $\|v\|^2_{L^2(\Gamma)}\gtrsim k$ from (3.4), the bound (3.15) holds and the proof is complete.

Proof (Proof of Theorem 1.3) If we can prove the bounds in (1.25), then the results (1.21) and (1.22) follow by combining (1.25) and (1.23).

To prove the first inequality in (1.25), we begin by choosing $\eta=k$ and writing the integral equation (1.1) as

$$\frac{1}{2}v+L_kv=f,$$

where $L_k:=D_k'-\mathrm{i}kS_k$. Since $L_k:L^2(\Gamma)\to H^1(\Gamma)$ when Γ is C^2 and $f\in H^1(\Gamma)$, we have that $v\in H^1(\Gamma)$. Using the triangle inequality

$$\frac{1}{2}\|v\|_{H^1(\Gamma)}\le\|f\|_{H^1(\Gamma)}+\|L_k\|_{L^2(\Gamma)\to H^1(\Gamma)}\|v\|_{L^2(\Gamma)}. \tag{3.25}$$

The definition of f (1.11) implies that, when $\eta=k$, $\|f\|_{L^2(\Gamma)}\sim k$ and $\|f\|_{H^1(\Gamma)}\sim k^2$, and then, using the bounds in Theorem 1.6 we have that

$$\|v\|_{H^1(\Gamma)}\lesssim k^2+k^{(d+1)/2}\|v\|_{L^2(\Gamma)}\sim k^{(d+1)/2}\left(k^{(3-d)/2}+\|v\|_{L^2(\Gamma)}\right).$$

The lower bound on $\|v\|_{L^2(\Gamma)}$ in (3.4) then implies the result on v in (1.25).

For the bound on ϕ in (1.25), we follow the proof of the bound on v and obtain the analogue of (3.25)

$$\|\phi\|_{H^1(\Gamma)}\lesssim\left\|\gamma_+u^I\right\|_{H^1(\Gamma)}+k^{(d+1)/2}\|\phi\|_{L^2(\Gamma)}.$$

Direct calculation shows that $\|\gamma_+u^I\|_{H^1(\Gamma)}\sim k$, and thus

$$\|\phi\|_{H^1(\Gamma)}\lesssim k^{(d+1)/2}\left(k^{(1-d)/2}+\|\phi\|_{L^2(\Gamma)}\right).$$

Using the lower bound (3.4) yields the bound on ϕ in (1.25).

4 Proofs of Theorems 1.4 and 1.5 (quasi-optimality for h–version of the BEM)

4.1 Proofs of Theorems 1.4 and 1.5

In this section we assume that $|\eta| \sim k$, and write the combined potential operators $A'_{k,\eta}$ and $A_{k,\eta}$ as $\lambda I + L_k$, where $\lambda = 1/2$ and L_k equals one of $D'_k - \mathrm{i}\eta S_k$ or $D_k - \mathrm{i}\eta S_k$. (Since $|\eta| \sim k$ the parameter η does not appear explicitly in the notation L_k.) Therefore, the integral equation (1.1) becomes

$$(\lambda I + L_k)v = f, \tag{4.1}$$

and (1.2) becomes $(\lambda I + L_k)\phi = g$. In the rest of this section we only consider the direct equation (4.1), but we note that the analysis for the indirect equation is identical.

We assume that Γ is C^2 and we consider the h–version of the Galerkin method, i.e. we seek $v_h \in \mathscr{V}_h$, the space of piecewise polynomials of degree p for some fixed $p \geq 0$ on shape regular meshes of diameter h, with h decreasing to zero. The Galerkin equations (1.16) can then be written as

$$\big((\lambda I + L_k)v_h, w_h\big)_{L^2(\Gamma)} = (f, w_h)_{L^2(\Gamma)} \qquad \text{for all } w_h \in \mathscr{V}_h. \tag{4.2}$$

If P_h denotes the orthogonal projection from $L^2(\Gamma)$ onto $\mathscr{V}_h$ then the Galerkin equations (4.2) are equivalent to the operator equation

$$(\lambda I + P_h L_k)\, v_h = P_h f \tag{4.3}$$

[3, §3.1.2].

We begin with a simple, classical lemma.

Lemma 4.1 *If*

$$\|(I - P_h)L_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \left\|(\lambda I + L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \frac{\delta}{1+\delta} \tag{4.4}$$

for some $\delta > 0$, then the Galerkin equations have a unique solution, v_h, which satisfies the quasi-optimal error estimate

$$\|v - v_h\|_{L^2(\Gamma)} \leq \lambda(1+\delta)\left\|(\lambda I + L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \inf_{w_h \in \mathscr{V}_h} \|v - w_h\|_{L^2(\Gamma)}. \tag{4.5}$$

Proof Since $\delta > 0$, the hypothesis (4.4) implies that

$$\left\|I - (\lambda I + L_k)^{-1}(\lambda I + P_h L_k)\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \left(\frac{\delta}{1+\delta}\right) < 1. \tag{4.6}$$

Using the fact that $(I - A)$ is invertible if $\|A\| < 1$ (with $\|(I-A)^{-1}\| \leq (1 - \|A\|)^{-1}$), the bound (4.6) implies that $(\lambda I + L_k)^{-1}(\lambda I + P_h L_k)$ is invertible, with

$$\left\|(\lambda I + P_h L_k)^{-1}(\lambda I + L_k)\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \frac{1}{1 - \delta/(1+\delta)} = 1 + \delta.$$

Therefore, $(\lambda I + P_h L_k)$ is invertible with

$$\left\|(\lambda I + P_h L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq (1+\delta)\left\|(\lambda I + L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)}. \tag{4.7}$$

Since

$$\begin{aligned} v - v_h &= v - (\lambda I + P_h L_k)^{-1} P_h f \\ &= (\lambda I + P_h L_k)^{-1}(\lambda v - P_h(f - L_k v)) \\ &= \lambda\,(\lambda I + P_h L_k)^{-1}\,(I - P_h)v, \end{aligned}$$

the result (4.5) follows from the bound (4.7).

The following corollary follows from Lemma 4.1 when we have an estimate of the smoothing power of L_k.

Corollary 4.1 *If*

$$N(k) := \|L_k\|_{L^2(\Gamma)\to H^1(\Gamma)} < \infty \tag{4.8}$$

then, for any $\delta > 0$, there exists a $C_\delta > 0$ such that the condition

$$hN(k)\left\|(\lambda I + L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C_\delta \tag{4.9}$$

ensures that the quasi-optimal estimate (4.5) holds.

Proof By the standard approximation theory result (1.23), we have that

$$\|(I-P_h)L_k\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim hN(k) \tag{4.10}$$

and so the result then follows from Lemma 4.1 (with C_δ taken to be $\delta/(1+\delta)$ divided by the hidden constant in (4.10)).

We now use Theorems 1.6 and 3.2 to get a k–explicit bound on the left-hand side of (4.9), and this proves Theorem 1.4.

Proof (Proof of Theorem 1.4) Since Ω_- is C^2, the bounds on S_k, D_k, and D'_k in Theorem 1.6 imply that, given $k_0 > 0$,

$$\|L_k\|_{L^2(\Gamma)\to H^1(\Gamma)} \lesssim k^{(d+1)/2} \tag{4.11}$$

for all $k \geq k_0$. Furthermore, since Ω_- is star-shaped, the bound (3.2) implies that, given $k_0 > 0$,

$$\left\|(\lambda I + L_k)^{-1}\right\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$$

for all $k \geq k_0$. Using these two bounds, we see that there exists a $C > 0$ such that if $hk^{(d+1)/2} \leq C$ then the condition (4.9) is satisfied, and the result follows.

To prove Theorem 1.5 we use the classical "superconvergence argument" for second kind integral equations; see, e.g., [9].

Lemma 4.2 *Suppose that both the conditions (4.8) and*

$$M(k) := \left\|(\lambda I + L_k^*)^{-1}L_k\right\|_{L^2(\Gamma)\to H^1(\Gamma)} < \infty \tag{4.12}$$

hold (where L_k^* is the adjoint of L_k). Then the condition (4.9) is sufficient to ensure that the Galerkin equations have a unique solution and furthermore there exists a C_0 independent of h and k such that if

$$C_0\, hC(k) \leq 1 \tag{4.13}$$

then

$$\inf_{w_h\in\mathscr{V}_h}\|v-w_h\|_{L^2(\Gamma)} \leq \|v-v_h\|_{L^2(\Gamma)} \leq \left[1 + C_0\, hC(k)\right]\inf_{w_h\in\mathscr{V}_h}\|v-w_h\|_{L^2(\Gamma)}, \tag{4.14}$$

where

$$C(k) = N(k) + \left(\lambda + \|L_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\right)M(k). \tag{4.15}$$

Proof If we apply P_h to (4.1) and subtract the resulting equation from (4.3) then we obtain

$$\lambda(v_h - P_h v) = P_h L_k (v - v_h). \tag{4.16}$$

Writing

$$\|v-v_h\|^2_{L^2(\Gamma)} = \left(v-v_h, v-P_h v\right)_{L^2(\Gamma)} + \left(v-v_h, P_h v - v_h\right)_{L^2(\Gamma)}, \tag{4.17}$$

we see that to prove (4.14) we essentially have to show that the second term on the right-hand side of (4.17) goes to zero more quickly than $\|v-v_h\|^2_{L^2(\Gamma)}$. This is done by taking the inner product of (4.16) with $v - v_h$ to obtain

$$\begin{aligned}\lambda\left(v-v_h, P_h v - v_h\right)_{L^2(\Gamma)} &= -\left(v-v_h, P_h L_k(v-v_h)\right)_{L^2(\Gamma)} \\ &= \left(v-v_h, (I-P_h)L_k(v-v_h)\right)_{L^2(\Gamma)} - \left(v-v_h, L_k(v-v_h)\right)_{L^2(\Gamma)}.\end{aligned} \tag{4.18}$$

Using the Cauchy-Schwarz inequality and (4.10), we estimate the first term on the right-hand side of (4.18) by

$$\left|\big(v-v_h,(I-P_h)L_k(v-v_h)\big)_{L^2(\Gamma)}\right| \lesssim hN(k)\,\|v-v_h\|^2_{L^2(\Gamma)}. \tag{4.19}$$

The second term on the right-hand side of (4.18) can be rewritten as

$$\begin{aligned}\big(v-v_h,L_k(v-v_h)\big)_{L^2(\Gamma)} &= \big((\lambda I+L_k)^{-1}(\lambda I+L_k)(v-v_h),L_k(v-v_h)\big)_{L^2(\Gamma)}\\ &= \big((\lambda I+L_k)(v-v_h),(\lambda I+L_k^*)^{-1}L_k(v-v_h)\big)_{L^2(\Gamma)}\\ &= \big((\lambda I+L_k)(v-v_h),(I-P_h)(\lambda I+L_k^*)^{-1}L_k(v-v_h)\big)_{L^2(\Gamma)},\end{aligned}$$

where the last line uses the Galerkin orthogonality (4.16), i.e. the fact that $P_h(\lambda I+L_k)(v-v_h)=0$. Hence, using again the Cauchy-Schwarz inequality and (4.10), we have that

$$\left|\big(v-v_h,L_k(v-v_h)\big)_{L^2(\Gamma)}\right| \lesssim h\left(\lambda+\|L_k\|_{L^2(\Gamma)\to L^2(\Gamma)}\right)M(k)\,\|v-v_h\|^2_{L^2(\Gamma)}. \tag{4.20}$$

Therefore, using (4.19) and (4.20) in (4.18) and using the definition of $C(k)$ (4.15), we obtain

$$\left|\big(v-v_h,P_hv-v_h\big)_{L^2(\Gamma)}\right| \lesssim hC(k)\,\|v-v_h\|^2_{L^2(\Gamma)}.$$

Finally, combining this with (4.17) and using the Cauchy-Schwarz inequality, we have that

$$\big(1-(C_0/2)C(k)h\big)\,\|v-v_h\|_{L^2(\Gamma)} \le \|v-P_hv\|_{L^2(\Gamma)}$$

for some constant C_0. If the threshold (4.13) holds, then we have the result (4.14).

Proof (Proof of Theorem 1.5) This follows from Lemma 4.2 if we can prove that

$$M(k)\lesssim k^d. \tag{4.21}$$

Indeed, using the bound (4.11) on L_k as a mapping from $L^2(\Gamma)\to H^1(\Gamma)$ and (4.21) we find that $hk^{(3d-1)/2}\to 0$ ensures that $C_0\,hC(k)\to 0$.

To bound $M(k)$, we consider u and g related by

$$(\lambda I+L_k^*)^{-1}L_ku=g. \tag{4.22}$$

This equation implies that if $\|g\|_{H^1(\Gamma)}\le c\|u\|_{L^2(\Gamma)}$ then $M(k)\le c$. Now, from (4.22), $(\lambda I+L_k^*)g=L_ku$, and therefore, using (4.11), we find that

$$\lambda\,\|g\|_{H^1(\Gamma)} \lesssim k^{(d+1)/2}\left(\|g\|_{L^2(\Gamma)}+\|u\|_{L^2(\Gamma)}\right). \tag{4.23}$$

We now need to bound $\|g\|_{L^2(\Gamma)}$ in terms of $\|u\|_{L^2(\Gamma)}$. To do this we use the bound (3.2) on $\|(\lambda I+L_k^*)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and the bound (1.36) on $\|L_k\|_{L^2(\Gamma)\to H^1(\Gamma)}$ to obtain $\|g\|_{L^2(\Gamma)}\lesssim\|L_ku\|_{L^2(\Gamma)}\lesssim k^{(d-1)/2}\|u\|_{L^2(\Gamma)}$. The result (4.21) follows from using this last bound in (4.23).

4.2 Comparison with the results of [4], [28], and [31]

As mentioned in §1.2.2, the papers [4] and [28] investigate quasi-optimality of the Galerkin method applied to (1.1) and (1.2) using a method that obtains sufficient conditions for quasi-optimality to hold in terms of how well the spaces $\mathscr{V}_N$ approximate the solution of certain adjoint problems. This method is often attributed to Schatz [41]; for examples of its use and further development see [20, §4], and the references therein.

We now compare the results of [4] and [28] to the analysis in §4.1. We focus on the indirect equation (1.2), since this allows us to keep the notation consistent with that in [28], and we write $A_{k,\eta}$ as $\lambda I+L_k$ when doing so links these results to the analysis in §4.1.

In [4], the method discussed above is used to prove that

$$\|\phi-\phi_N\|_{L^2(\Gamma)} \lesssim \left(\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\right)\inf_{w_N\in\mathscr{V}_N}\|\phi-w_N\|_{L^2(\Gamma)} \tag{4.24}$$

provided that

$$\big\|A_{k,\eta}\big\|_{L^2(\Gamma)\to L^2(\Gamma)}\big\|(I-P_N)(\lambda I+L_k^*)^{-1}L_k\big\|_{L^2(\Gamma)\to L^2(\Gamma)}\quad\text{is sufficiently small,}\tag{4.25}$$

where P_N denotes the orthogonal projection from $L^2(\Gamma)$ onto $\mathscr{V}_N$ [4, Corollary 3.3]. Choosing $\mathscr{V}_N=\mathscr{V}_h$ and using the approximation result (1.23), we see that the condition (4.25) becomes

$$h\big\|A_{k,\eta}\big\|_{L^2(\Gamma)\to L^2(\Gamma)}M(k)\quad\text{is sufficiently small,}\tag{4.26}$$

where $M(k)$ is defined by (4.12) (see also [4, Corollary 3.6]).

The presence of $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ in the error estimate (4.24) means that this estimate will not give us k–independent quasi-optimality for general domains (since for many domains $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ grows with k; see [11, §4], [12, §5.2.2]). However, when Γ is the circle or sphere and $\eta=k^{2/3}$, $\|A_{k,\eta}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$; see [4, Corollary 3.11], [12, Theorem 5.12]. Furthermore, there is some numerical evidence that, with this choice of η,

$$M(k)\lesssim k\tag{4.27}$$

[4, Figure 3.1]. Therefore, if Γ is the circle or sphere, $\eta=k^{2/3}$, and (4.27) holds, then (4.24) and (4.26) become

$$\|\phi-\phi_h\|_{L^2(\Gamma)}\lesssim\inf_{w_h\in\mathscr{V}_h}\|\phi-w_h\|_{L^2(\Gamma)}\quad\text{provided}\quad hk\lesssim 1.$$

The analysis in [28] treats $A_{k,\eta}$ as a perturbation of the k–independent, invertible operator $A_0:=1/2+D_0-\mathrm{i}S_0$, and employs the general method discussed above to obtain that

$$\|\phi-\phi_N\|_{L^2(\Gamma)}\lesssim\inf_{w_N\in\mathscr{V}_N}\|\phi-w_N\|_{L^2(\Gamma)}\tag{4.28}$$

provided that

$$\big\|(I-P_N)(A_{k,\eta}-A_0)\big\|_{L^2(\Gamma)\to L^2(\Gamma)}\quad\text{is sufficiently small}\tag{4.29}$$

and

$$\big\|(I-P_N)(A_{k,\eta}^*)^{-1}(A_{k,\eta}^*-A_0^*)\big\|_{L^2(\Gamma)\to L^2(\Gamma)}\quad\text{is sufficiently small}\tag{4.30}$$

(where the omitted constant in (4.28) contains $\|A_0\|_{L^2(\Gamma)\to L^2(\Gamma)}$) [28, Theorem 3.8 and Corollary 3.10].

The novel decompositions of $A_{k,\eta}$, $A_{k,\eta}^{-1}$, and their adjoints in [31] show that if $\mathscr{V}_N=\mathscr{V}_{h,p}$ (the space of piecewise polynomials of degree p on uniform meshes of mesh size h) and $\|(A_{k,\eta}^*)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ is bounded polynomially in k then the conditions (4.29) and (4.30) are satisfied when $p\gtrsim\log k$ and $hk\lesssim p$ [28, Corollary 3.18]. These conditions on h and p can be satisfied with the total number of degrees of freedom $\sim k^{d-1}$, and thus this result proves that the hp–BEM does not suffer from the pollution effect. (Note that the assumption that $\|(A_{k,\eta}^*)^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}=\|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ is bounded polynomially in k is ensured for nontrapping domains by the bound (3.3).)

Although the methods in [28] and [31] are geared towards the hp–BEM (with the underlying assumption that p will tend to infinity to obtain exponential convergence) we can take p to be constant and obtain a condition on h for quasi-optimality of the h–BEM. Indeed, taking p to be constant and assuming that $\|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$ (as it is when Ω_- is star-shaped by (3.2)), we find that [28, Theorem 3.17] implies that (4.28) holds when if $hk^6\lesssim 1$.

We can use the results of the present paper to obtain better bounds on the quantities in (4.29) and (4.30) for the h–BEM. Indeed, using the approximation theory result (1.23), we see that if $\|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$ then the condition (4.29) is almost identical to (4.9). The bounds in Theorem 1.6 therefore show that the condition (4.29) is satisfied if $hk^{(d+1)/2}\lesssim 1$ (and $\|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$). Similarly, we see that the condition (4.30) is essentially the condition that $hM(k)$ is sufficiently small, and then the bound on $M(k)$ (4.21) (valid when $\|A_{k,\eta}^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}\lesssim 1$) implies that this is satisfied when $hk^d\lesssim 1$.

In summary, the best result for the h–BEM that can be obtained from the results in [4], [28], and [31] is that if Ω_- is C^2 and star-shaped (in the sense of Definition 1.2) then the quantities in (1.5) are bounded independently of k if $hk^d\lesssim 1$; this is a weaker result than that of Theorem 1.4.

5 Numerical experiments concerning Question 2

Our numerical examples involve the direct operator $A'_{k,-k} = 1/2 + D'_k + \mathrm{i}kS_k$, where the coupling parameter η is taken to be $-k$. The geometries considered are polygons, and we use the canonical element maps to define the ansatz spaces $\mathscr{V}_{h,p}$ of piecewise polynomials of degree p on uniform meshes $\mathscr{T}_h$ of mesh size h. The BEM operators D'_k and S_k are set up with an hp–quadrature with 10 quadrature points in each direction per quadrature cell. Details of the fast quadrature technique employed are described in [27].

Denoting by $P_{\mathscr{T}_h,p} : L^2(\Gamma) \to \mathscr{V}_{h,p}$ the Galerkin projector, which is characterized by

$$(A'_{k,-k}(u - P_{\mathscr{T}_h,p}u), v)_{L^2(\Gamma)} = 0 \qquad \text{for all } v \in \mathscr{V}_{h,p},$$

we approximate the Galerkin error $\|I - P_{\mathscr{T}_h,p}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ by the formula

$$\|I - P_{\mathscr{T}_h,p}\|_{L^2(\Gamma)\to L^2(\Gamma)} \approx \sup_{0\neq v\in\mathscr{V}_{h,p_{max}}} \frac{\|v - P_{\mathscr{T}_h,p}v\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}, \qquad p_{max} = 5. \tag{5.1}$$

As described in [28, §4], we evaluate for $p \in \{0,1\}$ the expression

$$\sqrt{1+\gamma_p^2} := \sup_{0\neq v\in\mathscr{V}_{h,p_{max}}} \frac{\|v - P_{\mathscr{T}_h,p}v\|_{L^2(\Gamma)}}{\|v\|_{L^2(\Gamma)}}, \tag{5.2}$$

using an appropriate SVD; $\sqrt{1+\gamma_p^2}$ is therefore an approximation to the quantity involving v in (1.5) (see [28, Lemma 4.1]). At the same time, the Galerkin matrix corresponding to the space $\mathscr{V}_{h,p_{max}}$ is used to get estimates for the norms $\|A'_{k,-k}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ and $\|(A'_{k,-k})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$.

Example 5.1 The geometry is the rectangle $\Omega_- = (0,1/2)\times(0,5)$ and the numerical results are presented in Figure 5.1. By the star-shapedness of Ω_- we have $\|(A'_{k,-k})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$ (see Theorem 3.2) which is clearly visible in Figure 5.1. Furthermore, Figure 5.1 suggests an even better bound than the estimate $\|A'_{k,-k}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/2}$ given in Theorem 3.1. The values γ_0 and γ_1 computed according to (5.2) are obtained on uniform meshes keeping kh fixed. Specifically, with $L = 11$ being the length of Γ, the number of degrees of freedom per wavelength

$$N_\lambda := 2\pi\frac{N(p+1)}{Lk}$$

is $N_\lambda = 2\pi$ for $p = 0$ and 4π for $p = 1$. Despite using a uniform mesh (for a polygonal domain Ω_-), the values of γ_0 and γ_1 are practically constant over a large range of values of k. The value γ_1 is consistently smaller than γ_0, reflecting the better approximation properties of the space $\mathscr{V}_{h,1}$ over the space $\mathscr{V}_{h,0}$. ■

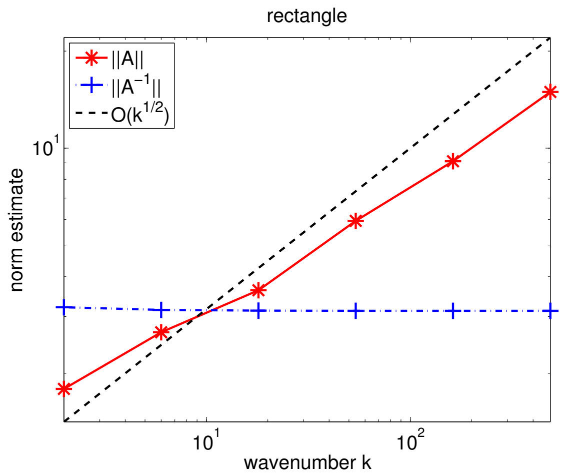
Example 5.2 The geometry is the C-shaped domain given by

$$\Omega_- = ((-r/2,r/2)\times(-r/3,r/3)) \setminus ((-r/6,r/6)\times(0,r/3)), \qquad r = 1/2.$$

For different values of the parmeter $m \in 3\mathbb{N}$, we select the number of elements N and the wavenumber k according to

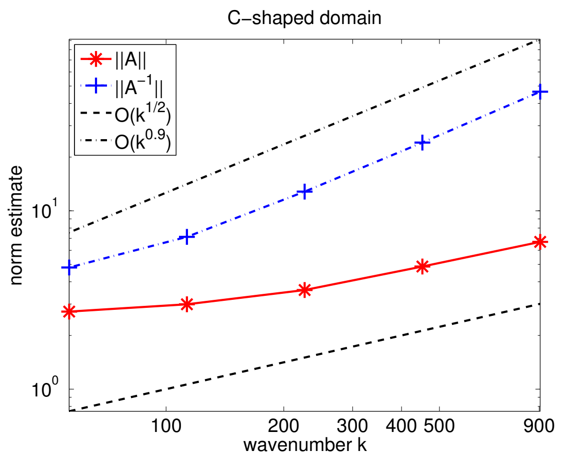
$$N = 20m, \qquad k = \frac{3m\pi}{r}.$$

The choice of these wavenumbers is motivated by the analysis in [11, §5] where it is shown that $\|(A'_{k,-k})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \gtrsim k^{0.9}$. Figure 5.2 suggests that this estimate is sharp. At the same time, it confirms the bound $\|A_{k,-k}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim k^{1/2}$. The table presents for the cases $p = 0$ and $p = 1$ the values of γ_p given by (5.2) when keeping kh fixed. Specifically, the number of degrees of freedom per wavelength $N_\lambda := 2\pi\frac{N(p+1)}{Lk}$ where $L = 4r$ is the length of Γ; then, $N_\lambda \approx 6.6$ for $p = 0$ and $N_\lambda \approx 13.2$ for $p = 1$. The values γ_p are practically constant as k is increased, so the condition $kh \sim 1$ appears to be sufficient for k–independent quasi-optimality. It is worth noting that, since $\|(A'_{k,-k})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$ grows with k, the analysis in §4 suggests more stringent conditions on the relation between h and k than in the case of the star-shaped geometry of Example 5.1 (although this analysis is only valid when Γ is C^2, and thus not when Ω_- is a polygon as in these examples). ■



N	k	γ_0	γ_1
22	2	0.368234	0.136623
66	6	0.334368	0.121106
198	18	0.337487	0.120028
594	54	0.335113	0.120023
1782	162	0.333687	0.12
5346	486	0.333559	0.119998

Fig. 5.1 Rectangular domain (see Example 5.1 for details)



m	k	N	γ_0	γ_1
3	56.5	120	0.480033	0.174585
6	113.1	240	0.487655	0.174454
12	226.2	480	0.51861	0.174301
24	452.4	960	0.527743	0.174264
48	904.8	1920	0.549879	0.174278

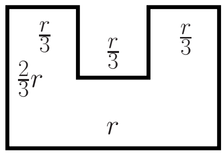


Fig. 5.2 C-shaped domain (see Example 5.2 for details)

Summary. The numerical experiments in these two examples (along with similar numerical results when Γ is a circle or an ellipse in [28, §4]) indicate that k–independent quasi-optimality holds when $hk \sim 1$, even in some situations where the norm of the solution operator (i.e. $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)}$) grows with k. This should be contrasted with the well-known fact that $hk \sim 1$ is not sufficient for k–independent quasi-optimality of the h–FEM, even when the solution operator is bounded independently of k. These observations about the quasi-optimality of the h–BEM have yet to be proved rigorously, however, with the analysis in §4 yielding the more restrictive condition $hk^{3/2} \lesssim 1$ (in 2–d) for k–independent quasi-optimality (under the assumption that $\|(A'_{k,\eta})^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \lesssim 1$).

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