Toric geometry for beginners II

Recap: $N \leq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

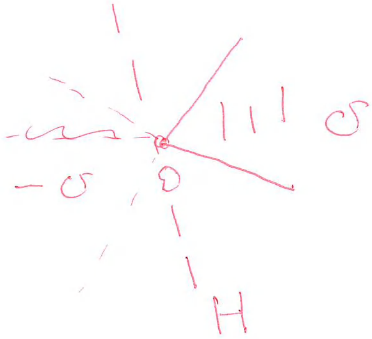
$\uparrow$ lattice

rational cone: $\sigma = \mathrm{Cone}(u_1, \dots, u_n) \quad u_i \in N$

strongly convex: $\sigma \cap -\sigma = \{0\}$

$\Updownarrow$

0 vertex



$M := \mathrm{Hom}(N, \mathbb{Z}) \subseteq M_{\mathbb{R}} := N_{\mathbb{R}}^*$ dual lattice

$T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^{\mathrm{rk}\, N}$ torus $\quad \mathbb{C}[T_N] = \mathrm{span}\{\chi^m \mid m \in M\}$

$M \cong$ char. lattice of T_N

$m \longmapsto \chi^m$

$N \cong$ 1PS lattice of T_N

$u \longmapsto \lambda^u$

$$\boxed{\chi^m(\lambda^u(z)) = z^{\langle m, u \rangle}}$$

$\sigma \left\{ \begin{array}{l} \text{str. convex} \\ \text{rational cone} \end{array} \right. \rightsquigarrow \check{\sigma} = \{ f \in M_{\mathbb{R}} \mid f(u) \geq 0 \ \forall u \in \sigma \}$

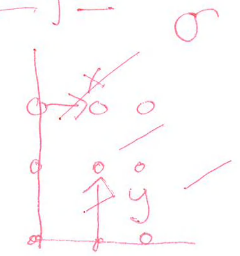
$S_\sigma := \check{\sigma} \cap M$ monoid

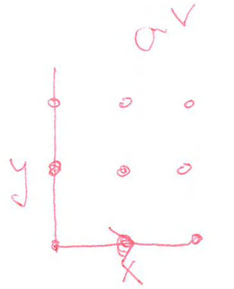
$U_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma])$ $\left\{ \begin{array}{l} \text{normal affine toric variety with } T_N \text{ dense} \\ + \text{all arise this way.} \end{array} \right.$

$\mathbb{C}[S_\sigma] = \mathrm{span}\{\chi^m \mid m \in S_\sigma\}$

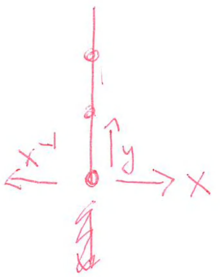
Examples

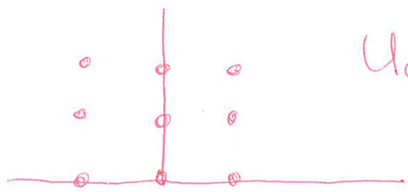
σ $\qquad$ $\sigma^\vee$



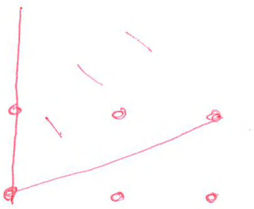


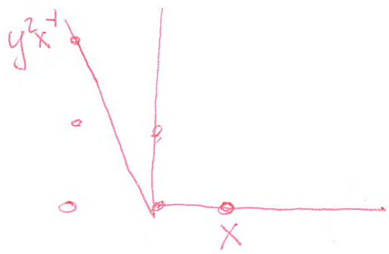
$U_\sigma = \mathrm{Spec}(\mathbb{C}[x,y]) = \mathbb{C}^2$





$U_\sigma = \mathrm{Spec}(\mathbb{C}[y,x,x^{-1}]) = \mathbb{C}^\times \times \mathbb{C}$





cone generators $\neq$ semigp generators ($\Rightarrow$ not smooth)

$U_\sigma \hookrightarrow \mathbb{C}^3$

~~(x, y, ...)~~ (x, y, y^2x^{-1})

$U_\sigma \cong V(xz-y^2)$

Geometry from Fans

A. Open sets ~~& gluing~~

$$\tau < \sigma \Rightarrow \sigma^\vee < \tau^\vee \Rightarrow S_\sigma \subset S_\tau$$

Ex $\quad \tau = \sigma \cap H_m \Rightarrow S_\tau = S_\sigma + \mathbb{Z}(-m)$

$H_m = \ker m$

Then: $\quad U_\tau = (U_\sigma)_{\chi^m} \subset U_\sigma$ $\quad$ (canonical inclusion $U_\tau \hookrightarrow U_\sigma$ helpful later on)

B. Torus orbits

Recall: $\{\text{pts in affine } V\} \Leftrightarrow \{\mathbb{C}\text{-alg homo } \mathbb{C}[V] \to \mathbb{C}\}$

$$p \quad \mapsto \quad f \mapsto f(p).$$

$\tau \leq \sigma \rightsquigarrow O_\tau := \{p \in U_\tau \mid \chi^m(p) \neq 0 \Leftrightarrow m \in \tau^\perp\}$

- T_N-invariant: $\chi^m(t \cdot p) = \underset{\substack{\neq \\ 0}}{\chi^m(t)}\, \chi^m(p)$

- single T_N-orbit: $p, \hat{p} \in O_\tau$

  Define t by $\chi^m(t) = \chi^m(\hat{p})/\chi^m(p) \in \mathbb{C}^\times$ if $m \in \tau^\perp$

  $= 1$ otherwise

  $\rightsquigarrow \mathbb{C}_{alg}\text{-homo } \mathbb{C}[M] \to \mathbb{C} \therefore t \in T_N$

Via limits $u \in \tau \Rightarrow \lim_{z \to 0} \lambda^u(z) q \in U_\tau \quad \forall q \in U_\tau$

indeed $f \chi^m \to \lim_{z \to 0} f \chi^m(\lambda^u(z) \cdot q) = \lim_{z\to 0} z^{\langle m,u \rangle} \chi^m(q)$

$$= \begin{cases} \chi^m(q) & \langle m,u\rangle = 0 \\ 0 & \text{otherwise} \end{cases}$$

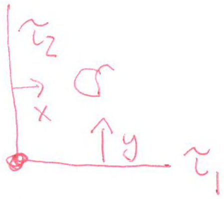
is $\mathbb{C}_{alg}$-homo $\mathbb{C}[U_\tau] \to \mathbb{C}$ ...

called $\lim_{z \to 0} \lambda^u(z) \cdot q$

Ex $u \in \operatorname{Relint} \tau = \{u \in \tau \mid \langle m,u \rangle > 0 \ \forall m \in \check{\tau} \setminus \tau^\perp\}$

& $q \in T_N$ then $\lim_{z \to 0} \lambda^u(z) q \in O_\tau$

Example



$\cancel{O_\sigma = \{0\}}$ $U_\sigma = \mathbb{C}^2$

$O_\sigma = \{0\}$

$O_{\tau_1} = \{y \neq 0, x = 0\} = \{0\} \times \mathbb{C}^\times$

$O_{\tau_2} = \mathbb{C}^\times \times \{0\}$

$O_0 = T_N$

N.B. Recover σ from orbit structure:

Thm • $U_\sigma = \bigcup_{\tau \leq \sigma} O_\tau$ • $\dim O_\tau = \operatorname{codim} \tau$

• $\sigma < \tau \iff O_\tau \subseteq \overline{O_\sigma}$

Remark: U_τ is smallest ~~affine~~ that contains O_τ

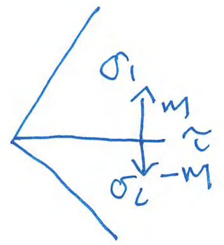
Fans + gluing

Def^n A fan in $N_{\mathbb{R}}$ is collection Σ of str. convex rational cones

- closed under faces
- $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 < \sigma_i \quad i = 1, 2.$

Support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$

$\sigma_1, \sigma_2 \in \Sigma \quad \tau = \sigma_1 \cap \sigma_2$



$$\begin{tikzcd} U_\tau \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\wr"] & (U_{\sigma_1})_{\chi^m} \arrow[dl, "\sim"', "g_{\sigma_2,\sigma_1}"] \\ (U_{\sigma_2})_{\chi^{-m}} & \end{tikzcd}$$

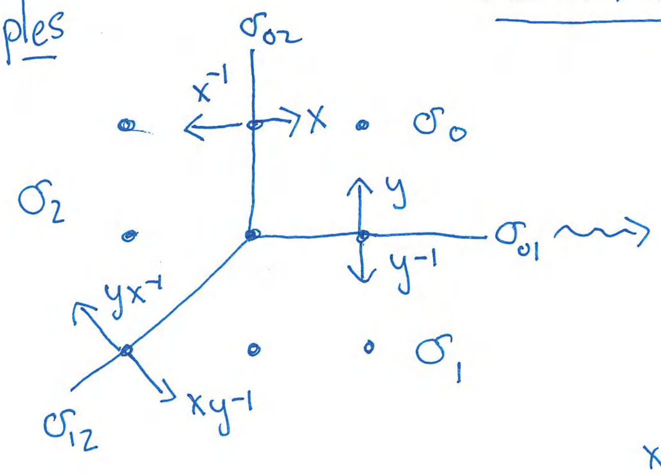
<u>FACT</u> $g_{\sigma_i \sigma_j}$ are T_N-equivariant & compatible: $\begin{cases} g_{\sigma_2\sigma_1} = g_{\sigma_1\sigma_2}^{-1} \\ g_{\sigma_3\sigma_1} = g_{\sigma_3\sigma_2}\, g_{\sigma_2\sigma_1} \end{cases}$
where defined $(U_{\sigma_1} \cap U_{\sigma_2} \cap U_{\sigma_3})$

$\therefore$ glue to build

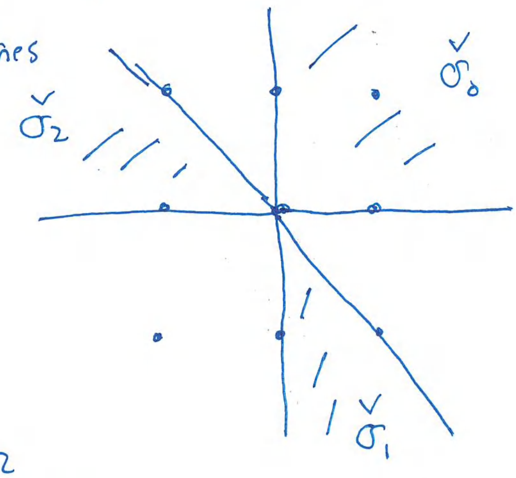
$$X_\Sigma = \coprod_{\sigma \in \Sigma} U_\sigma / \sim \qquad \left[\begin{matrix} x_i \in U_{\sigma_i} \\ x_j \in U_{\sigma_j} \end{matrix} \quad x_i \sim x_j \text{ if } x_j = g_{\sigma_j \sigma_i}(x_i) \ldots \right]$$

<u>normal toric variety</u>

<u>Examples</u>



3 affines



~~$U_{\sigma_0} = \mathrm{Spec}\, \mathbb{C}[x,y] = \mathbb{C}^2$~~ $\quad U_{\sigma_0} = \mathrm{Spec}\, \mathbb{C}[\overset{x}{\overset{\shortparallel}{s_0}}, \overset{y}{t_0}] = \mathbb{C}^2 \supset \mathbb{C} \times \mathbb{C}^\times = \mathrm{Spec}[x, y, y^{-1}] = U_{\sigma_{01}}$

~~$U_{\sigma_1} = \mathrm{Spec}\, \mathbb{C}[$~~ $\quad U_{\sigma_1} = \mathrm{Spec}\, \mathbb{C}[s_1, t_1] = \mathbb{C}^2$, $\quad s_1 = xy^{-1}$, $t_1 = y^{-1}$

i.e. on $U_{\sigma_{01}}$: $s_1 = s_0/t_0$, $t_1 = 1/t_0$

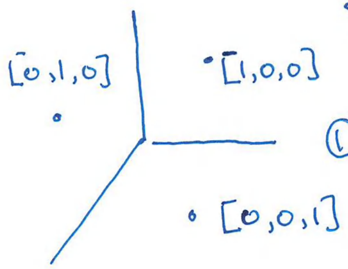
& simly $U_{\sigma_2} = \mathrm{Spec}\, \mathbb{C}[s_2, t_2]$, $\quad s_2 = x^{-1}$, $t_2 = yx^{-1}$ & $s_2 = 1/s_0$, $t_2 = t_0/s_0$

Punchline: 3 affine charts on $\mathbb{P}^2$:

$$[1, s_0, t_0] = [1/s_0, 1, t_0/s_0] = [1/t_0, s_0/t_0, 1]$$

$X_\Sigma \cong \mathbb{P}^2$

Orbit structure:

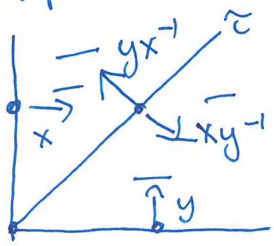


$3\times$ fixed pts

$\mathbb{C}^\times = [*, 0, *]$ $\quad \overline{\mathbb{C}^\times} = \mathbb{P}^1$

$\overline{\mathbb{C}^\times} = \mathbb{C}^\times \cup \{[1,0,0]\} \cup \{[0,0,1]\}$

Example (if time)



$\mathbb{C}^2$ coords s_0, t_0 $\quad \supset U_\tau = \{t_0 \neq 0\}$

$\phantom{\mathbb{C}^2 \text{ coords }} \overset{\shortparallel}{x} \quad \overset{\shortparallel}{yx^{-1}}$

$\mathbb{C}^2$ $\quad s_1, t_1$ $\quad \supset U_\tau = \{t_1 \neq 0\}$

$\phantom{\mathbb{C}^2} \quad y \quad xy^{-1}$

$1/t_0 = t_1$

$s_1 = s_0 t_0$

This is blowup of $\mathbb{C}^2$ at origin:

$$\{([z_0, z_1], (a,b)) \in \mathbb{P}^1 \times \mathbb{C}^2 \quad (a,b) \in [z_0, z_1]\}$$

$$a z_1 = b z_0$$

Charts: $([1, t_0], (s_0, s_0 t_0)) = ([t_1, 1], (s_1 t_1, s_1))$

Polytopes & proj. toric varieties

$A = \{m_1, \dots, m_s\} \subset M$ Two constructions:

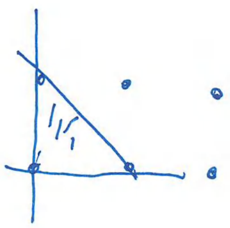
1/ · Recall

$$\begin{tikzcd} T_N \arrow[r, "\Phi_A"] \arrow[dr, "\Psi_A"'] & (\mathbb{C}^\times)^s \arrow[d] \\ & \mathbb{P}(\mathbb{C}^s) \end{tikzcd}$$

$\overline{\operatorname{Im}\Psi_A}$ toric for quotient of T_N

$P := \operatorname{Conv}(A) \subset M_{\mathbb{R}}$ lattice polytope

↑ convex hull



$u \in N_{\mathbb{R}}$ $b \in \mathbb{R}$

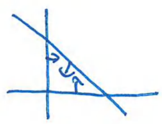
$H^+_{u,b} = \{m \mid \langle m,u\rangle \geq -b\}$

$\supseteq H_{u,b} = \{m \mid \langle m,u\rangle = b\}$

supporting if $P \subset H^+_{u,b}$

Face is $P \cap H^+_{u,b}$ (+ then can take $u \in N_{\neq 0}, b \in \mathbb{Z}$)

0-dim^l face = vertex



~~Assume P full : dim span A = $M_{\mathbb{R}}$~~

Normal fan: v vertex $\rightsquigarrow$ $\operatorname{Cone}((M \cap P) - v)$ $\rightsquigarrow$ σ_v dual cone

$\Sigma_P := \bigcup_v \{\sigma_v \text{ \& faces thereof}\}$ is normal fan

if P full (~~dim~~ span $P = M_{\mathbb{R}}$)

- $|\Sigma_P| = N_{\mathbb{R}}$
- Facets of P $\overset{1:1}{\longleftrightarrow}$ 1-dim subcones (rays) of Σ_P

$\therefore$ full lattice polytope $P \rightsquigarrow \Sigma_P \rightsquigarrow X_{\Sigma_P}$ toric variety.

P not determined by $\Sigma_P = \Sigma$

$\Sigma_P = \Sigma_{kP} = \Sigma_{P+m}$ ~~same fa~~ so what is extra data?

$\rho < \Sigma$ 1-dim^{l} $\rightsquigarrow \mathcal{O}_\rho \subset X_\Sigma$ co-dim 1 orbit

$\rightsquigarrow \overline{\mathcal{O}_\rho} = D_\rho \subset X_\Sigma$ divisor

u_ρ min. generator of ρ $\leftrightsquigarrow$ facet of P with minimal normal u_ρ

$$Q = P \cap H_{u_\rho, a_\rho} \qquad a_\rho \in \mathbb{Z}$$

$\therefore$ get divisor (T$_N$-invariant) $D_P = \sum a_\rho D_\rho$ $\therefore$ line bundle L_{D_P}

Cartier

FACT $\quad P \cap M \cong H^0(X_\Sigma, L_{D_P})$

So if L_{D_P} very ample (cond^{n} on P) get

proj. embedding ....