# Toric geometry for beginners me!



## 1. Tori

$$(\mathbb{C}^\times)^n = \{(z_1,\dots,z_n) \mid z_i \neq 0\}$$

abelian affine alg grp

- abelian grp
- affine alg var: affine open of $\mathbb{C}^n$: $\mathbb{C}^n \setminus V(z_1 \cdots z_n)$

$$\mathbb{C}[(\mathbb{C}^\times)^n] = \mathbb{C}[z_1^{\pm 1},\dots,z_n^{\pm 1}] = \mathbb{C}[z_1,\dots,z_n]_{z_1 \cdots z_n}$$

Laurent poly$^{\text{al}}$s

localise!

**Def$^n$** Torus is affine alg. variety $T \cong (\mathbb{C}^\times)^n$ some $n$.

free ab. grp finite rank

## Lattices

$T$ a torus

$$M := \mathrm{Hom}(T, \mathbb{C}^\times) \quad \text{character lattice}$$

$$T \cong (\mathbb{C}^\times)^n \rightsquigarrow \mathbb{Z}^n \cong M$$

$$m = (m_1, \dots, m_n) \longmapsto \chi^m : (z_1, \dots, z_n) \mapsto z_1^{m_1} \cdots z_n^{m_n}$$

$\therefore$ "$M$ for monomials". $\quad$ rank $M = \dim T$

Prompts:
- write $M$ additively
- write $m \longmapsto \chi^m$ for inclusion $M \hookrightarrow \mathbb{C}[T]$

Duality:

$$N := \mathrm{Hom}(\mathbb{C}^\times, T)$$

Pairing $\quad M \times N \xrightarrow{\text{composition}} \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \cong \mathbb{Z}$

$$z \mapsto z^m \mapsto m$$

If $T \cong (\mathbb{C}^\times)^n \quad \mathbb{Z}^n \cong N$

$$(n_1, \dots, n_n) \longmapsto \lambda_n : z \longmapsto (z^{n_1}, \dots, z^{n_n})$$

then pairing is dot product.

Set $\quad \left.\begin{array}{l} M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \\ N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \end{array}\right\}$ dual vector spaces$/\mathbb{R}$ $\quad$ dim = dim $T$

N.B. $M$ or $N$ determine $T$: $\quad N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong T$

$$\lambda \otimes z \longmapsto \lambda(z)$$

Aside (for later lecture):

$T$ is Lie grp with Lie alg $\mathfrak{t}$

$M \hookrightarrow \mathfrak{t}^*$

$m \mapsto d\chi_1^m$ $\quad \therefore \quad M_{\mathbb{R}} \leq (\mathfrak{t}^*)_{\mathbb{R}}$

$N \hookrightarrow \mathfrak{t}$

$n \to d\cancel{\lambda}\gamma_n(1)$

More: $T$ has unique max cpct subgrp $U \cong (S^1 \times \cdots \times S^1)^n$

Lie alg $\mathfrak{u} \cong i\mathbb{R}^n$

any $\chi, \lambda$ preserves these so $\chi : \mathfrak{u} \to i\mathbb{R}$

$\gamma : i\mathbb{R} \to \mathfrak{u}$

$\therefore \quad M_{\mathbb{R}} = i\mathfrak{u}^*$

$N_{\mathbb{R}} = i\mathfrak{u}$

## 2. Affine toric varieties

**Def<sup></sup>** (Affine) toric variety is

- $V$ irred (affine) alg. var
- $T \subset V$ Zariski open & dense
  ($\uparrow$ torus)

s.t. $T \times T \xrightarrow[\text{mult}^n]{} T$ extends to alg action $T \times V \to V$

Examples:

- $V = \mathbb{C}^n \supset T = (\mathbb{C}^\times)^n$
- $V = (\mathbb{C}^\times)^n \cong V(z_1 \cdots z_{n+1} - 1) \subset \mathbb{C}^{n+1}$

  $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, z_1^{-1} \cdots z_n^{-1})$

- $V = V(\cancel{xy - z^2}\ \ xz = y^2) \subset \mathbb{C}^3$

  $T = V \cap (\mathbb{C}^\times)^4 = \{(t_1, t_1 t_2, t_1 t_2^2) \mid t_i \in \mathbb{C}^\times\} \cong (\mathbb{C}^\times)^2$

More generally: given $T$ with char lattice $M$

& $\mathcal{A} = \{m_1, \dots, m_s\} \subset M$

$\varphi_\mathcal{A} : T \to (\mathbb{C}^\times)^s \subseteq \mathbb{C}^s$

$\varphi_\mathcal{A} = (\chi^{m_1}, \dots, \chi^{m_s})$

$Y_\mathcal{A} := \overline{\operatorname{Im} \varphi_\mathcal{A}}$ toric with torus $T_\mathcal{A} = \operatorname{Im} \varphi_\mathcal{A} \leq (\mathbb{C}^\times)^s$

& $M_{T_\mathcal{A}} = \mathbb{Z}\mathcal{A}$

Invariant viewpoint:

- $T$ Zariski dense $\therefore$ $\mathbb{C}[V] \leq \mathbb{C}[T] = \mathrm{span}_{\mathbb{C}}\{\chi^m \mid m \in M\}$
- $T$ acts on $\mathbb{C}[T]$: $(t \cdot f)(z) = f(t^{-1}z)$

  wt vectors are characters: $t \cdot \chi^m = \chi^{-m}(t)\chi^m$
- $T$ acts on $V$ $\therefore$ $\mathbb{C}[V]$ is $T$-submodule $\therefore$ spanned by wt spaces i.e.

$$\mathbb{C}[V] = \mathrm{span}_{\mathbb{C}}\{\chi^m \mid m \in S\}$$

for $S = \{m \mid \chi^m \in \mathbb{C}[V]\}$ ← chars extending to $V$

$\mathbb{C}[V]$ alg $\therefore$ $S$ semigrp (f.g since $\mathbb{C}[V]$ is)

↑ closed under addition.

Converse: $S \subseteq M$ f.g semigrp

$$\mathbb{C}[S] := \mathrm{span}_{\mathbb{C}}\{\chi^m \mid m \in S\} \leq \mathbb{C}[T]$$

is f.g. integral domain

$\therefore$ $\mathrm{Spec}(\mathbb{C}[S])$ affine toric for torus with char. lattice $\mathbb{Z}S$.

↑ max ideal spec

# 3. Cones & dual cones

Setting $\mathbb{R}^n \supseteq \mathbb{Z}^n$ (think $N_{\mathbb{R}} \supseteq N$)

$A = \{a_1, \dots, a_s\}$ finite

$\sigma = \mathrm{Cone}(A) := \{ \sum \lambda_i x_i \mid \lambda_i \in \mathbb{R}_{\geq 0} \}$ ← polyhedral cone ~~convex polyhedral cone~~ ⇒ convex

$\sigma$ rational if $\sigma = \mathrm{Cone}(A)$ $A \subset \mathbb{Z}^n$

Dual cone:

$\check{\sigma} := \{ f \in (\mathbb{R}^n)^* \mid f(u) \geq 0 \ \forall u \in \sigma \}$

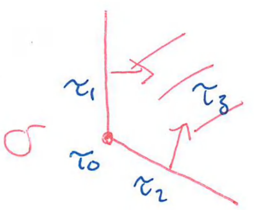
- $\sigma^\vee$ ~~convex~~ polyhedral cone
- $\sigma$ rational $\Leftrightarrow$ $\check{\sigma}$ rational
- $(\sigma^\vee)^\vee = \sigma$

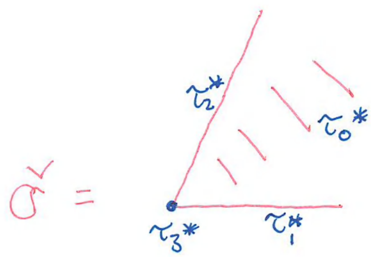
$f \in (\mathbb{R}^n)^* \rightsquigarrow$

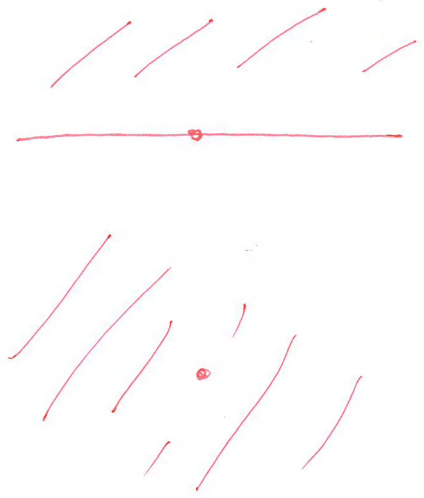
$H_f = \ker f$ : hyperplane if $f \neq 0$

$H_f^+ = \{ v \in \mathbb{R}^n \mid f(v) \geq 0 \}$ 1/2-space

$f \in \hat{\sigma} \Leftrightarrow \sigma \subseteq H_f^+$



Faces of $\sigma$ is $\tau = \sigma \cap H_f$ some $f \in \sigma^\vee$ write $\tau < \sigma$

~~Properties~~
- $\tau$ ~~convex~~ polyhedral cone
- $\tau' < \tau < \sigma \Rightarrow \tau' < \sigma$
- ~~$\tau \cap \tau'$~~ $\tau_1, \tau_2 < \sigma \Rightarrow \tau_1 \cap \tau_2 < \sigma$

Duality: $\tau < \sigma \rightsquigarrow \tau^{\perp} \subseteq (\mathbb{R}^n)^* = \{f \mid f|_{\tau} = 0\}$

$$\tau^* = \tau^{\perp} \cap \sigma^{\vee}$$

- $\tau^* < \sigma^{\vee}$
- $\tau \to \tau^*$ inclusion-reversing bijection $\{\text{faces of } \sigma\} \to \{\text{faces of } \sigma^{\vee}\}$
- $\dim \tau + \dim \tau^* = n$
  $\uparrow$
  dim span $\tau$

In particular: $\dim \check{\sigma} = n$ iff $0 < \sigma \iff \sigma \cap -\sigma = \{0\}$ (say $\sigma$ strongly convex)

~~*~~ <u>Application</u>

$\sigma \subset N_{\mathbb{R}}$ ~~cvx~~ rational ~~convex~~ polyhedral cone

~~Put~~ Let $S_\sigma := M \cap \check{\sigma}$

- semigroup (by convexity of $\check{\sigma}$)
- finitely generated (by $M \cap \{\sum \lambda_i \check{x}_i \mid \lambda_i \in [0,1]\}$ ← cpct, Gordon's Lemma; $\check{x}_i$ finite generators of $\check{\sigma}$)

$\therefore$ ~~and make~~ get affine toric variety

$$\cancel{X_\sigma} \; U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$$

~~Which toric~~

Q: Which affine toric arise this way?

Issue: $S_\sigma$ has <u>all</u> lattice pts in $\check{\sigma}$

If $S \subset M$ generated by $A$, put $\check{\sigma} = \operatorname{Cone}(A)$ ~~then~~

~~could miss~~ $\rightsquigarrow \sigma = (\sigma^{\vee})^{\vee} \subset N_{\mathbb{R}}$

$$\mathbb{C}[S] \subseteq \mathbb{C}[S_\sigma] \rightsquigarrow U_\sigma \xrightarrow[\text{normalisation}]{} \mathrm{Spec}\,\mathbb{C}[S]$$

$$\begin{array}{c} U_\sigma \\ \downarrow \\ \mathrm{Spec}\,\mathbb{C}[S] \end{array}$$