Fran Burstall Toric geometry for beginners II ①

Recap: $N \leq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

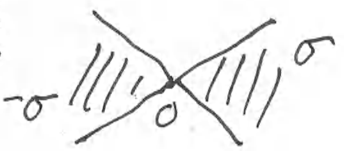
$\uparrow$ lattice (f.g. free abelian gp)

A rational cone $\sigma = \text{Cone}\{u_1, \dots u_n\}$ $u_i \in N$ is the set of all convex linear combinations of the u_i.

σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$

$(\Leftrightarrow)$ 0 is a vertex of σ

$(\Leftrightarrow$ σ contains no linear subspace$)$



$M := \text{Hom}(N, \mathbb{Z}) \leq M_{\mathbb{R}} = N_{\mathbb{R}}^*$ dual lattice.

$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong (\mathbb{C}^\times)^{\text{rk} N}$ is an affine alg. gp - a torus.

$M \cong$ character lattice of T_N

$m \mapsto \chi^m$

$N \cong$ 1parameter subgps of T_N

$u \mapsto \lambda^u$

Duality of M & N gives $\chi^m(\lambda^u(z)) = z^{\langle m, n \rangle}$

Rem: Coordinate ring $\mathbb{C}[T_N] = \text{span}\{\chi^m \mid m \in M\}$

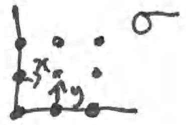
σ strongly convex rational $\rightsquigarrow \sigma^\vee = \{f \in M_{\mathbb{R}} \mid f(u) \geq 0 \ \forall u \in \sigma\}$

$\rightsquigarrow S_\sigma := \sigma^\vee \cap M$ a commutative monoid

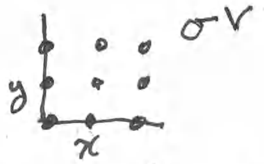
$\rightsquigarrow U_\sigma := \text{Spec}(\mathbb{C}[S_\sigma])$ $\text{span}\{\chi^m \mid m \in S_\sigma\}$

a normal affine algebraic toric variety & all such arise thusly.

Examples



$\rightsquigarrow$

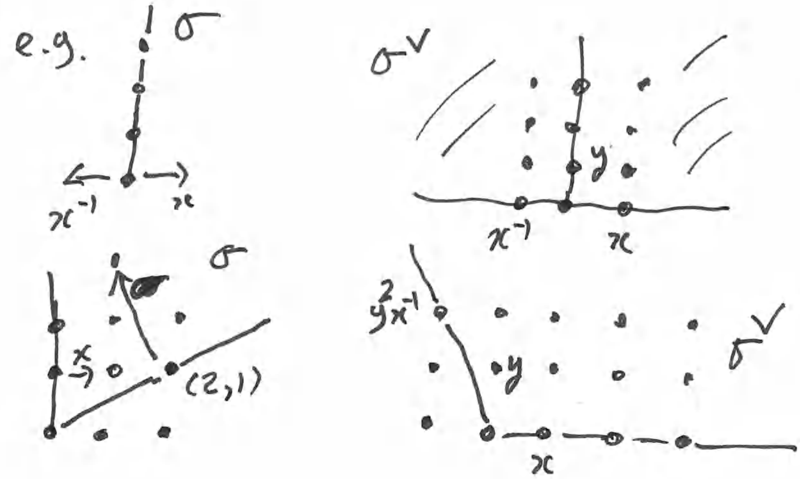


$U_\sigma = \text{Spec}(\mathbb{C}[x,y]) \cong \mathbb{C}^2$.

(Exercise: $\mathbb{C}^n = \text{Spec}(S_\sigma)$ with σ the nonnegative "quadrant orthant")

The cone σ need not have full dimension

② e.g.



$U_\sigma = \mathrm{Spec}(\mathbb{C}[y, x, x^{-1}])$
$= \mathbb{C}^\times \times \mathbb{C}$

cone generators
$\neq$ monoid generators
(y is not in integer span of x & yx^{-1})

$U_\sigma = \mathrm{Spec}(\mathbb{C}[x, y, y^2x^{-1}]) \hookrightarrow \mathbb{C}^3$

$(x, y, z) = (x, y, y^2x^{-1}) \iff xz = y^2$

So $U_\sigma \cong V(xy - z^2)$ is a cone over a conic
& is singular at 0

Geometry from faces

A. Open subsets $\tau < \sigma$ a face

Then $\sigma^\vee < \tau^\vee$ (linear fns +ve on σ are +ve on τ)

So $S_\sigma \subseteq S_\tau$.

Exercise If we write $\tau = \sigma \cap H_m$ $(H_m = \ker m,\ m \in \sigma^\vee)$
then $S_\tau = S_\sigma + \mathbb{Z}(-m)$ & hence $U_\tau = (U_\sigma)_{\chi^m} \subseteq U_\sigma$
is the localization at χ^m (pts where it is nonzero) $= \{p \in U_\sigma \mid \chi^m(p) \neq 0\}$

Punchline $\tau < \sigma \rightsquigarrow$ canonical inclusion $U_\tau \hookrightarrow U_\sigma$.
of a toric open set (~~with~~ under same torus)

Q: what if $\tau = \{0\}$? Then $\tau^\vee = M_\mathbb{R}$, so $U_\tau \cong T_N$
(the smallest T_N-invariant open subset)

B. Torus orbits

$\mathrm{Spec}(R) = \{\text{max ideals in } R\} \cong \{\mathbb{C}\text{-alg (nonzero) homomorphisms } R \to \mathbb{C}\}$
R a $\mathbb{C}$-alg.

Thus: affine $V \cong \{\mathbb{C}\text{-alg. homomorphisms } \mathbb{C}[V] \to \mathbb{C}\}$
$p \mapsto (f \mapsto f(p))$

$\tau \le \sigma \rightsquigarrow O_\tau := \{p \in U_\tau \mid \chi^m(p) \neq 0 \iff m \in \tau^\perp\}$

Then • O_τ is T_N-invariant:

$\forall t \in T_N$ $\chi^m(t\cdot p) = \underbrace{\chi^m(t)}_{\in \mathbb{C}^\times} \chi^m(p)$ so $\chi^m(t\cdot p)=0$
$\iff \chi^m(p)=0$.

• O_τ is a single orbit:

$p, \hat{p} \in O_\tau$ give a $t \in T_N$ (corresponding to the $\mathbb{C}$-alg hom with) $\chi^m(t) = \begin{cases} \chi^m(\hat{p})/\chi^m(p) & m \in \tau^\perp \\ 1 & \text{otherwise} \end{cases}$

Via limits $u \in \tau \Rightarrow \lim_{z \to 0} \lambda^u(z) q \in U_\tau$ $\forall q \in U_\tau$.

Indeed if $m \in S_\tau$, $\chi^m(\lambda^u(z)q) = \chi^m(\lambda^u(z))\chi^m(q)$
$= z^{\langle m,u\rangle}\chi^m(q)$, so consider $\chi^m \mapsto \lim_{z\to 0} \chi^m(\lambda^u(z)q)$

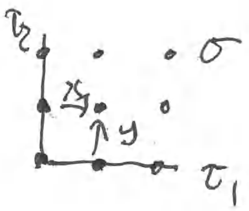
$= \lim_{z \to 0} z^{\langle m,u\rangle} \chi^m(q) = \begin{cases} 0 & \langle m,u\rangle > 0 \\ \chi^m(q) & \langle m,u\rangle = 0 \end{cases}$

So have $p = \lim_{z\to 0} \lambda^u(z) q \in U_\tau$ s.t. $\forall f \in \mathbb{C}[U_\tau]$

$f(p) = \lim_{z \to 0} f(\lambda^u(z)q)$ — we define p to make f cts!

Exercise $u \in$ rel. int. τ i.e. $\langle m,u\rangle > 0$ $\forall m \in \tau^\vee \setminus \tau^\perp$
& $q \in T_N \Rightarrow \lim_{z\to 0} \lambda^u(z) q \in O_\tau$.

Example



$U_\sigma = \mathbb{C}^2$ $\quad O_\sigma = \{0\}$

$U_{\tau_1} = \mathbb{C} \times \mathbb{C}^\times$ $\quad O_{\tau_1} = \{0\} \times \mathbb{C}^\times$

$U_{\tau_2} = \mathbb{C}^\times \times \mathbb{C}$ $\quad O_{\tau_2} = \mathbb{C}^\times \times \{0\}$.

Thm $U_\sigma = \bigcup_{\tau \le \sigma} O_\tau$, $\dim O_\tau = \operatorname{codim} \tau$,

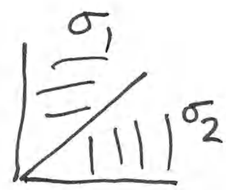
$\tau_1 < \tau_2 \iff O_{\tau_2} \subseteq \overline{O_{\tau_1}}$

④

Fans & gluing

Defn A fan in $N_{\mathbb{R}}$ is a collection Σ of strongly convex rational cones s.t.:

- Σ closed under faces
- $\sigma_1, \sigma_2 \in \Sigma \Rightarrow \sigma_1 \cap \sigma_2 < \sigma_1 \;\&\; \sigma_2$

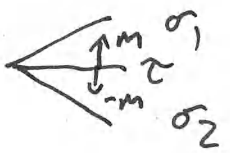


Example 

The support of Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

Plan: build a toric variety from Σ.

For $\sigma_1, \sigma_2 \in \Sigma$ let $\tau = \sigma_1 \cap \sigma_2$



Then $U_\tau \cong (U_{\sigma_1})_{\chi^m}$ (canonically)

$\cong (U_{\sigma_2})_{\chi^{-m}} \swarrow g_{\sigma_2\sigma_1}$ (an isomorphism)

Facts ~~...~~ $g_{\sigma_i\sigma_j}$ are T_N-invariant & compatible i.e.

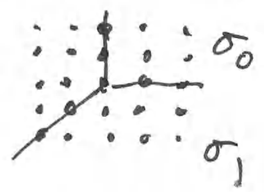
- $g_{\sigma_i\sigma_j} = g_{\sigma_i\sigma_j}^{-1}$ & $g_{\sigma_3\sigma_1} = g_{\sigma_3\sigma_2}\, g_{\sigma_2\sigma_1}$ where both sides make sense.

Hence can glue $U_\sigma : \sigma \in \Sigma$ together to get $X_\Sigma = \coprod_{\sigma \in \Sigma} U_\sigma /\sim$

where $x_i \sim x_j$ iff $x_i = x_j$ in $U_{\sigma_i \cap \sigma_j}$.

This is a normal toric variety (it has a dense open T_N)
(not immediately obvious it is separated)

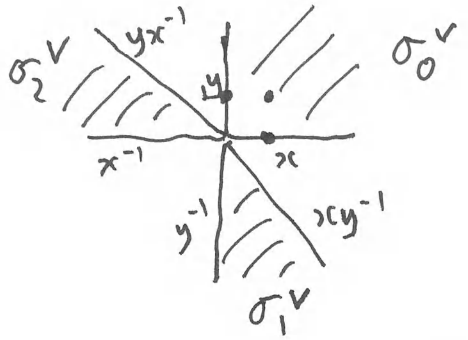
Examples



Can see all the torus orbits (3 points, 3 codimension ~~one~~ 1, one open)

Have normals $x, y, x^{-1}, y^{-1}, yx^{-1}, xy^{-1}$.

Dual cones do not fit together!



These correspond to opens

$$U_{\sigma_0} = \operatorname{Spec} \mathbb{C}[\underset{x}{s_0}, \underset{y}{t_0}] \cong \mathbb{C}^2$$

$$U_{\sigma_1} = \operatorname{Spec} \mathbb{C}[\underset{xy^{-1}}{s_1}, \underset{y^{-1}}{t_1}] \cong \mathbb{C}^2, \quad U_{\sigma_2} = \operatorname{Spec} \mathbb{C}[\underset{x^{-1}}{s_2}, \underset{yx^{-1}}{t_2}] \cong \mathbb{C}^2$$

& e.g. $s_1 = s_0/t_0$ & $t_1 = 1/t_0$ on $U_{\sigma_0 \cap \sigma_1}$

$s_2 = 1/s_0$ & $t_2 = t_0/s_0$ on $U_{\sigma_0 \cap \sigma_2}$

Punchline: 3 affine charts on $\mathbb{P}^2$

$$[1, s_0, t_0] = [1/s_0, 1, t_0/s_0] = [1/t_0, s_0/t_0, 1]$$

$$\phantom{[1, s_0, t_0] = {}} \quad\; [s_2, 1, t_2] = [t_1, s_1, 1]$$