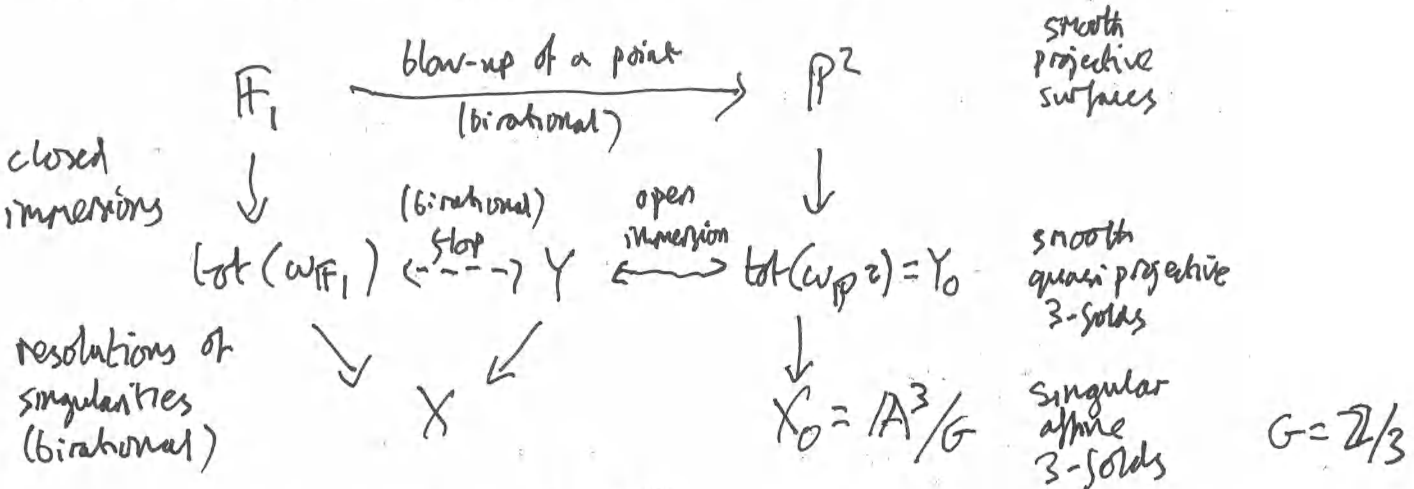
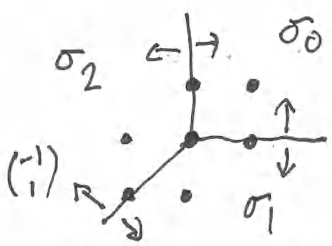
Ali Craw

A large commutative diagram of examples in toric geometry

(aka several species of small furry varieties gathered in a cave & grooving with a Pict – apologies to Pink Floyd)

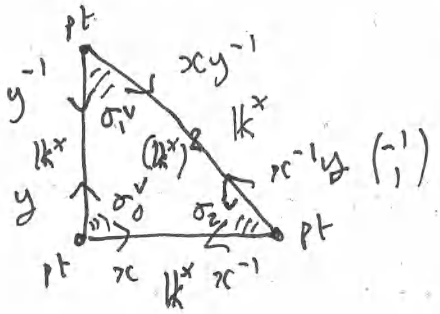


<u>Last time</u> Construction of $\mathbb{P}^2$ as a toric variety from a fan Σ:



$N_{\mathbb{R}}$ for a lattice N (gluing together semigroup algebras of dual cones)

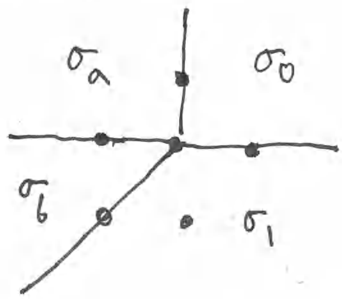
Useful representation of (translates of) dual cones



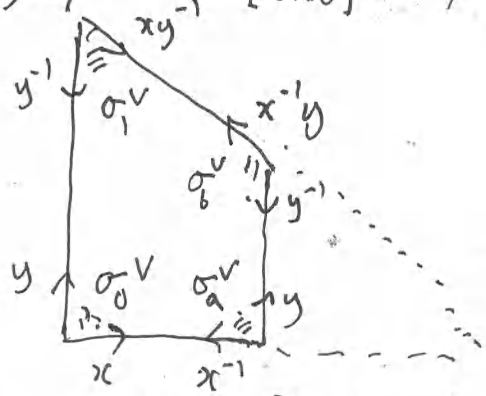
This reflects the decomposition of $\mathbb{P}^2$ into torus orbits: 3 points, 3 $\mathbb{k}^\times$'s & $(\mathbb{k}^\times)^2$.

Here $\mathbb{k} = \mathbb{C}$ or any alg. closed field.

Blow up of $\mathbb{P}^2$ at a point is $\mathbb{F}_1 = Bl_{[0:1:0]}(\mathbb{P}^2)$



$\rightsquigarrow$



The charts are $U_{\sigma_0} = \mathrm{Spec}\, \mathbb{k}[x,y]$, $U_{\sigma_1} = \mathrm{Spec}\, \mathbb{k}[xy^{-1}, y^{-1}]$,
$U_{\sigma_a} = \mathrm{Spec}\, \mathbb{k}[x^{-1}, y]$, $U_{\sigma_b} = \mathrm{Spec}\, \mathbb{k}[x^{-1}y, y^{-1}]$

$\mathbb{F}_1 = Bl_{[0:1:0]}(\mathbb{P}^2) = \{ ([z_0 : z_1 : z_2], [s,t]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid s z_0 = t z_2 \}$

This is the top row: now turn to the bottom row.

<u>Singularities & the role of the lattice</u>: $\mathbb{A}^3/G$

$$\mathbb{A}^3 = \operatorname{Spec} \mathbb{k}[\sigma^\vee \cap \bar{M}] = \operatorname{Spec} \mathbb{k}[x,y,z]$$

where $\bar{M} = \mathbb{Z}^3$, $\sigma \subseteq \bar{N} \otimes_{\mathbb{Z}} \mathbb{R}$ is $\left\{\sum_{i=1}^{3} \lambda_i e_i \mid \lambda_i \geq 0\right\}$

& $\bar{N} = \operatorname{Hom}(\bar{M}, \mathbb{Z})$.

Introduce an overlattice (i.e. with $\bar{N}$ as a sublattice)

$$N = \bar{N} + \mathbb{Z}\tfrac{1}{3}(1,1,1)$$

with dual
$$M = \{m \in \bar{M} \mid \langle m,n\rangle \in \mathbb{Z} \ \forall n \in N\}$$
$$= \{(m_1,m_2,m_3) \in \mathbb{Z}^3 \mid m_1+m_2+m_3 \in 3\mathbb{Z}\}$$

$$\therefore \ \mathbb{k}[\sigma^\vee \cap M] = \left\{\sum c_m x^{m_1} y^{m_2} z^{m_3} \mid m_1+m_2+m_3 \in 3\mathbb{Z},\ m_i \geq 0\right\}.$$

Compare $G = \left\langle \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \mid \omega = e^{2\pi i/3} \right\rangle \cong \mathbb{Z}/3$

which satisfies

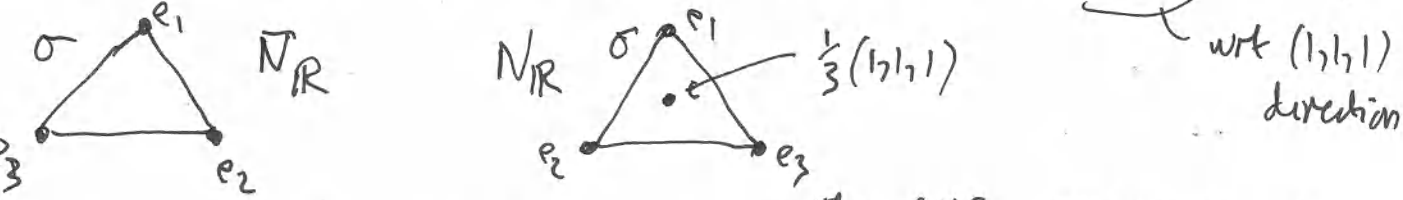
$$\mathbb{k}[x,y,z]^G := \{f \in \mathbb{k}[x,y,z] \mid g\cdot f = f\} \cong \mathbb{k}[\sigma^\vee \cap M]$$

Thus lattice M is exponents of Laurent monomials invariant under G.

Conclude: $U_\sigma := \operatorname{Spec} \mathbb{k}[\sigma^\vee \cap M] \cong \operatorname{Spec} \mathbb{k}[x,y,z]^G =: \mathbb{A}^3/G$.

(Here $\mathbb{A}^3/G$ is notation but one can prove its points do correspond to G-orbits)

Fan is a 3-dimensional cone with height 1 slice wrt $(1,1,1)$ direction



Note that in $N_{\mathbb{R}}$ (unlike $\bar{N}_{\mathbb{R}}$) the cone generators of σ ~~don't provide~~ don't provide a $\mathbb{Z}$-basis of N (unlike $\bar{N}$)

Lemma U_σ is nonsingular (smooth) $\iff$ cone generators of σ can be extended to a basis of N

In this case $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ for $\dim \sigma = k$

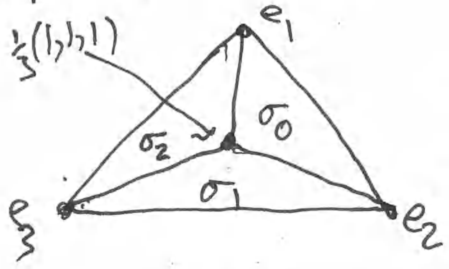
This can be seen by looking at the Zariski cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ to a point, which cannot have dimension $> n$ if U_σ ($\dim U_\sigma = n$) is nonsingular.

We have $[\chi^u] \in \mathfrak{m}_p/\mathfrak{m}_p^2$ for u primitive vector on ray generator of $\sigma^\vee$.

<u>Resolution of singularities</u> : $\mathrm{tot}(\omega_{\mathbb{P}^2})$

Given singular U_σ, the lemma suggests subdividing σ so that each cone in the resulting <u>fan</u> (NB) satisfying the condition of the lemma.

In this case



U_{σ_i} nonsingular for $i = 0,1,2$
e.g. $e_1, e_2, \frac{1}{3}(1,1,1)$ generate N.

$$\sigma_0 = \mathbb{R}_{\geq 0}\langle e_1, e_2, \tfrac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\rangle, \quad \sigma_0^\vee = \mathbb{R}_{\geq 0}\langle xz^{-1}, yz^{-1}, z^3\rangle$$

$$\mathbb{k}[\sigma_0^\vee \cap M] = \mathbb{k}[xz^{-1}, yz^{-1}, z^3] = \mathbb{k}[xz^{-1}, yz^{-1}] \otimes_{\mathbb{k}} \mathbb{k}[z^3]$$

$$\therefore U_{\sigma_0} = \operatorname{Spec} \mathbb{k}\left[\tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}\right] \times \operatorname{Spec} \mathbb{k}[z_0^3]$$

<u>Claim</u> $X_\Sigma \cong \mathrm{tot}(\omega_{\mathbb{P}^2})$ covered by 3 charts $\operatorname{Spec}\left(\bigoplus_{j\geq 0} \Gamma(U_i, \omega_{\mathbb{P}^2}^{-j})\right)$

& the canonical bundle $\omega_{\mathbb{P}^2}$ over $\mathbb{P}^2$ is $\mathcal{O}_{\mathbb{P}^2}(-3)$ so $\omega_{\mathbb{P}^2}^{-j} = \mathcal{O}_{\mathbb{P}^2}(3j)$

e.g. $U_0 \subseteq \mathbb{P}^2$: $\operatorname{Spec}\left(\Gamma(U_0, \omega_{\mathbb{P}^2}^{-j})\right) = \operatorname{Spec} \mathbb{k}\left[\tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}\right] \times \operatorname{Spec} \mathbb{k}[z_0^3]$
(U_0: $z_0 \neq 0$)

<u>NB</u> We can contract the zero section in $\mathrm{tot}\,\omega_{\mathbb{P}^2}$ to

$$\operatorname{Spec}\left(\bigoplus_{j\geq 0} \Gamma(\mathbb{P}^2, \omega_{\mathbb{P}^2}^{-j})\right) = \operatorname{Spec} \mathbb{k}[x,y,z]^G \quad \text{with } G = \mathbb{Z}/3 \text{ as before.}$$

Thus $\mathbb{P}^2 \to \mathrm{tot}\,\omega_{\mathbb{P}^2} \to \mathbb{A}^3/G$ is "exact"

[Aside: can generalize to $\bar{N} = \mathbb{Z}^n$, $N = \bar{N} + \mathbb{Z}\frac{1}{r}(a_1, \dots, a_n)$

$G = \left\langle \begin{pmatrix} \varepsilon^{a_1} & & 0 \\ & \ddots & \\ 0 & & \varepsilon^{a_n} \end{pmatrix} \,\middle|\, \varepsilon = e^{2\pi i/r} \right\rangle$ (or $\varepsilon^r = 1$ primitive)

$\cong \mathbb{Z}/r$

so $U_\sigma = \operatorname{Spec} \mathbb{k}[\sigma^\vee \cap M] \cong \operatorname{Spec} \mathbb{k}[x_1, \dots, x_n]^G = \mathbb{A}^n/G$.]

↑ Works for quotients by abelian groups (so toric varieties are "simple" as varieties in similar sense to abelian groups as groups)

Closed toric strata: V

Study closures of toric orbits

Defn Given a fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ & $\tau \in \Sigma$ we have:

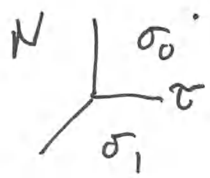
- a lattice $N(\tau) = N / \operatorname{Span}(\tau) \cap N$
- fan $\operatorname{Star}(\tau) = \{\text{image } \bar{\sigma} \subseteq N(\tau) \otimes \mathbb{R} \mid \tau \leq \sigma \in \Sigma\}$

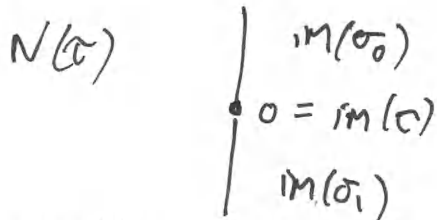
Then $V(\tau) = X_{\operatorname{Star}(\tau)}$ is a toric variety s.t.

(i) $V(\tau) = \overline{O_\tau}$ (ii) $V(\tau) = \bigcup_{\tau \leq \sigma} O_\sigma$

cf. $U_\sigma = \bigcup_{\tau \leq \sigma} O_\tau$

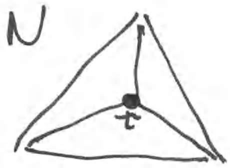
Examples ① $X_\Sigma = \mathbb{P}^2$





so $V(\tau) = \mathbb{P}^1$.

② For $X_\Sigma = \operatorname{tot}(\omega_{\mathbb{P}^2})$



Fan of $\mathbb{P}^2$ in an isomorphic lattice (A nicer one!)

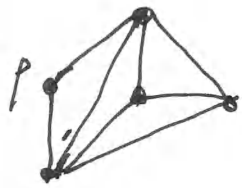
~~(Write ... for this ...)~~

This is the inclusion of $\mathbb{P}^2$ in $\operatorname{tot}(\omega_{\mathbb{P}^2})$

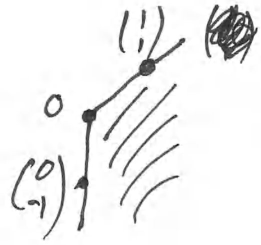
[Toric morphisms: Observe in this case each cone in the fan of $\mathrm{tot}(\omega_{\mathbb{P}^2})$ maps into a cone of the fan of $\mathbb{P}^2$.]

(3) Define a new fan Σ' from $\mathrm{tot}(\omega_{\mathbb{P}^2})$

with height 1 slice



$X_{\mathrm{Star}(p)} = V_p(p) = \mathbb{A}^2$ with Fan



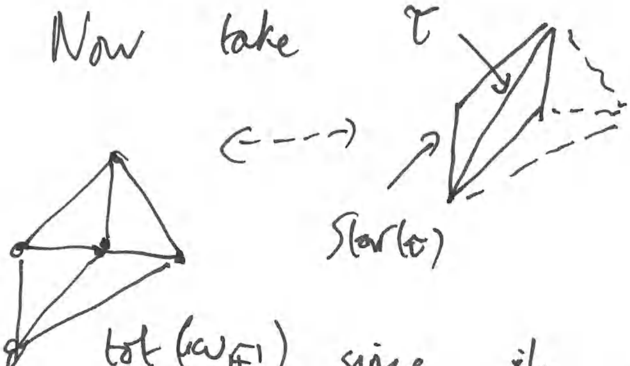
which is isomorphic to

Thus $X_{\Sigma'} = Y = \mathbb{A}^2 \sqcup \mathrm{tot}(\omega_{\mathbb{P}^2}) = \mathbb{A}^3 \cup \mathrm{tot}(\omega_{\mathbb{P}^2})/\sim$

↑ closed ↑ open

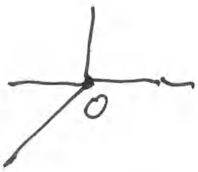
Gluing over $\mathbb{A}^3 \setminus \mathbb{A}^2 \cong \mathbb{C}^* \times \mathbb{C}^2$

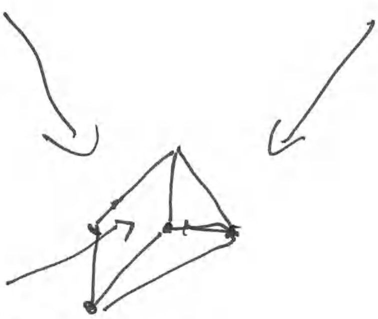
Now take



So $V(\tau) = \mathbb{P}^1$, which has a flop

$\mathrm{tot}(\omega_{\mathbb{P}^1})$ since it projects to





The classical Atiyah flop (only flop of his career)

Thus we have 2 resolutions of singularities here.

Toric geometry provide useful examples in the study of birational geometry.