Recap

$(k^{-2}\Delta - 1)u = f$ in $\mathbb{R}^d$ + radiation condition

reformulated as find $u_N \in \cancel{H^1(B_R)}\ \mathcal{H}_N$ s.t. $a(u_N, v_N) = F(v_N)$ $\forall v_N \in \cancel{H^1(B_R)}\ \mathcal{H}_N$

$$a(u,v) = \int_{B_R} \left(k^{-2}\nabla u \cdot \overline{\nabla v} - u\bar{v}\right) - k^{-1}\langle \mathrm{DtN}\, u, v\rangle_{\partial B_R}$$

Galerkin method: given $\mathcal{H}_N \subset H^1(B_R)$

Goal: find conditions on h and p s.t. $\|u - u_N\|_{H^1_k} \leq C_{qo} \min_{v_N \in \mathcal{H}_N} \|u - v_N\|_{H^1_k}$

indep of k "quasi optimality"

- $p = 1$, $hk^2 \leq C_0$ $\quad \#\text{dof} \sim \left(\frac{p}{h}\right)^d$
- $\frac{hk}{p} \leq C_1$, $p \geq C_2 \log k$ $\quad \#\text{dof} \sim k^d$

[Melenk + Sauter 2010]

$|a(u,v)| \leq C_{\text{cont}} \|u\|_{H^1_k} \|v\|_{H^1_k}$ $\forall u, v$

Gårding: $\operatorname{Re} a(v,v) \geq \|v\|^2_{H^1_k} - 2\|v\|^2_{L^2}$ $\forall v$

$\|u\|_{H^2_k} \leq C\, kR\, \|f\|_{L^2}$

Warm up: Q.O. when $a(\cdot,\cdot)$ is cts and coercive

$$|a(v,v)| \geq C_{coer} \|v\|^2_{H^1_k} \quad \forall v$$

Galerkin o/s:

$$a(u,v_N) = F(v_N) \quad \forall v_N \in \cancel{H^1} \mathcal{H}_N$$

$$a(u_N,v_N) = F(v_N) \quad \forall v_N \in \mathcal{H}_N$$

subtract $a(u-u_N, v_N) = 0 \quad \forall v_N \in \mathcal{H}_N$

need

$$\|u-u_N\|_{L^2} \leq \frac{1}{2}\|u-u_N\|_{H^1_k}$$

u_N exists by Lax-Milgram

$$\cancel{C} \|u-u_N\|^2_{H^1_k} \leq |a(u-u_N, u-u_N)| + 2\|u-u_N\|^2_{L^2}$$

$$= |a(u-u_N, u-v_N)| + 2\|u-u_N\|^2_{L^2} \quad \forall v_N \in \mathcal{H}_N \quad (\text{since } u_N - v_N \in \mathcal{H}_N)$$

$$\leq C_{cont} \|u-u_N\|_{H^1_k} \|u-v_N\|_{H^1_k} + 2\|u-u_N\|^2_{L^2}$$

$$\Rightarrow \boxed{\|u-u_N\|_{H^1_k} \leq \frac{C_{cont}}{\cancel{C}} \|u-v_N\|_{H^1_k} \quad \forall v_N \in \mathcal{H}_N}$$

[Céa's lemma]

How to get suff. condition for $\|u-u_N\|_{L^2} \le \frac{1}{2}\|u-u_N\|_{H^1_k}$ ("Aubin-Nitsche trick" "Schatz argument")

Def$^{\underline{n}}$ given $f \in L^2$ let $\mathcal{S}^* f \in H^1$ be sol$^{\underline{n}}$ of $a(v, \mathcal{S}^* f) = (v, f)_{L^2}$ $\forall v \in H^1$

Lemma $a(\overline{\mathcal{S}^* f}, v) = (\overline{f}, v)_{L^2}$ $\forall v \in H^1$

Lemma if $\eta(\mathcal{H}_N) := \sup\limits_{f \in L^2} \min\limits_{v_N \in \mathcal{H}_N} \dfrac{\|\mathcal{S}^* f - v_N\|_{H^1_k}}{\|f\|_{L^2}} \le \dfrac{1}{2C_{cont}}$

then $\|u-u_N\|_{L^2} \le \frac{1}{2}\|u-u_N\|_{H^1_k}$ and $\|u-u_N\|_{H^1_k} \le 2C_{cont} \min\limits_{v_N \in \mathcal{H}_N} \|u-v_N\|_{H^1_k}$

how well sol$^{\underline{n}}$s of |adjoint eq$^{\underline{n}}$| |Helmholtz eq$^{\underline{n}}$| are approximated in $\mathcal{H}_N$

$$\|u-u_N\|_{L^2}^2 = a(u-u_N, \mathcal{S}^*(u-u_N)) \quad \text{by def}^{\underline{n}} \text{ of } \mathcal{S}^*$$

$$\left(a(\underbrace{v}_{u-u_N}, \mathcal{S}^* \underbrace{f}_{u-u_N}) = (\underbrace{v}_{u-u_N}, \underbrace{f}_{u-u_N}) \ \forall v \in H^1 \right)$$

$$= a(u-u_N, \mathcal{S}^*(u-u_N) - v_N) \quad \forall v_N \in \mathcal{H}_N$$

by Galerkin o/g

$$\leq C_{cont} \|u-u_N\|_{H^1_k} \underbrace{\|\mathcal{S}^*(u-u_N) - v_N\|_{H^1_k}}$$

by def$^{\underline{n}}$ of $\eta(\mathcal{H}_N)$ $\exists\, v_N \in \mathcal{H}_N$ st

$$\leq \eta(\mathcal{H}_N) \|u-u_N\|_{L^2}$$

$$\boxed{\eta(\mathcal{H}_N) := \sup_{f \in L^2} \min_{v_N \in \mathcal{H}_N} \frac{\|\mathcal{S}^* f - v_N\|_{H^1_k}}{\|f\|_{L^2}}}$$

$$\Rightarrow \|u-u_N\|_{L^2} \leq C_{cont}\, \eta(\mathcal{H}_N) \|u-u_N\|_{H^1_k}$$

$$\therefore \text{need } \eta(\mathcal{H}_N) \leq \frac{1}{2C_{cont}}$$

Piecewise polynomial approx. theory

$$\|v - I_h v\|_{H^m(D)} \le C\, h^{s-m}\, |v|_{H^s(D)} \quad \text{if } p \ge s-1$$

interpolation operator $I_h v \in \mathcal{V}_N$

e.g. $m=0$ $\quad p = s-1$

Taylor series $(s-1)$ terms $\quad$ remainder $\underline{h^s (\partial^s v)}$

given $v \in H^s$

$$\min_{w_N \in \mathcal{V}_N} \|v - w_N\|_{H^1_k} \le C_{approx} \left(\frac{hk}{p}\right)^{s-1} \left(1 + \frac{hk}{p}\right) \underbrace{k^{-1} |v|_{H^s}}_{|v|_{H^s_k}} \quad \text{if } p \ge s-1$$

C_{approx} depends on s

use $p=1,\ s=2$

bound on $\eta(\mathcal{V}_N)$

$$\eta(\mathcal{V}_N) := \sup_{f \in L^2} \frac{1}{\|f\|_{L^2}} \min_{v_N \in \mathcal{V}_N} \|\mathcal{S}^* f - v_N\|_{H^1_k} \le \sup_{f \in L^2} \frac{1}{\|f\|_{L^2}} C_{approx}\, hk\, (1 + hk)\, \underbrace{k^{-2} |\mathcal{S}^* f|_{H^2}}_{\le C\, kR\, \|f\|_{L^2}}$$

$$\le \underbrace{C_{approx}\, C\, hk \cdot kR\, (1 + hk)}$$ "hk^2 suff. small"

Theorem (Melenk + Sauter 2010) choose $k_0 > 0$

$$u\big|_{B_R} = u_{H^2} + u_{\mathcal{A}}$$

where $\|u_{H^2}\|_{H^2_k(B_R)} \leq C_1 \|f\|_{L^2}$ $\forall k \geq k_0$

one power of kR better

and $\|(k^{-1}\partial)^\alpha u_{\mathcal{A}}\|_{L^2(B_R)} \leq C_2\, kR\, (C_3)^{|\alpha|} \|f\|_{L^2(B_R)}$ $\forall k \geq k_0$, $\forall \alpha$

same kR dependence

(recall $\|u\|_{H^2_k(B_R)} \leq CkR\, \|f\|_{L^2(B_R)}$)

Use splitting to bound $\eta(\mathcal{M}_N)$

$$\eta(\mathcal{M}_N) \leq \sup_f \left(\min_{v_N^{(1)} \in \mathcal{M}_N} \frac{\|u_{H^2} - v_N^{(1)}\|_{H^1_k}}{\|f\|_{L^2}} + \min_{v_N^{(2)} \in \mathcal{M}_N} \frac{\|u_{\mathcal{A}} - v_N^{(2)}\|_{H^1_k}}{\|f\|_{L^2}} \right)$$

$$\leq C \frac{hk}{p}\left(1 + \frac{hk}{p}\right)$$

can show $\leq CkR\left(\frac{hk}{\sigma p}\right)^p$

suff small if $\frac{hk}{p} \leq C_1$, $p \geq C_2 \log k$