Fran Burstall   <u>Toric geometry for beginners</u>

<u>References</u> Cox–Little–Schenk Toric varieties (compatible notation)

J. P. Brasselet: Introduction to toric varieties

Gelfand–Kapranov–Zelevinsky: Multideterminants Ch. 5.

<u>One sentence summary</u>: Varieties with a complex torus action with a dense open orbit.

<u>1. Tori</u> $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ multiplicative group

$(\mathbb{C}^\times)^n = \{(z_1, \dots z_n) \mid z_i \neq 0\}$ is (compatibly!)

- an abelian group under mult^n
- an affine algebraic variety & an affine open subset $\mathbb{C}^n$

(it is $\mathbb{C}^n \setminus V(z_1 \cdots z_n)$, the complement of $z_1 \cdots z_n = 0$)

NB. Can understand affine varieties via their coordinate rings (a contravariant equivalence of categories)

Here $\mathbb{C}[(\mathbb{C}^\times)^n] = \underset{\text{Laurent polynomials}}{\mathbb{C}[z_1^{\pm 1}, \dots z_n^{\pm 1}]} = \underset{\text{localization}}{\mathbb{C}[z_1, \dots z_n]_{z_1 \cdots z_n}}$

Compatability (group mult^n etc. is ~~algebraic~~ algebraic) means $(\mathbb{C}^\times)^n$ is an affine algebraic group.

<u>Defn</u> A <u>torus</u> is an affine alg. grp $T \cong (\mathbb{C}^\times)^n$ for some n.

<u>Lattices</u> (free abelian groups) T a torus has:

- $M := \mathrm{Hom}(T, \mathbb{C}^\times)$, the <u>character lattice</u>

$T \cong (\mathbb{C}^\times)^n$ gives $M \cong \mathbb{Z}^n$

$$\left[(z_1, \dots z_n) \mapsto z_1^{m_1} \dots z_n^{m_n}\right] \leftarrow\!\shortmid (m_1, \dots m_n)$$

Thus the characters $m \in M$ are Laurent <u>monomials</u>

Henceforth:
- Write M additively
- Write $m \mapsto \chi^m$ for $M \hookrightarrow \mathbb{C}[T]$

[This picks out an interesting basis]

& T dually has

- $N := \mathrm{Hom}(\mathbb{C}^\times, T)$ 1-parameter subgroups

with a pairing $M \times N \to \mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) \cong \mathbb{Z}$

$$(z \mapsto z^m) \leftarrow\!\shortmid\; m$$

$T \cong (\mathbb{C}^\times)^n$ gives $\mathbb{Z}^n \cong N$

$$(n_1, \dots n_n) \mapsto \overbrace{\left[z \mapsto (z^{n_1}, \dots z^{n_n})\right]}^{\lambda_n}$$

and the pairing with $M \cong \mathbb{Z}^n$ is dot product.

Set $\left.\begin{aligned} M_{\mathbb{R}} &= M \otimes_{\mathbb{Z}} \mathbb{R} \\ N_{\mathbb{R}} &= N \otimes_{\mathbb{Z}} \mathbb{R} \end{aligned}\right\}$ dual vector spaces of dimension $n = \operatorname{rank} M = \dim T$.

N.B. M or N determine T: $N \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong T$

$$\lambda \otimes z \mapsto \lambda(z)$$

<u>Aside</u>: T is a complex Lie group with complex Lie algebra $\mathfrak{t}$ &

$$M \hookrightarrow \mathfrak{t}^*$$
$$m \mapsto (d\chi^m_1 : \mathfrak{t} \to \mathbb{C})$$

so $\mathfrak{t}^* \cong M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

& $N \hookrightarrow \mathfrak{t}$

$$n \mapsto (d\lambda_n)_1(1)$$

Furthermore, T has a unique max. compact subgroup $U \cong (S^1)^n$ with Lie alg. $\mathfrak{u} \cong i\mathbb{R}^n$ so χ, λ give $M_{\mathbb{R}} = i\underline{u}^*$ & $N_{\mathbb{R}} = i\underline{u}$.

2. <u>Affine toric varieties</u>

<u>Defn</u> An <u>affine toric variety</u> is

- V irreducible affine variety
- $T \subseteq V$ Zariski open $\therefore$ dense

s.t. $T \times T \to T$ extends to an action $T \times V \to V$.

[Or: V has an action of T with a Zariski open orbit but the above defn picks $1 \in T \subseteq V$.]

<u>Examples</u> • $V = \mathbb{C}^n \supseteq T = (\mathbb{C}^\times)^n$

• $V = (\mathbb{C}^\times)^n \cong V(z_1 z_2 \cdots z_{n+1} - 1) \subseteq \mathbb{C}^{n+1}$

$(z_1, \dots z_n) \mapsto (z_1, \dots z_n, z_1^{-1} z_2^{-1} \cdots z_n^{-1})$

• $V = V(xz - y^2) \subseteq \mathbb{C}^3$

$T = V \cap (\mathbb{C}^\times)^3 = \{ (t_1, t_1 t_2, t_1 t_2^2) \mid t_i \in \mathbb{C}^\times \} \subseteq (\mathbb{C}^\times)^3$

More generally for T with character lattice M & any $A = \{m_1, \dots m_s\} \subseteq M$ define

$\varphi_A : T \to (\mathbb{C}^\times)^s \subseteq \mathbb{C}^s$ by $\varphi_A = (\chi^{m_1}, \dots \chi^{m_s})$

& $Y_A = \overline{\operatorname{im} \varphi_A}$ (Zariski closure)

(Strictly $A = (m_1, \dots, m_s) : \{1, \dots, s\} \to M$ is a list, not a subset!)

This is toric with torus $T_A = \operatorname{im} \varphi_A \leq (\mathbb{C}^\times)^s$

& $M_{T_A} = \mathbb{Z}A$. This involved choices!

<u>Invariant viewpoint:</u>

• T Zariski dense in V $\therefore$ $\mathbb{C}[V] \subseteq \mathbb{C}[T] = \operatorname{span}_{\mathbb{C}} \{\chi^m \mid m \in M\}$
by restriction

• T acts on $\mathbb{C}[T]$: $(t \cdot f)(p) = f(t^{-1} p)$.

• The weight vectors are the characters: $t \cdot \chi^m = \chi^m(t)\, \chi^m$

• T acts on $\mathbb{C}[V] \leq \mathbb{C}[T]$ i.e. $\mathbb{C}[V]$ is a submodule, hence spanned by weight vectors: $\mathbb{C}[V] = \operatorname{span}_{\mathbb{C}} \{\chi^m \mid m \in S\}$
where $S = \{ m \mid \chi^m \in \mathbb{C}[V] \} \subseteq M$.

• $\mathbb{C}[V]$ is an algebra $\therefore$ S is a (abelian) monoid under addition
(Above A is a set of generators for $S = \mathbb{N}A$)

<u>Converse:</u> $S \subseteq M$ f.g. monoid $\Rightarrow$ $\mathbb{C}[S] := \operatorname{span}_{\mathbb{C}} \{\chi^m \mid m \in S\}$
$\leq \mathbb{C}[T]$ is f.g. integral domain $\therefore$ $\operatorname{Spec}(\mathbb{C}[S])$ is affine
a max ideal spectrum

& an affine toric variety for a torus T_S with character lattice $\mathbb{Z}S$.

Thus everything is driven by S: the characters of T which extend, as functions, to V.

3. Cones & duals

Setting: $\mathbb{R}^n \supseteq \mathbb{Z}^n$ (think of $N_{\mathbb{R}} \supseteq N$)

$A = \{a_1, \dots, a_s\} \subseteq \mathbb{R}^n$ finite

$\sigma = \mathrm{Cone}(A) = \{\sum \lambda_i a_i \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$, a polyhedral convex cone.

e.g.

Say σ rational if $\sigma = \mathrm{Cone}(A)$ with $A \subseteq \mathbb{Z}^n$.

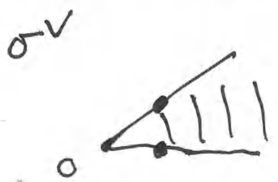
Dual cone $\sigma^\vee := \{f \in \mathbb{R}^{n*} \mid f(u) \geq 0 \ \forall u \in \sigma\}$

Then
- $\sigma^\vee$ polyhedral with $(\sigma^\vee)^\vee = \sigma$
- σ rational $\Leftrightarrow$ $\sigma^\vee$ rational

$f \in (\mathbb{R}^n)^*$, $f \neq 0$ $\rightsquigarrow$ $H_f = \ker f$ a hyperplane

& $H_f^+ = \{v \in \mathbb{R}^n \mid f(v) \geq 0\}$

Then $f \in \sigma^\vee \Leftrightarrow \sigma \subseteq H_f^+$.



$\sigma^\vee$ is cone on the inward normals to σ.

Note $\sigma = \{0\} \Leftrightarrow \sigma^\vee = \mathbb{R}^{n*}$

A face of σ is $\tau = \sigma \cap H_f$ for some $f \in \sigma^\vee$.

(Codimension 1 faces – facets – generate $\sigma^\vee$)

Faces τ of σ, written $\tau \prec \sigma$

- τ polyhedral cones.
- $\tau' \prec \tau \prec \sigma \implies \tau' \prec \sigma$
- $\tau_1, \tau_2 \prec \sigma \implies \tau_1 \cap \tau_2 \prec \sigma$

<u>Application</u> $\sigma \subseteq N_{\mathbb{R}}$ rational polyhedral

$S_\sigma = M \cap \sigma^\vee$ is a monoid ($\sigma^\vee$ convex

- finitely generated (by $M \cap \{ \sum \lambda_i x_i^\vee \mid \lambda_i \in [0,1]\}$

where $x_1^\vee, \dots x_n^\vee \in M$ generate $\sigma^\vee$: this is discrete $\cap$ compact hence finite)

- $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ is an affine toric variety.

Which affine toric varieties arise this way?

<u>Problem</u> 2 & 3 generate a submonoid of $\mathbb{N}$ which does not contain 1.

More generally if $\sigma^\vee$ is a cone generated by a monoid S then S need not be $\sigma^\vee \cap M$.

Thus an affine toric variety V arises in this way if its coordinate ring is <u>integrally closed</u> i.e. V is a normal variety.

Hence U_σ is a normal variety & in general the construction of U_σ from an arbitrary V is the normalization.

e.g. problem example $z \mapsto (\underset{x}{z^2}, \underset{y}{z^3}) \subseteq V(x^3 - y^2)$

Missing function is $z = y/x$ which is rational but satisfies an integral polynomial eqn. The normalization is $\operatorname{Spec} \mathbb{C}[z]$