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A commutative diagram of examples in toric geometry.

Goal: to study the commutative diagram

$$
\begin{tikzcd}
\mathbb{F}_1 \arrow[rr, "\text{blow-up}"] \arrow[d, hook] & & \mathbb{P}^2 \arrow[d] \\
Y' = \mathrm{tot}(\omega_{\mathbb{F}_1}) \arrow[r, dashed, leftrightarrow, "\text{flop}"] \arrow[dr] & Y \arrow[d] & \mathrm{tot}(\omega_{\mathbb{P}^2}) = Y_0 \arrow[l, hook', "\text{open}"', "\text{immersion}"] \arrow[d] \\
& X & X_0 = \mathbb{A}^3/G
\end{tikzcd}
$$

closed immersions

(birational) projective resolutions

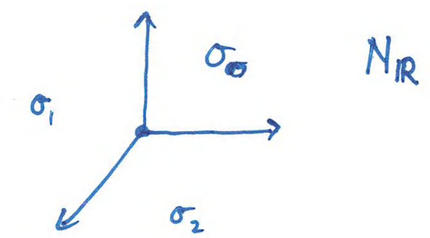
smooth proj surfaces

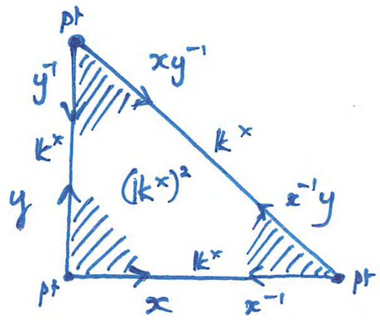
smooth, quasi-proj. 3-folds

singular affine 3-folds

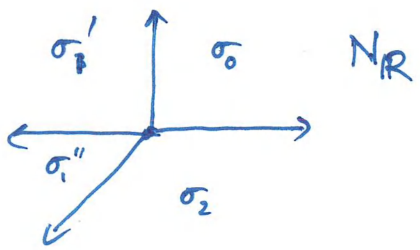
Last time

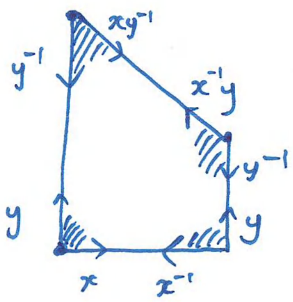
Fran constructed $\mathbb{P}^2$ from fan:





orbit decomp





and these four charts

Spec $\mathbb{k}[x,y]$, Spec $\mathbb{k}[x^{-1},y]$, Spec $\mathbb{k}[x^{-1}y, y^{-1}]$, Spec $\mathbb{k}[y^{-1}, xy^{-1}]$

glue to give

$$\mathbb{F}_1 = \mathrm{Bl}_{[0:1:0]}\mathbb{P}^2 = \left\{ ([z_0 : z_1 : z_2], [s:t]) \in \mathbb{P}^2 \times \mathbb{P}^1 \;\middle|\; z_0 t = z_2 s \right\}.$$

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Singularities and the role of the lattice: $\mathbb{A}^3/G$.

Write $\mathbb{A}^3 = \operatorname{Spec} \Bbbk[x,y,z]$
$\cong \operatorname{Spec} \Bbbk[\sigma^\vee \cap \overline{M}]$ for $\sigma \subset (\overline{N})_{\mathbb{R}}$ positive octant.
and $\overline{M} = \overline{N}^\vee$.

The overlattice
$$N := \overline{N} + \mathbb{Z}\cdot\tfrac{1}{3}(1,1,1)$$
determines
$$\begin{aligned} M &:= \{ m \in \overline{M} \mid \langle m, n\rangle \in \mathbb{Z} \ \ \forall n \in N\} \\ &= \{ (m_1,m_2,m_3) \in \overline{M} \mid m_1+m_2+m_3 \in 3\mathbb{Z}\} \end{aligned}$$

$$\therefore \ \Bbbk[\sigma^\vee \cap M] = \left\{ \sum c_{\underline{m}}\, x^{m_1} y^{m_2} z^{m_3} \in \Bbbk[x,y,z] \ \middle|\ m_1+m_2+m_3 \in 3\mathbb{Z} \right\}$$

For $G = \mathbb{Z}/3 = \left\langle \omega\begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix} \ \middle|\ \omega = e^{2\pi i/3} \right\rangle$
then $\Bbbk[\sigma^\vee \cap M] \cong \Bbbk[x,y,z]^G$ so

$$U_\sigma := \operatorname{Spec} \Bbbk[\sigma^\vee \cap M] = \operatorname{Spec} \Bbbk[x,y,z]^G = \mathbb{A}^3/G.$$

N.B: primitive cone gens of σ don't form basis of N, ie
$\tfrac{1}{3}(1,1,1) \notin \operatorname{Span}(e_1,e_2,e_3) = \overline{N}$.

Lemma U_σ non-singular $\iff$ primitive cone gens can be extended to a $\mathbb{Z}$-basis of lattice N.

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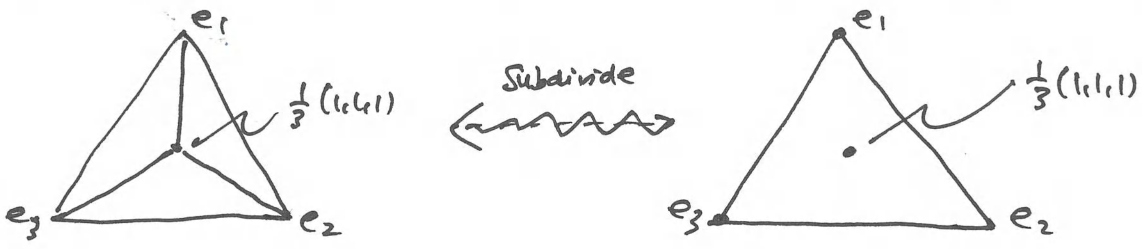
Resolution of singularities $\mathrm{tot}(\omega_{\mathbb{P}^2})$

Given a singular toric var U_σ, subdivide the cone ~~by~~ to produce a fan s.t. ~~each~~ primitive cone gens in each cone of the fan can be completed to a $\mathbb{Z}$-basis of N.

$U_\sigma = \mathbb{A}^3/G$: cross-section of σ and of a refinement Σ:



$$Y_0 := X_\Sigma \longrightarrow U_\sigma = \mathbb{A}^3/G$$

Claim:
In this example, $Y_0 = \mathrm{tot}(\omega_{\mathbb{P}^2}) = \underline{\mathrm{Spec}} \bigoplus_{j\geq 0} \omega_{\mathbb{P}^2}^{-j}$

(obtained by gluing $\left\{ \mathrm{Spec} \bigoplus_{j\geq 0} \Gamma(U_i, \omega_{\mathbb{P}^2}^{-j}) \right\}_{i=0,1,2}$ over $\mathbb{P}^2$).

On $U_0 = \mathrm{Spec}\, \mathbb{k}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}\right] \subseteq \mathbb{P}^2$

$$\Gamma(U_0, \mathcal{O}_{\mathbb{P}^2}(3)) = \langle z_0^3 \rangle \quad \text{as} \quad \mathcal{O}_{\mathbb{P}^2}(U_0) = \mathbb{C}\left[\tfrac{z_1}{z_0}, \tfrac{z_2}{z_0}\right]\text{-mod}$$

(similarly for $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^2}(3)) = \langle z_i^3 \rangle$) and

$$\mathrm{tot}(\omega_{\mathbb{P}^2}) \supset U_i \times \mathbb{A}^1 = \mathrm{Spec}\, \mathcal{O}_{U_i}[z_i^3]$$

with gluing $\quad z_0^3 \cdot \left(\frac{z_1}{z_0}\right)^3 = z_1^3$, ie

$\mathrm{tot}(\omega_{\mathbb{P}^2})$ covered by $\quad \mathrm{Spec}\, \mathbb{k}\left[\frac{z_1}{z_0}, \frac{z_2}{z_0}, z_0^3\right] + \text{sim};$

Compare:

$$\sigma_0 = \mathbb{R}_{\geq 0}\left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \tfrac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} \right\rangle \quad \sigma_0^\vee = \mathbb{R}_{\geq 0}(xz^{-1}, yz^{-1}, z^3) \text{ so}$$

$$U_{\sigma_0} = \mathrm{Spec}\, \mathbb{k}\left[\tfrac{x}{z}, \tfrac{y}{z}, z^3\right].$$

Rmk: $Y_0 = X_\Sigma$ is a resolution of $\mathbb{A}^3/G$; alternatively, can contract the zero-section of $\mathrm{tot}(\omega_{\mathbb{P}^2})$ to

$$\mathrm{Spec} \bigoplus_{j\geq 0} \Gamma(\omega_{\mathbb{P}^2}^{-j}) = \mathrm{Spec} \bigoplus_{j\geq 0} \mathbb{k}[x,y,z]_{3j} \cong \mathbb{A}^3/G.$$

Closed toric strata : Y

Study closures of torus orbits in toric variety X_Σ.

Def: Fan $\Sigma \subseteq N\otimes_{\mathbb{Z}} \mathbb{R}$ and $\tau \in \Sigma$, define
- lattice $N(\tau) = N/(\mathrm{Span}(\tau)\cap N)$
- fan $\mathrm{Star}(\tau) = \{\mathrm{image}(\sigma) \subset N(\tau)\otimes_{\mathbb{Z}}\mathbb{R} \mid \tau < \sigma \in \Sigma\}$

Then

$$\boxed{V(\tau) := X_{\mathrm{Star}(\tau)}}$$ is toric var s.t.

(i) $V(\tau) = \overline{O(\tau)}$ so $\dim V(\tau) = n - \dim(\tau)$

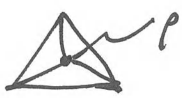
(ii) $V(\tau) = \bigcup_{\sigma \geq \tau} O(\sigma)$

Special case : $D_\rho := V(\rho)$ for toric divisor (codim 1) in X_Σ defined by the ray $\rho \in \Sigma(1) = \{\rho \in \Sigma \mid \dim \rho = 1\}$

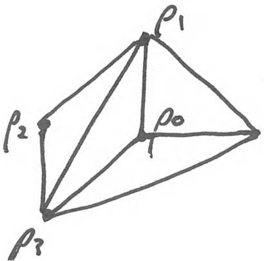
Examples

① $X_\Sigma = \mathbb{P}^2$ 

$\mathrm{Star}(\tau)$  $D_\rho \cong \mathbb{P}^1$

② $\mathrm{tot}(\omega_{\mathbb{P}^2})$ 

$\mathrm{Star}(\rho)$  $D_\rho \cong \mathbb{P}^2$

③ Define Σ' 

$D_{\rho_2} \cong \mathbb{A}^2$

$X_{\Sigma'} = D_{\rho_2} \sqcup \mathrm{tot}(\omega_{\mathbb{P}^2}) =: Y.$

Cox GIT construction of toric vars : Y and Y'

Assume now that X_Σ has no $\mathbb{k}^\times$-factors, i.e. fan Σ not contained in a hyperplane of $N \otimes_{\mathbb{Z}} \mathbb{R}$. Set

$$\mathbb{Z}^{\Sigma(1)} := \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho$$

– the lattice of torus-invariant (Weil) divisors on X_Σ. If $v_\rho \in N$ is primitive ~~vector~~ lattice point on ray $\rho \in \Sigma(1)$, the $\mathbb{Z}$-linear map

$$\mathrm{div} : M \longrightarrow \mathbb{Z}^{\Sigma(1)} : m \longmapsto \sum_{\rho \in \Sigma(1)} \langle m, v_\rho \rangle D_\rho$$

is injective (as our assumption on X_Σ implies that the dual map to div is surjective), and cokernel (IF X_Σ smooth as is the case for us) is

$$\mathrm{Pic}(X_\Sigma) := \{ \mathcal{O}_{X_\Sigma}(D) \mid D \text{ is Cartier divisor} \} /_{\sim \text{isom}}$$

where for $\mathcal{U} \subset X_\Sigma$ open,

$$\Gamma(\mathcal{U}, \mathcal{O}_{X_\Sigma}(D)) = \{ f \in \mathbb{k}(X_\Sigma)^* \mid (\mathrm{div}(f) + D)|_{\mathcal{U}} \geq 0 \} \cup \{0\}$$

Hence $\exists$ short exact sequence

$$0 \longrightarrow M \xrightarrow{\mathrm{div}} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Pic}(X_\Sigma) \longrightarrow 0$$
$$D \longmapsto \mathcal{O}_X(D).$$

<u>Rmk</u>

If X_Σ not smooth, replace $\mathrm{Pic}(X_\Sigma)$ by the Class group of rank one reflexive sheaves

$$\mathrm{Cl}(X_\Sigma) := \{ \mathcal{O}_{X_\Sigma}(D) \mid D \text{ is Weil divisor} \} /_{\sim \text{isom.}}$$

Th'm

For $L \in \mathrm{Pic}(X_\Sigma)$ [true also for $\mathrm{Cl}(X_\Sigma)$]

$$\Gamma(X_\Sigma, L) \cong \bigoplus_{D \in \mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L)} \mathbb{C}\cdot \cancel{\cong} D \qquad \mathbb{C} = \mathbb{k}!$$

$$\cong \bigoplus_{\substack{D \text{ effective} \\ \mathcal{O}_{X_\Sigma}(D) = L}} \mathbb{C}\cdot D. \qquad \left(\begin{array}{c}\text{integral points of polytopal} \\ \text{slice } \mathbb{N}^{\Sigma(1)} \cap \deg^{-1}(L).\end{array}\right)$$

Examples

(i) $\mathbb{P}^2$

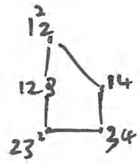
$$0 \longrightarrow \underset{\mathbb{Z}^2}{\overset{M}{\wr\|}} \xrightarrow{\begin{pmatrix} +1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{(1,1,1)} \underset{\mathbb{Z}}{\overset{\mathrm{Pic}(\mathbb{P}^2)}{\wr\|}} \to 0$$

and $\mathbb{k}[z_0, z_1, z_2]_d \cong \Gamma(\mathcal{O}_{\mathbb{P}^2}(d))$ spanned by monomials of deg d.

(ii) $\mathbb{F}_1$

$$0 \longrightarrow \underset{\mathbb{Z}^2}{\overset{M}{\wr\|}} \xrightarrow{\begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}} \underset{\mathbb{Z}^2}{\overset{\mathrm{Pic}(\mathbb{F}_1)}{\wr\|}} \longrightarrow 0$$

and $\Gamma(\mathcal{O}_{\mathbb{F}_1}\binom{1}{1}) \cong \mathbb{k}^5$ has basis $D_1 + D_4$, $D_3 + D_4$, $2D_1 + D_2$, $D_1 + D_2 + D_3$, $D_2 + 2D_3$.



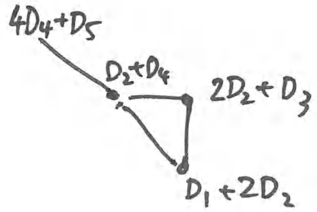
Which are the integral points of the picture on p1 of these notes.

(iii) Y and Y':

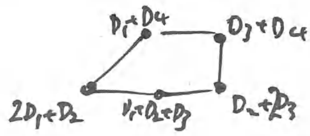
$$0 \longrightarrow \underset{\mathbb{Z}^3}{\overset{M}{\wr\|}} \xrightarrow{\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

and

- $\Gamma(\mathcal{O}_Y\binom{-1}{2})$ has module gens



- $\Gamma(\mathcal{O}_Y\binom{1}{1})$



- $\Gamma(\mathcal{O}_Y\binom{0}{0})$ = coord ring of $X = \mathbb{k}[z_3 z_4^2 z_5,\ z_1 z_4^2 z_5,\ z_1^2 z_2^2 z_5,\ z_2^2 z_3^3 z_5]$.