SUBMANIFOLD GEOMETRY IN GENERALIZED FLAG MANIFOLDS

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Based on lecture notes taken by Dalibor Smid

Abstract. These notes aim to explain why some unashamedly modern machinery has something to say about classical differential geometry. They concern the study of the induced geometry on submanifolds in generalized flag manifolds G/P, where G is a semisimple Lie group and P a parabolic subgroup. The prototype and motivating example is the case that G/P is the conformal n-sphere, which is the space of null lines in $\mathbb{R}^{n+1,1}$ viewed as a homogeneous space for $O(n+1,1)$. Submanifolds of the conformal n-sphere are studied in detail and this study is related to the classical treatment of conformal submanifolds using sphere congruences. Hypersurfaces in projective space provide another example, which we study in less detail, again with comparison to the classical work.

Introduction: classical and modern geometry

Submanifold geometry is an important and long-standing theme in differential geometry. The classical theory of surfaces in euclidean 3-space was a triumph of 19th century mathematics, which continued to have important ramifications both in geometry and in the theory of integrable systems through the 20th century to the present day. The classical work on submanifolds was not just confined to the euclidean theory, however. Mathematicians such as Bianchi, Darboux, Lie and Blaschke were also very interested in the conformal and projective geometry of surfaces, i.e., those aspects of surface geometry invariant under the larger groups of conformal or projective transformations. From the point of view of Klein, this means the natural theory of surfaces in the conformal 3-sphere S^3 (the projective light cone in $\mathbb{R}^{4,1}$) under the group $O_+(4,1)$ of conformal transformations, or in the projective 3-space $\mathbb{R}P^3$ under the group $PSL(4,\mathbb{R})$ of projective transformations.

The theory of such surfaces—their description in terms of geometric invariants—is very different from the euclidean case simply because the ambient geometries of S^3 and $\mathbb{R}P^3$ are different from the geometry of $\mathbb{R}^3$. Unlike the euclidean group, the groups $O_+(4,1)$ and $PSL(4,\mathbb{R})$ are semisimple (in fact simple) and the stabilizer of a point in S^3 or $\mathbb{R}P^3$ is a parabolic subgroup. In other words, S^3 and $\mathbb{R}P^3$ are *generalized flag manifolds.* The fact that the stabilizer is not reductive makes the study of geometric invariants in these geometries much harder than in euclidean geometry, where the stabilizer of a point is a simple group (the orthogonal group).

The first problem in the projective and conformal geometry of surfaces is to find an analogue of the *Gauss map*, which for surfaces in euclidean 3-space is the unit normal vector, but is perhaps better thought of here as the *tangent plane congruence* of the surface, i.e., the map which assigns, to each point on the surface, its tangent plane, viewed as an element of the grassmannian $Gr_2(\mathbb{R}^3)$ of planes in $\mathbb{R}^3$. (The sphere to which the unit normal vector belongs is a double cover of this grassmannian.)

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The Gauss map is an effective tool because it provides the best approximation of the surface by the simplest surfaces in $\mathbb{R}^3$, the totally geodesic planes. The (entirely classical) idea now is to replace the planes by a different family of simple model surfaces, on which the group of conformal or projective transformations act transitively, preferably so that a typical model surface has a reasonably nice stabilizer (in the conformal or projective group).

• For surfaces in the conformal 3-sphere, the (totally umbillic) 2-spheres provide such a model family: $O_+(4,1)$ acts transitively on this space of 2-spheres (which is the grassmannian of $(3,1)$-planes in $\mathbb{R}^{4,1}$), and the stabilizer of a given 2-sphere is isomorphic to $O_+(3,1)$.

• For surfaces in projective 3-space, the 2-planes themselves appear as the most obvious first choice of model family, but because the projective group is larger than the euclidean group, a better choice is available, at least for surfaces with nondegenerate second fundamental form: the quadrics. Actually the space of quadrics has two components under $PSL(4,\mathbb{R})$: the ellipsoids, stabilized by $O_+(3,1)$, and the hyperboloids, stabilized by $O_+(2,2)$. Ellipsoids or hyperboloids can approximate a surface with definite or signature $(1,1)$ second fundamental form (respectively) more closely than planes can.

In conformal or projective geometry one therefore studies *sphere congruences* or *congruences of quadrics*, i.e., maps from the given surface to the grassmannian of 2-spheres or (the appropriate component of) the space of quadrics. In either case there is not an obvious 'best' approximation at each point, but a family of such approximations. However, a more subtle normalization condition picks out a distinguished choice.

• An *enveloped sphere congruence* is a sphere congruence on a surface S^3 such that each sphere is tangent (1st order contact) to the surface at the corresponding point. Although there is a 1-parameter family of spheres tangent to the surface at any point, there is a unique sphere with the same mean curvature as the surface at the given point (i.e., with a bit of second order contact), and this gives rise to the so-called *central sphere congruence* of the surface.

• An *enveloped congruence of quadrics* is a congruence of quadrics on a surface in $\mathbb{R}P^3$ such that each quadric is tangent (1st order contact) at the corresponding point and the second fundamental form of the quadric agrees with that of the enveloping surface (2nd order contact). At each point there is a family of such quadrics, but a normalization condition (which nearly amounts to 3rd order contact) picks out a distinguished one, leading to the so-called *congruence of Lie quadrics.*

If one reads the theory in the classical literature, one cannot fail to be impressed (if not baffled!) by the ingenuity of the constructions. The goal of these notes is to consider the geometry of submanifolds in generalized flag manifolds as a way to bring a modern perspective to bear on the classical theory, with the aim of demystifying and extending some of this work, albeit at the expense of introducing a certain amount of algebraic machinery. In particular we obtain a simple unifying explanation of the normalization conditions mentioned above.

The astute reader has undoubtedly noticed that in both of the theories described above, not only is the ambient geometry a generalized flag manifold, but also the model surfaces are generalized flag manifolds (2-spheres under the Möbius group $O_+(3,1)$, or the lorentzian analogue, where the group is $O_+(2,2)$). A parabolic subgeometry is a submanifold of a generalized flag manifold G/P modelled on a homogeneous inclusion of a generalized flag manifold $H/Q \hookrightarrow G/P$ (in the sense that the model submanifolds are the images of H/Q under the natural G action).

We develop enough of the theory of parabolic subgeometries to show that conformal submanifold geometry and projective hypersurface geometry are special cases. Cartan connections provide a natural tool to do this: indeed the notion of a Cartan connection is inspired by the idea of a family of model geometries osculating a manifold [11]. A general theory could in principle be developed for arbitrary homogeneous inclusions between homogeneous spaces (see [15]). In the parabolic case, however, there is an additional ingredient, an algebraic homology theory called Lie algebra homology, which provides an explanation of the normalization conditions used to fix the sphere congruence or congruence of quadrics in the classical theory. To be precise, the normalization amounts to requiring that a generalized 'shape operator' is a 1-cycle for this homology theory. This is a pleasing analogue of the normalization condition imposed on the Cartan connections in parabolic geometries, such as conformal geometry, in which the curvature of the 'normal' Cartan connection is required to be a 2-cycle.

We caution the reader that the general theory of parabolic subgeometries is not fully developed: a number of technical conditions are needed to make the theory run smoothly, and it is not yet clear what the best framework is, or how many other nontrivial examples can be covered by the theory. We can at least mention one further example: the Lie sphere geometry of surfaces, interpreted appropriately, is another real form of the geometry of surfaces in projective 3-space.

The notes begin by introducing some basic ideas and concepts. First, in section 1, we discuss the euclidean geometry of submanifolds of $\mathbb{R}^n$ and give a straightforward proof of the Bonnet theorem which states that the geometry may be described by certain induced data on the submanifold satisfying the so-called Gauss–Codazzi-Ricci equations: a manifold equipped with such Gauss–Codazzi–Ricci data (satisfying these equations) may be locally immersed in $\mathbb{R}^n$ with these induced data, uniquely up to a euclidean motion.

The standard approach to conformal submanifold geometry builds on the euclidean theory, by considering how these data change under a conformal rescaling of the ambient metric. Our approach, by contrast, builds in conformal invariance from the start. To achieve this, we need a good understanding of what a conformal manifold is, in manifestly invariant terms. We develop this in section 2, where we explain the natural realization of the conformal sphere as a projective light cone and also how this leads to the idea of a conformal Cartan connection. The key point here is that a manifold with a flat conformal Cartan connection has a natural development into the conformal sphere, i.e., there are local diffeomorphisms from the manifold to the conformal sphere which induce this conformal Cartan connection, unique up to conformal transformation. We view this as a 'codimension zero' conformal Bonnet theorem, since the immersion of submanifolds given by the Bonnet theorem can be regarded as a generalization to higher codimension of the notion of development. Our goal then, is to extend the conformal Bonnet theorem to higher codimension.

We make a first attempt at such a generalization in section 3, where we show how the notion of a sphere congruence provides a conformal analogue of the Gauss–Codazzi–Ricci equations and hence a Bonnet theorem for submanifolds of the conformal sphere. However, although these equations are conceptually simple, it is difficult to understand their meaning, especially as they depend on the auxiliary choice of a sphere congruence (breaking the manifest conformal invariance which we seek). Although these difficulties can be resolved by ad hoc means, it is the parabolic nature of conformal geometry that underlies the remedies.

Therefore, in sections 4–6 we develop some parabolic machinery: Lie algebra homology, the BGG calculus of multilinear differential operators, and the A-infinity equation. In section 7 we revisit conformal geometry, both to illustrate some of the differential operators arising in the general theory, and also to show how more conventional definitions of conformal manifolds correspond to conformal Cartan connections satisfying a homological normalization condition.

In section 8 we sketch the general theory of parabolic subgeometries, showing how, under suitable technical assumptions, there is an induced choice of a normalized congruence of model geometries, and that the A-infinity equation provides a suitable generalization of the Gauss–Codazzi–Ricci equations. We then (section 9) revisit conformal submanifold geometry with the general theory in mind, explaining the normalization of the sphere congruence and the induced equations in detail. The theory is particularly interesting for surfaces, as isothermic surfaces show. We also indicate in section 10 how the classical theory of sphere congruences and transformations of surfaces has a straightforward development in our approach.

We end in section 11 by turning our eyes to projective hypersurface geometry, as another example of the general machine in action, although as in the conformal case, we develop the detail in a more self-contained way.

These notes are based on forthcoming work [5, 6] to which we refer the reader for further information. Also, we have not given many references here: we refer the reader to these papers for a more extensive bibliography.

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1. Prologue: the Bonnet theorem in euclidean submanifold geometry

We begin with a brief review of the classical theory of submanifolds in euclidean space. Let Σ be an m-dimensional manifold immersed via a map f into $\mathbb{R}^n$. The tangent bundle $T\mathbb{R}^n$ restricted to Σ is a trivial bundle $\underline{\mathbb{R}}^n := \Sigma \times \mathbb{R}^n$ and df a 1-form on Σ with values in this trivial bundle, which therefore splits as the direct sum of (the image under df of) $T\Sigma$, the tangent bundle of Σ, and its orthogonal complement $N\Sigma$, the normal bundle of Σ.

The geometry of the immersion may be described by two symmetric bilinear forms: the induced metric (first fundamental form) $g(X,Y) = \langle df(X), df(Y)\rangle$ and the second fundamental form $\mathrm{II}(X,Y) = (\nabla df_{X,Y})^\perp \in N\Sigma$, where ∇ is any connection on $T\Sigma$ (coupled to the trivial connection d on $\underline{\mathbb{R}}^n$) and $^\perp$ denotes the normal component: this is independent of the connection ∇ and is symmetric in X and Y. The second fundamental form, viewed as a 1-form with values in $\mathrm{Hom}(T\Sigma, N\Sigma)$, determines the off-diagonal part of the trivial connection d on $\underline{\mathbb{R}}^n$ with respect to the direct sum decomposition $\underline{\mathbb{R}}^n = T\Sigma \oplus N\Sigma$. More precisely, we have

$$d = \begin{pmatrix} \nabla^g & S \\ \mathrm{II} & \nabla^\perp \end{pmatrix},$$

where ∇^g is the Levi-Civita connexion of g, $\nabla^\perp$ the metric connection on $N\Sigma$ and $S = -\mathrm{II}^T$ is the shape operator, a 1-form with values in $\mathrm{Hom}(N\Sigma, T\Sigma)$. The flatness of d is now expressed by the following equations, called the Gauss–Codazzi–Ricci (GCR) equations.

Gauss: $$R^{\nabla^g} + S \wedge \mathrm{II} = 0$$

Codazzi: $$d^{\nabla^g,\nabla^\perp}\mathrm{II} = 0$$

Ricci: $$R^{\nabla^\perp} + \mathrm{II} \wedge S = 0$$

It is natural to ask whether these data suffice to recover the immersion of Σ in $\mathbb{R}^n$, a problem which was solved by O. Bonnet in the 1860s.

1.1. **Theorem** (Bonnet). *Let (Σ, g) be a riemannian manifold, $(N\Sigma, \nabla^\perp)$ a metric vector bundle with a metric connection, and $\mathrm{II} \in S^2T^*\Sigma \otimes N\Sigma$. Suppose that these data satisfy the GCR equations. Then there is locally an immersion of Σ into $\mathbb{R}^n$, unique up to a euclidean motion, with this induced data.*

Proof. The GCR equations imply that

$$\begin{pmatrix} \nabla^g & S \\ \mathrm{II} & \nabla^\perp \end{pmatrix}$$

(with $S = -\mathrm{II}^T$) is a flat metric connection on $T\Sigma \oplus N\Sigma$ and therefore we have a local isomorphism $T\Sigma \oplus N\Sigma \equiv \Sigma \times \mathbb{R}^n$ unique up to an orthogonal transformation of $\mathbb{R}^n$. The inclusion $i\colon T\Sigma \hookrightarrow \Sigma \times \mathbb{R}^n$ may be viewed as a 1-form α on Σ with values in $\mathbb{R}^n$. If we write $\alpha \in \Omega^1(\Sigma, T\Sigma \oplus N\Sigma)$ in block matrix form, it is

$$\alpha = \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix} \in \begin{pmatrix} T^*\Sigma \otimes T\Sigma \\ T^*\Sigma \otimes N\Sigma \end{pmatrix}$$

We now compute $d\alpha$ using the block decomposition of the flat connection on $\mathbb{R}^n$ and obtain

$$d\alpha = \begin{pmatrix} d^{\nabla^g}\mathrm{id} \\ \mathrm{II} \wedge \mathrm{id} \end{pmatrix}.$$

Using the definition of the exterior derivative coupled to ∇^g, we have $d^{\nabla^g}\mathrm{id}(X,Y) = \nabla^g_X(\mathrm{id}(Y)) - \nabla^g_Y(\mathrm{id}(X)) - \mathrm{id}([X,Y])$ which vanishes because the Levi-Civita connection ∇^g on $T\Sigma$ is torsion-free. The other term $\mathrm{II} \wedge \mathrm{id}$ also vanishes because $(\mathrm{II} \wedge \mathrm{id})(X,Y) = \mathrm{II}(X,Y) - \mathrm{II}(Y,X)$ and II is symmetric.

Thus $d\alpha = 0$ and there is locally a function $f\colon \Sigma \to \mathbb{R}^n$, unique up to translation, such that $\alpha = df$. Since α is injective, f is an immersion, and since α was determined up to an orthogonal transformation, f is determined up to a euclidean motion. □

As a first attempt to understand conformal submanifold geometry, suppose that we rescale the flat metric on $\mathbb{R}^n$ by e^{2u} for a smooth function u on $\mathbb{R}^n$. This new metric and its Levi-Civita connection induce the conformally equivalent metric $\tilde{g} = e^{2u}g$ on Σ and the connection $(X,\xi) \mapsto d_X\xi + du(X)\xi + du(\xi)X - \langle \xi, X\rangle du^\sharp$ on $\underline{\mathbb{R}}^n$. With respect to the direct sum decomposition into tangential and normal parts this connection is

$$\begin{pmatrix} \nabla^{\tilde{g}} & \tilde{S} \\ \tilde{\mathrm{II}} & \tilde{\nabla}^\perp \end{pmatrix},$$

where $\nabla^{\tilde{g}}$ is the Levi-Civita connection of $\tilde{g}$, $\tilde{\nabla}^\perp = \nabla^\perp + du|_{T\Sigma} \otimes id$, $\tilde{\mathrm{II}}(X,Y) = \mathrm{II}(X,Y) + g(X,Y)(du^\sharp)^\perp$ and $\tilde{S} = -\tilde{\mathrm{II}}^T$. It follows that the conformal class of the metric g, the trace-free part II^0 of the second fundamental form, and the metric connection on the 'weightless normal bundle' are unchanged by a conformal rescaling. One might hope that these data would suffice to recover the immersion up to conformal transformations of $\mathbb{R}^n$. However, this turns out not to be true for immersed curves and surfaces.

For this reason, and to understand the conformal invariants more intrinsically, we adopt a different point of view.

2. Conformal geometry

We wish to investigate immersions of manifolds into S^n, the conformal sphere. We could regard S^n as the round sphere in $\mathbb{R}^{n+1}$ equipped with the conformal class of the induced metric. However, there is a more natural point of view available which makes more manifest the conformal symmetry group of the sphere.

In this point of view, the conformal sphere S^n is described as the 'celestial sphere' in an $n+2$ dimensional spacetime, i.e., the projective light-cone of $\mathbb{R}^{n+1,1}$, an $n+2$ dimensional vector space equipped with the quadratic form $(v,v) = -v_0^2 + v_1^2 + \cdots + v_{n+1}^2$. To be precise, we denote the light-cone $\mathcal{L} = \{v \in \mathbb{R}^{n+1,1} : (v,v) = 0, v \neq 0\}$ and $P(\mathcal{L}) = \mathcal{L}/\mathbb{R}^\times$ its projectivisation (the space of null lines in $\mathbb{R}^{n+1,1}$). There is a 'tautological' line bundle Λ over $P(\mathcal{L})$, whose fibre over U (a point in $P(\mathcal{L})$ and thus a null line in $\mathbb{R}^{n+1,1}$) is the line U itself. Note that $\mathcal{L}$ itself is a fibre subbundle of Λ given by removing the zero section of Λ. By choosing a time orientation of $\mathbb{R}^{n+1,1}$ we distinguish the future lightcone $\mathcal{L}^+ \subset \mathcal{L} \subset \Lambda$. As the following diagram suggests, it is straightforward to see that $P(\mathcal{L})$ is diffeomorphic to an n-sphere.

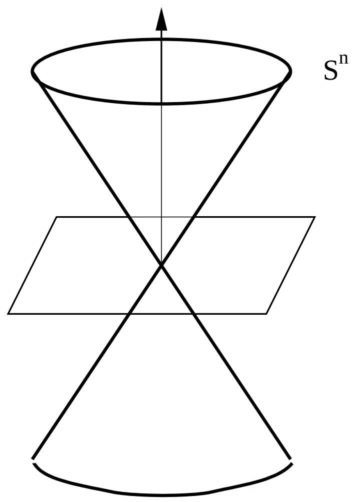

Figure 1. The conformal sphere as the projective lightcone.

The positive sections σ of Λ, i.e., those valued in $\mathcal{L}^+$, will be called gauges. Let us choose such a gauge $\sigma\colon P(\mathcal{L}) \to \mathcal{L}^+$. If d denotes the flat derivative on the trivial bundle $\underline{\mathbb{R}}^{n+1,1} := P(\mathcal{L}) \times \mathbb{R}^{n+1,1}$ then $d\sigma$ maps $TP(\mathcal{L})$ to $\Lambda^\perp$. This is because $(\sigma,\sigma) = 0$ implies that $(d_X\sigma, \sigma) = 0$ for all $X \in TP(\mathcal{L})$, and σ generates Λ. If we pull back Λ and $\Lambda^\perp$ along the bundle projection $p\colon \mathcal{L} \to TP(\mathcal{L})$, we see that $T\mathcal{L}$ is isomorphic to $p^*\Lambda^\perp$ and the tangent map dp projects this onto $TP(\mathcal{L})$ with $p^*\Lambda$ being the kernel.

Thus $d\sigma \mod \Lambda\colon TP(\mathcal{L}) \to \Lambda^\perp/\Lambda$ is an isomorphism, algebraic linear in σ, that gives us a canonical isomorphism $\Lambda \otimes TP(\mathcal{L}) \cong \Lambda^\perp/\Lambda$. The bundle on the right side inherits a metric from $\underline{\mathbb{R}}^{n+1,1}$ and it is easy to see that this metric is positive definite. We have therefore constructed, in a canonical way, a metric on $\Lambda \otimes TP(\mathcal{L})$, i.e., a positive definite section of $S^2T^*P(\mathcal{L})\otimes\Lambda^{-2}$. We refer to such a section as a *conformal metric*: we can regard it as a metric on $TP(\mathcal{L})$ with values in a line bundle.

In more concrete terms, a choice of gauge σ allows us to define a riemannian metric g_σ in the usual sense by $g_\sigma(X, Y) = (d_X\sigma, d_Y\sigma)$. If we rescale σ by a positive function f, we find that $g_{f\sigma} = f^2 g_\sigma$, since $(\sigma,\sigma) = 0$ and hence also $(d\sigma,\sigma) = 0$. Thus $P(\mathcal{L})$ is equipped naturally with a conformal equivalence class of riemannian metrics, which are just the contractions of the conformal metric with σ^2 for gauges σ. By considering conic sections of $\mathcal{L}$ (i.e., those gauges defined by hyperplane sections of the positive light-cone) we see that the round metric on S^n belongs to this conformal class. Thus $P(\mathcal{L})$ is an n-sphere with its usual conformal structure. The advantage of working with conformal metrics rather than conformal equivalence classes is that the former are tensorial objects which can be more easily manipulated, differentiated, etc.

Let $O_+(n + 1, 1)$ be the group of orthogonal transformations of $\mathbb{R}^{n+1,1}$ which preserve the time orientation. This group preserves the light cone $\mathcal{L}$ and maps null lines to null lines. Since the conformal metric on $P(\mathcal{L})$ was constructed in a manifestly invariant way, it follows that $O_+(n + 1, 1)$ acts on S^n by conformal diffeomorphisms and it is easy to see that this action is transitive. The stabilizer of a null line is a subgroup P that is in fact a parabolic subgroup of $O_+(n + 1, 1)$. Thus $P(\mathcal{L}) \cong O_+(n + 1, 1)/P$.

We refer to $O_+(n + 1, 1)$ as the group of Möbius transformations, since when $n = 1$ or 2, the identity component is isomorphic to $PSL(2, \mathbb{R})$ or $PSL(2, \mathbb{C})$ acting by fractional linear transformations on $\mathbb{R}P^1 \cong S^1$ or $\mathbb{C}P^1 \cong S^2$. For $n \geqslant 2$, $O_+(n + 1, 1)$ is the group of all global conformal diffeomorphisms of S^n, and for $n \geqslant 3$, any conformal diffeomorphism from an open subset of S^n to S^n is the restriction of a Möbius transformation. The fact that this last statement does not hold for $n = 1$, 2 means that the conformal metric does not suffice to capture the Möbius geometry of S^n in these low dimensions, a fact which will be of critical importance in the sequel.

We now want to define a conformal (or Möbius) manifold to be a 'curved version' of S^n. The above description of S^n as the projective light cone may be summarized as follows.

- We have a metric vector bundle $\underline{\mathbb{R}}^{n+1,1}$ over S^n with a flat metric connection d.
- A null line subbundle Λ of $\underline{\mathbb{R}}^{n+1,1}$ is given.
- For any nonzero section σ of Λ, $d\sigma \mod \Lambda$ is an isomorphism.

The idea, due to Cartan, is to replace the flat derivative on $\underline{\mathbb{R}}^{n+1,1}$ by a more general connection.

2.1. **Definition.** A *conformal Cartan connection* on an n-manifold M is defined by:

- the Cartan vector bundle $V \to M$ that is a rank $n + 2$ bundle with a signature $(n + 1, 1)$ metric and a metric connection $\mathcal{D}$;
- the tautological line bundle $\Lambda \to M$ that is an oriented null line subbundle of V.

The connection $\mathcal{D}$ is required to satisfy the following **Cartan condition:**

$$\pi \circ \mathcal{D}|_{\Lambda} \text{ is an isomorphism } TM \otimes \Lambda \to \Lambda^{\perp}/\Lambda,$$

where $\pi\colon V \to V/\Lambda$ is the natural projection.

Note the flavour of this definition: we have a G-bundle with a G-connection, where $G = O_+(n+1,1)$, a reduction of this structure to the parabolic subgroup P, and an open condition relating the G-connection to the reduction.

Also note that at each point $x \in M$ we can form the projective light cone in the fibre V_x, which is an n-sphere with a distinguished point Λ_x and (by the Cartan condition) an identification of the tangent space at this point with T_xM. This is the motivation behind Cartan's idea: we have replaced the 'global' n-sphere by an infinitesimal n-sphere attached to each point of M, with a connection to tell us how to 'roll' these n-spheres over M; in a sense which will become clearer when we study conformal submanifolds, these n-spheres are 'tangent' to M.

2.2. *Remark.* The Cartan vector bundle is also called the *standard tractor bundle.* The isomorphism $\pi \circ \mathcal{D}|_{\Lambda}$ (or minus this) is called the *solder form* of the connection $\mathcal{D}$, because it 'glues' V to M.. (The word 'solder' comes from electronics: it refers to the gluing of wires onto contacts, and also to the metal used to do this.) We shall generally identify $TM \otimes \Lambda$ with $\Lambda^{\perp}/\Lambda$ using $-\pi \circ \mathcal{D}|_{\Lambda}$.

The following result shows that the geometry of the conformal sphere is essentially characterized by its flat conformal Cartan connection $(\underline{\mathbb{R}}^{n+1,1}, d, \Lambda)$. The result was proven in various ways by Cartan [11], Thomas [17], Gauduchon [14], Bailey–Eastwood–Gover [1], Čap and Gover [9].

2.3. **Theorem.** *If M is equipped with a flat conformal Cartan connection, then M is locally isomorphic to $S^n = P(\mathcal{L})$—in fact there are 'developments' (local diffeomorphisms) of M into S^n unique up to Möbius transformations.*

Proof. Let $\Omega \subset M$ be simply connected, then the inclusion $(\Lambda \hookrightarrow V)|_{\Omega}$ defines a map Φ that smoothly assigns to each point x a parallel line in V_x, i.e., a point of S^n. Cartan condition ensures that Φ is a local diffeomorphism. This identifies $(V, \Lambda, \mathcal{D})$ locally with $S^n = P(\mathcal{L})$ with its standard flat conformal Cartan connection. ☐

We regard this result as a codimension 0 analogue of the Bonnet theorem in conformal geometry, since it solves the problem of when an n-dimensional manifold M can be locally immersed into S^n as a (codimension 0) conformal submanifold. Our aim is to generalize this result to submanifolds of arbitrary codimension. At the same time, we also want to understand better what it means for a manifold to have a conformal Cartan connection.

3. Conformal submanifold geometry

Let Σ be an m-dimensional manifold. We shall study its immersions into $P(\mathcal{L})$, an n-dimensional conformal sphere. Imitating previous constructions, we can form a trivial bundle $\underline{\mathbb{R}}^{n+1,1} = \Sigma \times \mathbb{R}^{n+1,1}$, d being its flat differentiation and recover the immersion from a null line subbundle $\Lambda \leqslant \underline{\mathbb{R}}^{n+1,1}$ such that $d\sigma \mod \Lambda : T\Sigma \hookrightarrow \Lambda^{\perp}/\Lambda$ is injective (a Cartan condition).

The problem now is that for $m < n$, $(\underline{\mathbb{R}}^{n+1,1}, \Lambda, d)$ is not a conformal Cartan connection: the bundle is too big. We want to attach an m-sphere to each point of Σ, not an n-sphere. We therefore seek a subbundle $V \leqslant \underline{\mathbb{R}}^{n+1,1}$ of signature $(m+1,1)$, i.e., an *m-sphere* *congruence* $\Sigma \to \{m\text{-spheres in } S^n\}$, which is a family of spheres in S^n parameterized by Σ. Equivalently we can speak about the subbundle $V^\perp \leqslant \underline{\mathbb{R}}^{n+1,1}$ of signature $(n-m, 0)$, or equivalently the map $\Sigma \mapsto \mathrm{Gr}^+_{n-m}(\mathbb{R}^{n+1,1})$ (sending x to $V_x^\perp$) into the grassmannian of spacelike $n-m$ planes in $\mathbb{R}^{n+1,1}$.

We wish this m-sphere congruence to define a conformal Cartan connection on Σ. The condition for this to hold (motivated by our intuition about conformal Cartan connections) is that the m-spheres should be tangent to Σ at corresponding points.

3.1. **Definition.** Let $\Lambda \leqslant \underline{\mathbb{R}}^{n+1,1}$ be a line subbundle defining the immersion of Σ into $P(\mathcal{L})$ and $V \leqslant \underline{\mathbb{R}}^{n+1,1}$ be a sphere congruence. We say that the sphere congruence is *enveloped* by the immersion (or that the immersion *envelopes* the sphere congruence) if $\Lambda \leqslant V$ and $d\Lambda \leqslant V$.

The second condition means that for any gauge $\sigma \in \Lambda$ and any $X \in T\Sigma$, $d_X\sigma \in V$. This is the tangency condition.

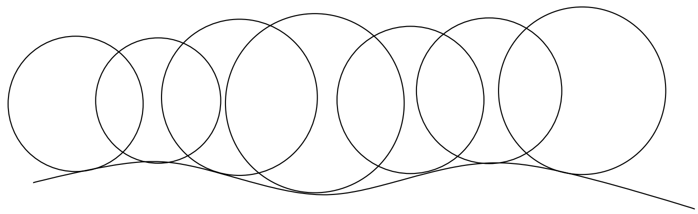

FIGURE 2. An enveloped sphere congruence.

If we have an immersion of Σ and an enveloped sphere congruence V then flat connection d on $\underline{\mathbb{R}}^{n+1,1}$ splits then in a similar way as in the euclidean submanifold theory:

$$d = \mathcal{D}^V + \nabla^V + \mathcal{N},$$

where $\mathcal{D}^V$ is a conformal Cartan connection on V, ∇^V is a metric connection on $V^\perp$, and $\mathcal{N} \in \Omega^1(\mathrm{Hom}(V, V^\perp) \oplus \mathrm{Hom}(V^\perp, V))$ with $\mathcal{N}^T = -\mathcal{N}$. Flatness of d gives us a conformal analogue of the GCR equations:

Gauss: $$R^{\mathcal{D}^V} + \mathcal{N} \wedge \mathcal{N}|_V = 0$$

Codazzi: $$d^{\mathcal{D}^V, \nabla^V} \mathcal{N} = 0$$

Ricci: $$R^{\nabla^V} + \mathcal{N} \wedge \mathcal{N}|_{V^\perp} = 0.$$

These equations are conceptually simple and provide a first attempt to characterize conformal immersions in terms of induced data on the submanifold. However, they depend essentially on the choice of V (enveloped sphere congruences are not unique!) and so cannot really be said to provide a conformally invariant characterization. Furthermore, the meaning of these equations in terms of classical differential geometry is not clear. In order to tackle these problems we shall embed conformal geometry in the larger context of parabolic geometries, and develop some machinery for studying these geometries.

4. Parabolic geometries and Lie algebra homology

A parabolic geometry is a Cartan geometry modelled on a generalized flag manifold G/P, where G is a semisimple Lie group and P is a parabolic subgroup. In this section we explain this definition and begin to explain why the geometry of Cartan connections and the algebra of generalized flag manifolds fit harmoniously together.

To begin with, let P be an arbitrary Lie subgroup of a Lie group G. Then the quotient $G \to G/P$ has a structure of a principal P-bundle which is equipped with the Maurer-Cartan form $\eta\colon TG \to \mathfrak{g}$. We want to get a 'curved version' of this over a manifold M of the same dimension as G/P.

4.1. **Definition.** A *Cartan geometry* of type $(\mathfrak{g}, P)$ on M is a principal P-bundle $\mathcal{G} \xrightarrow{\pi} M$ with a P-equivariant 1-form $\eta\colon T\mathcal{G} \to \mathfrak{g}$ such that for $u \in \mathcal{G}$, $\eta_u\colon T_u\mathcal{G} \to \mathfrak{g}$ is an isomorphism restricting to the canonical isomorphism $V_u\mathcal{G} \to \mathfrak{p}$ on the vertical bundle.

The Cartan connection $\eta\colon T\mathcal{G} \to \mathfrak{g}$, once projected onto $\mathfrak{g}/\mathfrak{p}$ becomes horizontal, and therefore induces an isomorphism $\eta_M\colon TM \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$. This is the *solder form* that glues $\mathcal{G}$ to M.

Using the associated bundle $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$, the above definition can be rephrased in a way that relates more easily to the conformal Cartan connections we have met already. Indeed:

- $\tilde{\mathcal{G}}$ is a principal G-bundle with a G-connection ω;
- $\mathcal{G} \subset \tilde{\mathcal{G}}$ is a reduction of this bundle to a principal P-bundle;

and $\eta = \omega|_{\mathcal{G}}$ satisfies the open condition that it is an isomorphism on each tangent space. The definition we gave earlier for conformal Cartan connections is really just a linear representation of the principal bundle definition.

Whether one works with principal bundles or vector bundles is a matter of taste. We prefer the latter, as we understand connections and their curvature more easily in terms of covariant derivatives on vector bundles. Therefore, let us consider a linear representation $\mathbb{W}$ of G and form the associated vector bundle $W = \tilde{\mathcal{G}} \times_G \mathbb{W}$ over M equipped with the induced linear connection (covariant derivative) $\mathcal{D}\colon C^\infty(W) \to C^\infty(T^*M \otimes W)$. Its curvature

$$R^{\mathcal{D}}_{X,Y} \cdot w = \mathcal{D}_X(\mathcal{D}_Y w) - \mathcal{D}_Y(\mathcal{D}_X w) - \mathcal{D}_{[X,Y]} w$$

is an element of $C^\infty(\wedge^2 T^*M \otimes \mathfrak{g}_M)$, i.e., a 2-form with values in the *adjoint bundle* $\mathfrak{g}_M = \mathcal{G} \times_P \mathfrak{g}$, where the Lie algebra $\mathfrak{g}$ carries the adjoint representation of G restricted to P.

The connection $\mathcal{D}$ is the first operator in the twisted deRham sequence

$$C^\infty(W) \xrightarrow{\mathcal{D}} C^\infty(T^*M \otimes W) \xrightarrow{d^{\mathcal{D}}} C^\infty(\wedge^2 T^*M \otimes W) \xrightarrow{d^{\mathcal{D}}} \cdots$$

and $(d^{\mathcal{D}})^2 = R^{\mathcal{D}} \wedge$, where $\wedge$ contains implicitly also the Lie algebra action of $\mathfrak{g}_M$ on the values of the differential forms. The curvature satisfies the Bianchi identity $d^{\mathcal{D}} R^{\mathcal{D}} = 0$.

We now turn to the class of homogeneous spaces G/P that we shall restrict attention to in the sequel: the generalized flag manifolds associated to parabolic subgroups.

4.2. **Definition.** Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{p}$ a subalgebra and $\mathfrak{p}^\perp$ its orthogonal complement with respect to the Killing form of $\mathfrak{g}$. We call $\mathfrak{p}$ *parabolic* iff $\mathfrak{p}^\perp$ is the nilradical (maximal nilpotent ideal) of $\mathfrak{p}$.

For $\mathfrak{p}$ parabolic there is a (non-canonically) split short exact sequence

$$0 \to \mathfrak{p}^\perp \to \mathfrak{p} \to \mathfrak{g}_0 \to 0 \tag{4.1}$$

where $\mathfrak{g}_0$ is reductive. We thus have $\mathfrak{p} = \mathfrak{g}_0 \ltimes \mathfrak{p}^\perp$ (depending on a choice of splitting) and $\mathfrak{p}^\perp = (\mathfrak{g}/\mathfrak{p})^*$.

For a semisimple Lie group G and a parabolic subgroup P (by which we mean the stabilizer of $\mathfrak{p}$ in the adjoint representation of G on $\mathfrak{g}$) the homogeneous space G/P is called a *generalized flag manifold.*

4.3. *Examples.* 1. The conformal sphere $S^n = P(\mathcal{L})$ is a generalized flag manifold, with $G = O_+(n+1,1)$ and $P = CO(n) \ltimes (\mathbb{R}^n)^*$. The 'flag' here is $0 \leqslant \Lambda_x \leqslant \Lambda_x^\perp \leqslant \mathbb{R}^{n+1,1}$, where $x \in S^n$ and Λ_x is the corresponding null line. G acts on $\mathbb{R}^{n+1,1}$ by matrix multiplication and P is precisely the group that preserves the flag at the identity coset $x = eP$.

2. The projective space $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$ is also a generalized flag manifold is the projective space $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$. The group that acts here is $PSL(n+1)$ and the 'flag' is $0 \leqslant l_x \leqslant \mathbb{R}^{n+1}$, where l_x is the span of v, $[v] = x \in \mathbb{R}P^n$. The subgroup that preserves this flag at the identity coset $x = eP$ is then $P = GL(n) \ltimes (\mathbb{R}^n)^*$.

The key feature of parabolic representation theory that we shall exploit in the sequel is a tool called Lie algebra (or Chevalley–Eilenberg) homology. This is the homology of a Koszul chain complex

$$0 \xleftarrow{\partial} \mathbb{W} \xleftarrow{\partial} \mathfrak{p}^\perp \otimes \mathbb{W} \xleftarrow{\partial} \wedge^2\mathfrak{p}^\perp \otimes \mathbb{W} \xleftarrow{\partial} \cdots$$

where $\partial\colon \wedge^k\mathfrak{p}^\perp \otimes \mathbb{W} \to \wedge^{k-1}\mathfrak{p}^\perp \otimes \mathbb{W}$ is given by

$$\partial(\beta \otimes w) = \sum_i \frac{1}{2}\varepsilon^i \cdot (e_i \lrcorner \beta) \otimes w + e_i \lrcorner \beta \otimes \varepsilon^i \cdot w,$$

where e_i, ε_i are elements of mutually dual bases of $(\mathfrak{p}^\perp)^*, \mathfrak{p}^\perp$ respectively. In particular $\partial w = 0$, $\partial(\alpha \otimes w) = \alpha \cdot w$. Calculation shows that this prescription ensures indeed $\partial^2 = 0$. We can verify this easily for the case that $\mathfrak{p}^\perp$ is abelian, since then

$$\partial(\beta \otimes w) = \sum e_i \lrcorner \beta \otimes \varepsilon^i \cdot w$$
$$\partial^2(\beta \otimes w) = \sum e_i \lrcorner e_j \lrcorner \beta \otimes \varepsilon^i \cdot \varepsilon^j \cdot w$$

But $e_i \lrcorner e_j \lrcorner \beta$ is skew in i, j, we get $\frac{1}{2}[\varepsilon_i, \varepsilon_j] \cdot w$ on the right, which is zero.

We therefore have Lie algebra homology groups $H_k(\mathfrak{p}^\perp, \mathbb{W}) = \mathit{ker}\,\partial / \mathit{im}\,\partial$

A *parabolic geometry* is a Cartan geometry of type $(\mathfrak{g}, P)$ with P a parabolic subgroup of a semisimple Lie group G. Dualizing the solder form η_M, we have an isomorphism $T^*M \simeq \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^* \simeq \mathcal{G} \times_P \mathfrak{p}^\perp$. Therefore the bundle associated to the P-module $\wedge^k\mathfrak{p}^\perp \otimes \mathbb{W}$ arising in the Koszul chain complex is isomorphic to the bundle $\wedge^k T^*M \otimes W$ of W-valued differential forms arising in the twisted deRham sequence.

Since the Lie algebra homology boundary operator ∂ is P-equivariant it induces a boundary operator $\wedge^k T^*M \otimes W \to \wedge^{k-1} T^*M \otimes W$ and we get a chain complex bundle

$$0 \xleftarrow{\partial} W \xleftarrow{\partial} T^*M \otimes W \xleftarrow{\partial} \Lambda^2 T^*M \otimes W \xleftarrow{\partial} \cdots \tag{4.2}$$

and homology bundles $H_k(T^*M, W) = \mathit{ker}\,\partial / \mathit{im}\,\partial$, which are isomorphic to the associated bundles $\mathcal{G} \times_P H_k(\mathfrak{p}^\perp, \mathbb{W})$.

This construction is the main reason that parabolic representation theory and Cartan geometry fit together so well. In particular, it gives rise to a normalization condition for Cartan geometries: we say that a Cartan geometry is *normal* if the curvature $R^{\mathcal{D}} \in C^\infty(\wedge^2 T^*M \otimes \mathfrak{g}_M)$ is a 2-cycle, $\partial R^{\mathcal{D}} = 0$. We shall see by example that this restricted class of Cartan geometries corresponds more closely to classical notions in differential geometry.

We shall also see that Lie algebra homology classes in $H_k(T^*M, W)$ have canonical differential representatives in $\wedge^k T^*M \otimes W$ and the curvature of a normal Cartan geometry is the differential representative of its homology class.

5. The differential lift and BGG machinery

We are now in a position of having operators in both directions between W-valued differential k and $k-1$ forms: the twisted deRham differentials and Lie algebra codifferentials. We wish to use the deRham differentials to induce differential operators $d^{\mathcal{D}}_{BGG}$ between the Lie algebra homology bundles, first constructed by Čap, Slovak and Souček [10]. To do this we shall define a differential lift and a differential projection from the Lie algebra homology bundles to the twisted deRham bundles and vice versa, so that we can define $d^{\mathcal{D}}_{BGG}$ in terms of $d^{\mathcal{D}}$.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & W & \underset{\partial}{\overset{d^{\mathcal{D}}}{\rightleftarrows}} & T^*M\otimes W & \underset{\partial}{\overset{d^{\mathcal{D}}}{\rightleftarrows}} & \Lambda^2T^*M\otimes W & \underset{\partial}{\overset{d^{\mathcal{D}}}{\rightleftarrows}} & \cdots \\
 & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\
 & & H_0(W) & \xrightarrow{d^{\mathcal{D}}_{BGG}} & H_1(W) & \xrightarrow{d^{\mathcal{D}}_{BGG}} & H_2(W) & \xrightarrow{d^{\mathcal{D}}_{BGG}} & \cdots
\end{array}$$

Let us introduce the *quabla* operator $\Box_{\mathcal{D}} = d^{\mathcal{D}}\partial + \partial d^{\mathcal{D}}$. We have the following result.

5.1. Theorem. [8] *The operator $\Box_{\mathcal{D}}$ is invertible on the image of ∂ with differential inverse.*

This theorem may seem surprising at first, but as the identity $(\mathrm{id} + d)^{-1} = \mathrm{id} - d$ shows, a differential operator can have a differential inverse. The proof is not difficult: one shows that $\Box_{\mathcal{D}}$ differs from an algebraic quabla operator due to Kostant by a nilpotent first order differential operator. Since Kostant's operator is invertible on the image of ∂, we obtain an inverse by a finite geometric series. (In fact, as stated, a mild 'regularity' condition on the curvature of the Cartan geometry is needed for this argument to work.)

It follows that $Q = \Box_{\mathcal{D}}^{-1}\partial$ is well defined and we can therefore define $\Pi := \mathrm{id} - Qd^{\mathcal{D}} - d^{\mathcal{D}}Q$. We note immediately the following properties of this operator.

5.2. Lemma.

(i) $\Pi\alpha \in \mathit{ker}\,\partial$.
(ii) $\Pi|_{\mathit{im}\,\partial} = 0$.
(iii) *For* $\alpha \in \ker\partial$, $[\Pi(\alpha)] = [\alpha]$.

Proof. We first remark that $Q\partial = 0$ and $\partial Q = 0$ (the latter because $\Box_{\mathcal{D}}$ preserves the image of ∂).

(i) $\partial\Pi = \partial - \partial Q d^{\mathcal{D}} - \partial d^{\mathcal{D}} Q = \partial - \Box_{\mathcal{D}}\Box_{\mathcal{D}}^{-1}\partial = 0$, where we used $\partial d^{\mathcal{D}} Q = \Box_{\mathcal{D}} Q$.
(ii) $\Pi\partial = \partial - Qd^{\mathcal{D}}\partial - d^{\mathcal{D}}Q\partial = \partial - \Box_{\mathcal{D}}^{-1}\Box_{\mathcal{D}}\partial = 0$, where we used $Qd^{\mathcal{D}}\partial = \Box_{\mathcal{D}}^{-1}\Box_{\mathcal{D}}\partial$.
(iii) If $\alpha \in \mathit{ker}\,\partial$, then $\Pi\alpha = \alpha + Qd^{\mathcal{D}}\alpha$ differs from α by an element of $\mathit{im}\,\partial$. ☐

If $\alpha \in \mathit{ker}\,\partial$, the operator Π can be understood as the addition of a suitable $\beta \in \mathit{im}\,\partial$ such that $\Box_{\mathcal{D}}\Pi\alpha = \Box_{\mathcal{D}}(\alpha + \beta) = 0$. There is only one such β since $\Box_{\mathcal{D}}$ is invertible on $\mathit{im}\,\partial$. Thus $\Pi\alpha$ gives a canonical differential lift of the homology class $[\alpha]$.

We can now define the linear differential operators $d^{\mathcal{D}}_{BGG}\colon C^\infty(H_k(W)) \to C^\infty(H_{k+1}(W))$ of [10] by $d^{\mathcal{D}}_{BGG}[\alpha] = [\Pi d^{\mathcal{D}}\Pi\alpha]$. Moreover the wedge product on forms together with any G-equivariant map $\phi\colon \mathbb{W}_1 \otimes \mathbb{W}_2 \to \mathbb{W}_3$ gives rise to the bilinear differential operators

$\mathrm{C}^\infty(H_k(W_1)) \times \mathrm{C}^\infty(H_l(W_2)) \to \mathrm{C}^\infty(H_{k+l}(W_3))$ of [8] by $[\alpha] \sqcup [\beta] = [\Pi(\Pi\alpha \wedge \Pi\beta)]$, where $\wedge$ contains implicitly also the operation ϕ on the values of α and β.

More generally, for any $k \geqslant 2$ and any G-equivariant map $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_k \mapsto \mathbb{W}$ there is a k-linear differential operator

$$\mu_k \colon \mathrm{C}^\infty(H_{l_1}(\mathbb{W}_1)) \times \cdots \times \mathrm{C}^\infty(H_{l_k}(\mathbb{W}_k)) \mapsto \mathrm{C}^\infty(H_l(\mathbb{W})),$$

with $l = l_1 + \cdots + l_k + 2 - k$, given by

$$\mu_k(A_1, \dots A_k) = [\Pi\lambda_k(\Pi A_1, \dots \Pi A_k)],$$

where λ_k is defined inductively by

$$\lambda_k(\alpha_1, \dots \alpha_k) = \sum_{\substack{i+j=k\\ i,j\geqslant 1}} Q\lambda_i(\alpha_1, \dots \alpha_i) \wedge Q\lambda_j(\alpha_{i+1}, \dots \alpha_k)$$

and we set (formally) $Q\lambda_1 = -\,\mathrm{id}$. For instance

$$\begin{aligned} \lambda_2(\alpha_1, \alpha_2) &= \alpha_1 \wedge \alpha_2 \\ \lambda_3(\alpha_1, \alpha_2, \alpha_3) &= Q(\alpha_1 \wedge \alpha_2) \wedge \alpha_3 + \alpha_1 \wedge Q(\alpha_2 \wedge \alpha_3) \end{aligned}$$

from which we recover the bilinear operators $[\alpha_1] \sqcup [\alpha_2] = [\Pi(\Pi\alpha_1 \wedge \Pi\alpha_2)]$ and trilinear operators $\langle [\alpha_1], [\alpha_2], [\alpha_3] \rangle = [\Pi(Q(\Pi\alpha_1 \wedge \Pi\alpha_2) \wedge \Pi\alpha_3 + \Pi\alpha_1 \wedge Q(\Pi\alpha_2 \wedge \Pi\alpha_3))]$.

6. The A_∞ equation

In this section, we present the main piece of machinery that reduces the question of the flatness of a connection to a homological equation.

Suppose that the Cartan connection $\mathcal{D}$ is *normal*, i.e., $\partial R^{\mathcal{D}} = 0$ and so, since $d^{\mathcal{D}} R^{\mathcal{D}} = 0$, we have $\Box_{\mathcal{D}} R^{\mathcal{D}} = 0$ and $R^{\mathcal{D}}$ is the differential lift of its homology class $K^{\mathcal{D}} := [R^{\mathcal{D}}]$. Taking this homology class together with the multilinear operators defined above, we have a family of k-linear operators associated to G-equivariant maps $\mathbb{W}_1 \otimes \cdots \otimes \mathbb{W}_k \mapsto \mathbb{W}$:

$$\mu_k(A_1, \dots A_k) = \begin{cases} K^{\mathcal{D}} = [R^{\mathcal{D}}] & k = 0 \\ d^{\mathcal{D}}_{BGG} A_1 = [\Pi d^{\mathcal{D}} \Pi A_1] & k = 1 \\ A_1 \sqcup A_2 = [\Pi(\Pi A_1 \wedge \Pi A_2)] & k = 2 \\ [\Pi\lambda_k(\Pi A_1, \dots \Pi A_k)] & k \geqslant 2 \end{cases}$$

where we take $\mathbb{W} = \mathfrak{g}$ for $k = 0$.

These operators formally satisfy the identities of an A_∞-algebra with curvature (zero order term). The first few such identities read

$$\begin{aligned} d^{\mathcal{D}}_{BGG} K^{\mathcal{D}} &= 0 \\ d^{\mathcal{D}}_{BGG} d^{\mathcal{D}}_{BGG} A &= K^{\mathcal{D}} \sqcup A \\ d^{\mathcal{D}}_{BGG}(A_1 \sqcup A_2) &= d^{\mathcal{D}}_{BGG} A_1 \sqcup A_2 \pm A_1 \sqcup d^{\mathcal{D}}_{BGG} A_2 \\ &\quad + \langle K^{\mathcal{D}}, A_1, A_2 \rangle \pm \langle A_1, K^{\mathcal{D}}, A_2 \rangle \pm \langle A_1, A_2, K^{\mathcal{D}} \rangle, \end{aligned}$$

where the signs depend on the parities of the homology classes in a complicated way that can only be remembered by understanding the statement that an A_∞-algebra is a differential coderivation on the tensor coalgebra (with shifted grading). We shall not need to know the signs here and refer to [8] for further information.

Suppose now that $\mathcal{D}+\alpha$ a new Cartan connection, where $\alpha \in C^\infty(T^*M \otimes \mathfrak{g}_M)$ with $\partial\alpha = 0$. Its curvature is $R^{\mathcal{D}} + d^{\mathcal{D}}\alpha + \alpha\wedge\alpha$. When does this vanish, i.e., when is the new Cartan connection flat?

We first observe that if $\mathcal{D}+\alpha$ is flat, it is certainly normal, and this implies that

$$0 = \partial d^{\mathcal{D}}\alpha + \partial(\alpha\wedge\alpha) = \Box_{\mathcal{D}}\alpha + \partial(\alpha\wedge\alpha).$$

We claim that

$$\alpha + Q(\alpha\wedge\alpha) = \Pi\alpha. \tag{6.1}$$

Indeed, since $\Box_{\mathcal{D}}$ is invertible on $im\,\partial$ and $Q = \Box_{\mathcal{D}}^{-1}\partial$, both sides express the only lift of $A = [\alpha]$ that is in $ker\,\Box_{\mathcal{D}}$. We can solve this equation formally in terms of an infinite series

$$\alpha = -\sum_{m=1}^{\infty} Q\lambda_m(\Pi A, \dots \Pi A) = \Pi A - \sum_{m=2}^{\infty} Q\lambda_m(\Pi A, \dots \Pi A).$$

In good situations, this formal series is finite, and is the unique solution to (6.1). To check this formal solution, we substitute to obtain

$$\alpha\wedge\alpha = \sum_{m=2}^{\infty}\sum_{j+k=m} Q\lambda_j(\Pi A, \dots \Pi A)\wedge Q\lambda_k(\Pi A, \dots \Pi A) = \sum_{m=2}^{\infty}\lambda_m(\Pi A, \dots \Pi A),$$

using the inductive definition of λ_m. Applying Q, we recover $\Pi\alpha - \alpha$ as required.

We now have the main result of this section.

6.1. **Lemma** (A_∞-equation). *Suppose that $\alpha = \Pi A - \sum_{m=2}^{\infty} Q\lambda_m(\Pi A, \dots \Pi A)$ is well defined and lies in $C^\infty(T^*M\otimes\mathfrak{p}_M) \leqslant C^\infty(T^*M\otimes\mathfrak{g}_M)$. Then $R^{\mathcal{D}+\alpha} = 0$ if and only if*

$$0 = \sum_{m=0}^{\infty}\mu_m(A, \dots A) = K^{\mathcal{D}} + d^{\mathcal{D}}_{BGG}A + A\sqcup A + \cdots,$$

which is called the A_∞-equation.

Proof. We compute

$$\begin{aligned} R^{\mathcal{D}} + d^{\mathcal{D}}\alpha + \alpha\wedge\alpha &= R^{\mathcal{D}} + d^{\mathcal{D}}\Pi A + (\mathrm{id} - d^{\mathcal{D}}Q)\sum_{m=2}^{\infty}\lambda_m(\Pi A, \dots \Pi A) \\ &= R^{\mathcal{D}} + d^{\mathcal{D}}\Pi A + (\Pi + Qd^{\mathcal{D}})\sum_{m=2}^{\infty}\lambda_m(\Pi A, \dots \Pi A) \end{aligned}$$

where we first use the expression for α, then the definition of Π. $R^{\mathcal{D}} + d^{\mathcal{D}}\Pi A$ differs from $\Pi R^{\mathcal{D}} + \Pi d^{\mathcal{D}}\Pi A$ only by an element of $im\,\partial$. The term $Qd^{\mathcal{D}}\sum\lambda_m(\Pi A, \dots \Pi A)$ lies in $im\,\partial$ as well. Thus $R^{\mathcal{D}} + d^{\mathcal{D}}\alpha + \alpha\wedge\alpha$ differs from the differential lift

$$\Pi R^{\mathcal{D}} + \Pi d^{\mathcal{D}}\Pi A + \Pi\sum_{m=2}^{\infty}\lambda_m(\Pi A, \dots \Pi A)$$

of $\sum_{m=0}^{\infty}\mu_m(A, \dots A)$ by an element in $im\,\partial$.

We conclude that the A_∞-equation $\sum_{m=0}^{\infty}\mu_m(A, \dots A) = 0$ is equivalent to the condition that $R^{\mathcal{D}+\alpha} \in im\,\partial$. Now since $d^{\mathcal{D}+\alpha}R^{\mathcal{D}+\alpha} = 0$ by the Bianchi identity, $R^{\mathcal{D}+\alpha}$ is an element of $ker\,\Box^{\mathcal{D}+\alpha}$, in other words $R^{\mathcal{D}+\alpha} = \Pi[R^{\mathcal{D}+\alpha}]$. But $R^{\mathcal{D}+\alpha} \in im\,\partial \Leftrightarrow [R^{\mathcal{D}+\alpha}] = 0 \Leftrightarrow R^{\mathcal{D}+\alpha} = 0$. We must be careful in this argument, however, since we need the solder forms of $\mathcal{D}$ and $\mathcal{D}+\alpha$ to agree, in order that ∂ does not change. Recall that the soldering isomorphism

$\eta_M\colon TM \to \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ is induced by $\eta\colon T\mathcal{G} \to \mathfrak{g}$ by passing to the quotient $\mathfrak{g} \mapsto \mathfrak{g}/\mathfrak{p}$. If α takes values in $\mathfrak{p}_M \leqslant \mathfrak{g}_M$, as assumed, then the solder form does not change. □

This conditions on α are not very natural, and may be too strong in some examples. It would be better to obtain such a result assuming only that A has 'negative geometric weight'. We shall meet this issue again later, but exploring it here would take us beyond our present understanding.

7. Conformal geometry revisited

Let $(V, \mathcal{D}, \Lambda)$ be a conformal Cartan connection on M, i.e., V is vector bundle over M with an $(n+1,1)$ metric $(\cdot,\cdot)$ and metric connection $\mathcal{D}$, and Λ is a null line subbundle satisfying the Cartan condition

$$\pi \circ \mathcal{D}|_\Lambda : TM \otimes \Lambda \mapsto \Lambda^\perp/\Lambda \text{ is an isomorphism.}$$

Moreover, let us define:

- $\mathfrak{so}(V)$ — the Lie algebra bundle of $(\cdot,\cdot)$-skew endomorphisms;
- $\mathfrak{stab}(\Lambda)$ — the bundle whose fibre at x is the parabolic subalgebra of $\mathfrak{so}(V)$ that stabilizes Λ_x;
- $\mathfrak{stab}(\Lambda)^\perp \cong T^*M$ — the bundle whose fibres are the (abelian) nilradicals of the fibres of $\mathfrak{stab}(\Lambda)$.

We note that there are canonically given filtrations

$$0 \leqslant \Lambda \leqslant \Lambda^\perp \leqslant V$$
$$0 \leqslant \mathfrak{stab}(\Lambda)^\perp \leqslant \mathfrak{stab}(\Lambda) \leqslant \mathfrak{so}(V)$$

and cofiltrations

$$V \xrightarrow{\pi} V/\Lambda \xrightarrow{\pi} V/\Lambda^\perp \to 0$$
$$\mathfrak{so}(V) \to \mathfrak{so}(V)/\mathfrak{stab}(\Lambda)^\perp \to \mathfrak{so}(V)/\mathfrak{stab}(\Lambda) \to 0.$$

It is convenient to split these filtrations and cofiltrations by splitting $V = \hat{\Lambda} \oplus V_0 \oplus \Lambda$, so that $\Lambda^\perp = V_0 \oplus \Lambda$. Such a splitting is called a *Weyl structure* and can be given either by choosing any null line subbundle $\hat{\Lambda}$ distinct from Λ or by choosing V_0 a complement to Λ in $\Lambda^\perp$. The action on V (and splitting) of $\mathfrak{so}(V)$ can be then described as

$$\begin{pmatrix} \lambda & p & 0 \\ -q & A & p^T \\ 0 & -q^T & -\lambda \end{pmatrix} \begin{pmatrix} w \\ v \\ \hat{w} \end{pmatrix},$$

where $w \in \Lambda$, $v \in V_0$, $\hat{w} \in \hat{\Lambda}$ and $p \in T^*M$, $q \in TM$, $A \in \mathfrak{so}(V_0)$, $\lambda \in \mathbb{R}$ and the inner product on V is $(u_1, u_2) = (v_1, v_2) - (w_1, \hat{w}_2) - (w_2, \hat{w}_1)$ where we have written $u_i = (w_i, v_i, \hat{w}_i)$ with respect to the direct sum decomposition. The parabolic subalgebra $\mathfrak{stab}(\Lambda)$ contains matrices with $q = 0$, while the elements of $\mathfrak{stab}(\Lambda)^\perp$ moreover satisfy $A = 0$, $\lambda = 0$.

The splitting of $\mathfrak{so}(V)$ gives also a splitting of the connection $\mathcal{D}$:

$$\mathcal{D} = r^{D,\mathcal{D}} + D^{\mathcal{D}} + \mathrm{id}$$

where $r^{D,\mathcal{D}}$ is a T^*M valued 1-form, $D^{\mathcal{D}}$ a $\mathfrak{co}(TM)$ connection and id is the identity, viewed as a TM-valued 1-form (this is really the solder form). The curvature of $\mathcal{D}$ splits in a similar way

$$\begin{aligned} R^{\mathcal{D}} &= R^{D^{\mathcal{D}}} + d^{D^{\mathcal{D}}}(r^{D,\mathcal{D}} + \mathrm{id}) + (r^{D,\mathcal{D}} + \mathrm{id}) \wedge (r^{D,\mathcal{D}} + \mathrm{id}) \\ &= d^{D^{\mathcal{D}}} r^{D,\mathcal{D}} + (R^{D^{\mathcal{D}}} + [r^{D,\mathcal{D}} \wedge \mathrm{id}]) + d^{D^{\mathcal{D}}}\,\mathrm{id}, \end{aligned}$$

where we think of $r^{D,\mathcal{D}}$ and id as matrix valued 1-forms and define $(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \beta(Y)\alpha(X) = [\alpha(X), \beta(Y)]$ and $[\alpha \wedge \beta](X,Y) = [\alpha(X), \beta(Y)] - [\beta(Y), \alpha(X)] = 2[\alpha(X), \beta(Y)]$. The terms $r^{D,\mathcal{D}} \wedge r^{D,\mathcal{D}}$ and $\mathrm{id} \wedge \mathrm{id}$ vanish, because $T^*M, TM \leqslant \mathfrak{so}(V)$ are (bundles of) abelian subalgebras.

The last term $d^{D^{\mathcal{D}}}\,\mathrm{id} = \pi^2 R^{\mathcal{D}}$ is the torsion of $D^{\mathcal{D}}$. If it vanishes then $D^{\mathcal{D}}$ must be the unique torsion-free conformal connection D^c induced by $D = D^{\mathcal{D}}|_{\Lambda}$ and the conformal metric $c(X,Y) = \sigma^{-2}(\mathcal{D}_X\sigma, \mathcal{D}_Y\sigma)$ (analogous to the Levi-Civita connection of riemannian geometry).

7.1. **Proposition.** *Let us denote $L := H_0(V) = V/\Lambda^{\perp} \simeq \Lambda^*$. There is a unique linear map $j^{\mathcal{D}} : \mathrm{C}^\infty(M,L) \mapsto \mathrm{C}^\infty(M,V)$ such that*

- *$\pi^2(j^{\mathcal{D}}l) = l$, i.e., $j^{\mathcal{D}}l$ is a lift of l to $V \xrightarrow{\pi^2} V/\Lambda^{\perp}$.*
- *$\partial\mathcal{D}j^{\mathcal{D}}l := \sum \varepsilon^i \cdot \mathcal{D}_{e_i} j^{\mathcal{D}} l = 0$*

Proof. $j^{\mathcal{D}}$ is in fact an instance of one of the differential lifts Π constructed via the 'BGG machinery', since the homology class of $j^{\mathcal{D}}l$ is l (first condition) and $j^{\mathcal{D}}l$ is in $\ker \Box_{\mathcal{D}}$ (second condition)—indeed, $\partial\mathcal{D} = \Box_{\mathcal{D}}$, since $\partial = 0$ on 0-forms. We can thus either appeal the general theory, or directly calculate with a Weyl structure. Let us try the latter way. Since $\pi^2(j^{\mathcal{D}}l)$ has to be l, we have

$$\begin{aligned} j^{\mathcal{D}}l = \begin{pmatrix} \sigma \\ \theta \\ l \end{pmatrix} \implies \sum \varepsilon^i \cdot \mathcal{D}_{e_i} j^{\mathcal{D}} l &= \sum \varepsilon^i \cdot \begin{pmatrix} D_{e_i} & r^{D,\mathcal{D}}_{e_i} & 0 \\ -e_i & D^{\mathcal{D}}_{e_i} & r^{D,\mathcal{D}}_{e_i} \\ 0 & -e_i & D_{e_i} \end{pmatrix} \begin{pmatrix} \sigma \\ \theta \\ l \end{pmatrix} \\ &= \sum \begin{pmatrix} 0 & \varepsilon_i & 0 \\ 0 & 0 & \varepsilon_i^T \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} D_{e_i}\sigma + r^{D,\mathcal{D}}_{e_i}(\theta) \\ -e_i\sigma + D^{\mathcal{D}}_{e_i}\theta + r^{D,\mathcal{D}}_{e_i} l \\ D_{e_i} l - \theta_{e_i} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{tr}_c\, D^{\mathcal{D}}\theta - n\sigma + \mathrm{tr}_c\, r^{D,\mathcal{D}} l \\ Dl - \theta \\ 0 \end{pmatrix}. \end{aligned}$$

This has to be zero, so $\theta = Dl$ and $\sigma = \frac{1}{n}\,\mathrm{tr}_c(D^{\mathcal{D}}Dl + r^{D,\mathcal{D}}l)$. □

We can now define *Möbius operators*

$$\begin{aligned} \mathcal{H}^{\mathcal{D}}l &= d^{\mathcal{D}}_{BGG} l = \mathit{sym}\,\pi\mathcal{D}j^{\mathcal{D}}l = \mathrm{sym}_0(D^{\mathcal{D}}Dl + r^{D,\mathcal{D}}l) \\ \mathcal{S}^{\mathcal{D}}l &= (l \sqcup l) = (j^{\mathcal{D}}l, j^{\mathcal{D}}l) = (Dl, Dl) - \frac{2}{n}(l\triangle^D l + \mathrm{tr}_c\, r^{D,\mathcal{D}} l^2) \end{aligned}$$

where the last terms are expressions with respect to a Weyl structure. By the brackets around $l \sqcup l$ we denote that the cup product here is contracted by the pairing $(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ given by the scalar product on V.

We recall that a Cartan connection is called *normal* iff $\partial R^{\mathcal{D}} = 0$. Together with Bianchi identity $d^{\mathcal{D}}R^{\mathcal{D}} = 0$ it implies that $R^{\mathcal{D}} = \Pi K^{\mathcal{D}}$ and $d^{\mathcal{D}}_{BGG} K^{\mathcal{D}} = 0$, where $K^{\mathcal{D}} := [R^{\mathcal{D}}]$.

In the conformal case this implies in particular that $\mathcal{D}$ is torsion-free. In terms of a Weyl structure we then have

$$R^{\mathcal{D}} = d^{D^c} r^{D,\mathcal{D}} + W^{\mathcal{D}},$$

where $W^{\mathcal{D}} = R^{D^c} + [r^{D,\mathcal{D}} \wedge \mathrm{id}]$ is the 'Weyl curvature' of $\mathcal{D}$. We write here R^{D^c} instead of $R^{D^{\mathcal{D}}}$ to emphasise that the torsion-freeness of $D^{\mathcal{D}}$ identifies it with D^c. Now

$$\partial R^{\mathcal{D}} = 0 \Leftrightarrow 0 = \partial W^{\mathcal{D}} = \sum \varepsilon^i \cdot W^{\mathcal{D}}_{e_i} \in T^*M \otimes T^*M$$

This is the 'Ricci trace' of $W^{\mathcal{D}}$. For $n \geqslant 3$ this condition means that $W^{\mathcal{D}}$ is the usual Weyl curvature and $r^{D,\mathcal{D}}$ the usual normalized Ricci tensor. For $n \geqslant 4$ $K^{\mathcal{D}} = W^{\mathcal{D}}$, but for $n = 2, 3$, $W^{\mathcal{D}} = 0$ and $K^{\mathcal{D}} = [d^{D^c} r^{D,\mathcal{D}}]$. For $n = 1$ there is no curvature at all.

A normal Cartan connection is uniquely determined up to isomorphism by

- the conformal metric $c \in \mathrm{C}^\infty(\Lambda^{-2} \otimes S^2 T^*M)$, $c(X,Y)\sigma^2 = g_\sigma(X,Y) = (\mathcal{D}_X\sigma, \mathcal{D}_Y\sigma)$
- the operator $\mathcal{H}^{\mathcal{D}}$ if $n = 2$.
- the operator $\mathcal{S}^{\mathcal{D}}$ if $n = 1$.

There is also an existence result given these data. Let us explain why the extra data appear for $n = 1, 2$. If $\mathcal{E} \in T^*M \otimes T^*M \leqslant T^*M \otimes \mathfrak{so}(V)$ then $\mathcal{D} + \mathcal{E}$ has the same solder form as $\mathcal{D}$, hence the same c, and it is still torsion-free, but it is not isomorphic. Since $r^{D,\mathcal{D}+\mathcal{E}} = r^{D,\mathcal{D}} + \mathcal{E}$, we have $W^{\mathcal{D}+\mathcal{E}} = W^{\mathcal{D}} + [\mathcal{E} \wedge \mathrm{id}]$ and if we claim normality of both $W^{\mathcal{D}+\mathcal{E}}$ and $W^{\mathcal{D}}$, we get that

$$\partial[\mathcal{E} \wedge \mathrm{id}] = -(n-2)\, sym_0\, \mathcal{E} - \frac{(n-1)}{n}\, tr_c\, \mathcal{E} \cdot c - \frac{n}{2}\, alt\, \mathcal{E}$$

For $n \geqslant 3$ this implies that $\mathcal{E} = 0$, but this doesn't follow for $n = 1$ and $n = 2$.

As a by-product of this computation, we observe that for any torsion-free Cartan connection $\mathcal{D}$ there is a unique $\mathcal{E} \in im\, \partial \leqslant T^*M \otimes T^*M$ such that $\mathcal{D} + \mathcal{E}$ is normal. Here the condition that $\mathcal{E}$ lies in $im\, \partial$ places no additional constraint for $n \geqslant 3$, but it implies that $sym_0\, \mathcal{E} = 0$ for $n = 2$ and $\mathcal{E} = 0$ for $n = 1$ (when any Cartan connection is flat, therefore normal).

8. Parabolic subgeometries

Let G be a semisimple Lie group and H a subgroup which is also semisimple and suppose that P is a parabolic subgroup of G such that $Q := H \cap P$ is a parabolic subgroup of H. Then the generalized flag manifold H/Q is a submanifold of G/P and we have the following commutative diagram of principal bundles.

$$\begin{array}{ccc} H & \hookrightarrow & G \\ \downarrow Q & & P \downarrow \\ H/Q & \hookrightarrow & G/P \end{array}$$

On the level of Lie algebras we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ (so is an H-module) and $\mathfrak{q} = \mathfrak{h} \cap \mathfrak{p}$.

8.1. *Examples.* We have already encountered an instance of such a situation: the inclusion of a conformal sphere S^m into S^n. In this case we have

$$\mathfrak{g} = \mathfrak{so}(n+1,1), \qquad \mathfrak{p} = \mathfrak{co}(n) \ltimes (\mathbb{R}^n)^*$$

$$\mathfrak{h} = \mathfrak{so}(m+1,1) \times \mathfrak{so}(n-m), \qquad \mathfrak{q} = [\mathfrak{co}(m) \ltimes (\mathbb{R}^m)^*] \times \mathfrak{so}(n-m).$$

The embedding of a conformal sphere S^m into the projective space $P(\mathbb{R}^{m+1,1})$ provides another example, with

$$\begin{aligned} \mathfrak{g} &= \mathfrak{sl}(m+2,\mathbb{R}), & \mathfrak{p} &= \mathfrak{gl}(m+1,\mathbb{R}) \ltimes (\mathbb{R}^{m+1})^*, \\ \mathfrak{h} &= \mathfrak{so}(m+1,1), & \mathfrak{q} &= \mathfrak{co}(m) \ltimes (\mathbb{R}^m)^*. \end{aligned}$$

A parabolic subgeometry is a curved version of such a homogeneous inclusion between generalized flag manifolds.

$$\begin{array}{ccc} \mathcal{H} & \hookrightarrow & \mathcal{G} \\ \big\downarrow Q & & P \big\downarrow \\ \Sigma & \hookrightarrow & M \end{array}$$

We are especially interested in the case that $\mathcal{G} \to M$ is flat. Given a submanifold $\Sigma \hookrightarrow M$ we shall study reductions of $\mathcal{G}|_\Sigma$ to Q.

8.2. **Definition.** A *congruence* is a reduction $\mathcal{H}$ of $\mathcal{G}|_\Sigma$ to Q. Then $\mathcal{D}^{\mathfrak{g}}|_\Sigma = \mathcal{D}^{\mathfrak{h}} + \mathcal{N}$ with $\mathcal{N} \in C^\infty(\Sigma, T^*\Sigma \otimes \mathfrak{m}_\Sigma)$, where $\mathfrak{g}_\Sigma = \mathfrak{h}_\Sigma \oplus \mathfrak{m}_\Sigma$ and $\mathcal{D}^{\mathfrak{h}}$ is the induced H-connection. We say that the congruence is *enveloped* by Σ if $\mathcal{N}$ takes values in $\mathfrak{m}_\Sigma \cap \mathfrak{p}_\Sigma$ and that it is *normal* iff it is enveloped and $\partial\mathcal{N} = 0$.

The enveloping condition ensures that $\mathcal{D}^{\mathfrak{g}}$ and $\mathcal{D}^{\mathfrak{h}}$ induce the same solder form on Σ. If $M = G/P$ then a congruence is a family of model subgeometries (isomorphic to H/Q) parameterized by Σ and the enveloping condition amounts to a certain order of contact with Σ at each point, generalizing the enveloping condition for sphere congruences in conformal geometry.

Technical point 1: Do enveloped congruences exist? In the simplest cases this places only an open condition on Σ, but the general story is not yet clear. In fact, the enveloping condition given here is probably too strong in general (see the remark below): in particular it implies that $\mathcal{D}^{\mathfrak{h}}$ is torsion-free, which is too restrictive to cover some examples.

The gauge group $\mathcal{A} = C^\infty(\Sigma, \mathcal{G}|_\Sigma \times_{Ad} P)$ acts transitively on congruences (at least locally). However, the action is not free, nor does it preserve the class of enveloped congruences (since it does not preserve the solder form).

Technical point 2: Find a subgroup of $\mathcal{A}$ acting freely and transitively on enveloped congruences.

In order to tackle this point, it is helpful to change our viewpoint: instead of fixing the flat connection $\mathcal{D}^{\mathfrak{g}}$ and gauging the congruence, we fix the reduction $\mathcal{H}$ of $\mathcal{G}|_\Sigma$ and gauge the flat connection. This point of view has a two advantages: first, we can write $\mathcal{A} = C^\infty(\Sigma, \mathcal{H} \times_{Ad|_Q} P)$; second, the action of gauge transformations on connections is easy to compute (d changes by $g^{-1}dg$). In particular, we can ignore the subgroup $C^\infty(\Sigma, \mathcal{H} \times_{Ad} Q)$, since this just gauges the induced H-connection.

In good examples the subgroup we seek will be the exponential image of a subspace $\mathfrak{m}_0$ of $\mathfrak{m}_\Sigma$ lying in a nilpotent (perhaps even abelian) Lie subalgebra of $\mathfrak{p}_\Sigma$.

8.3. **Proposition.** *In good examples, normal congruences are unique modulo elements of $H_0(T^*\Sigma, \mathfrak{m}_\Sigma)$ represented by 0-chains in $\mathfrak{m}_0$. Moreover any enveloping congruence can be normalized by a gauge transformation.*

To show this (somewhat vague) claim, we compute the change of $\mathcal{D}^{\mathfrak{g}}$ by the action of $g = \exp\gamma$, which is

$$g^{-1}\mathcal{D}^{\mathfrak{g}}g = \sum \frac{(-1)^n}{(n+1)!}(ad\gamma)^n \cdot \mathcal{D}^{\mathfrak{g}}\gamma.$$

Let us suppose (for simplicity) that $\mathfrak{m}_0$ lies in an abelian subalgebra of $\mathfrak{p}$.

$$\mathcal{N} \to \mathcal{N} + \mathcal{D}^{\mathfrak{h}}\gamma$$
$$\partial\mathcal{N} \to \partial\mathcal{N} + \partial\mathcal{D}^{\mathfrak{h}}\gamma$$

Thus the first of the claims follows from $\partial\mathcal{D}^{\mathfrak{h}}\gamma = 0 \Leftrightarrow \gamma = \Pi[\gamma]$. We also see that we can always normalize by $\gamma = -\Box_{\mathcal{D}}^{-1}\partial\mathcal{N} \in im\,\partial$.

In the non-abelian case we must work our way down the central descending series of a Lie subalgebra of $\mathfrak{p}_\Sigma$ containing $\mathfrak{m}_0$.

The final step is to normalize $\mathcal{D}^{\mathfrak{h}}$. Since this connection is torsion-free, one can show that there is a section $\mathcal{E} \in im\,\partial \subset \mathrm{C}^\infty(\Sigma, T^*\Sigma \otimes \mathfrak{h}_\Sigma)$ such that $\mathcal{D} = \mathcal{D}^{\mathfrak{h}} + \mathcal{E}$ is a normal Cartan connection (i.e., $\partial R^{\mathcal{D}} = 0$).

Technical point 3*:* We actually need $\mathcal{E}$ to be $\mathfrak{q}_\Sigma$-valued in order to preserve the solder form. This is automatically satisfied if $\mathfrak{q}_\Sigma^\perp$ is abelian or 2-step nilpotent.

Thus, up to technical considerations and a subspace of $H_0(T^*\Sigma, \mathfrak{m}_\Sigma)$ we have a canonical decomposition

$$\mathcal{D}^{\mathfrak{g}} = \mathcal{D} + \mathcal{N} + \mathcal{E}$$
$$\text{with } \partial(\mathcal{N} + \mathcal{E}) = 0 \text{ and } [\mathcal{N}] = [\mathcal{N} + \mathcal{E}]$$

A slight generalization of the arguments of section 6 shows that the flatness of $\mathcal{D}^{\mathfrak{g}}$ is then equivalent to the A_∞ equation

$$0 = K^{\mathcal{D}} + d^{\mathcal{D}}_{BGG}[\mathcal{N}] + [\mathcal{N}] \sqcup [\mathcal{N}] + \langle [\mathcal{N}], [\mathcal{N}], [\mathcal{N}] \rangle + \cdots$$

(The fact that $\mathcal{D}^{\mathfrak{g}}$ is not an H-connection does not cause a problem.)

In the special case of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ (i.e., G/H is a symmetric space) this equation splits as

Gauss–Ricci: $\quad 0 = K^{\mathcal{D}} + [\mathcal{N}] \sqcup [\mathcal{N}] + \cdots$

Codazzi: $\quad 0 = d^{\mathcal{D}}_{BGG}[\mathcal{N}] + \langle [\mathcal{N}], [\mathcal{N}], [\mathcal{N}] \rangle + \cdots$

More generally, we can still split the flatness condition into $\mathfrak{h}$ and $\mathfrak{m}$ components, but after the leading terms, multilinear operators of arbitrary degree could appear in both equations.

8.4. *Remark.* Imposing conditions on the values of differential forms is not natural in parabolic geometries as soon as the nilradical $\mathfrak{q}^\perp$ is non-abelian. Instead, there are natural filtrations of Q-modules by a 'homogeneity' or 'geometric weight', given by the action of a 'grading element', and it is better to impose conditions that the geometric weight of a differential form is negative or non-positive rather than imposing conditions on its values. For instance, a Cartan connection is said to be *regular* if the curvature has negative geometric weight, but this is a weaker assumption than the torsion-free condition (which is a condition on the values of the curvature). Similarly, it may be more natural to define enveloped congruences to be those for which $\mathcal{N}$ has negative geometric weight. This condition means that the 'graded' solder forms of $\mathcal{D}^{\mathfrak{g}}$ and $\mathcal{D}^{\mathfrak{h}}$ agree, rather than the solder forms themselves.

However, the development of this more natural theory requires considerable work, which would take us too far afield to discuss here, and is still in progress.

9. Conformal submanifold geometry revisited

The model for the study of conformal submanifolds is the map $S^m \hookrightarrow S^n$ between conformal spheres induced by an isometric inclusion $\mathbb{R}^{m+1,1} \hookrightarrow \mathbb{R}^{n+1,1}$. We then have $\mathfrak{g} = \mathfrak{so}(n+1,1)$, $\mathfrak{h} = \mathfrak{so}(m+1,1) \times \mathfrak{so}(n-m)$, and the orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$ is $\mathfrak{m} = \mathfrak{so}(n+1,1) \cap (\mathrm{Hom}(\mathbb{R}^{m+1,1}, \mathbb{R}^{n-m}) \oplus (\mathrm{Hom}(\mathbb{R}^{n-m}, \mathbb{R}^{m+1,1}))$. The parabolic subalgebras are $\mathfrak{p} = \mathfrak{co}(n) \ltimes \mathbb{R}^{n*}$ and $\mathfrak{q} = \mathfrak{co}(m) \ltimes \mathbb{R}^{m*}$.

A smooth map $\Sigma^m \hookrightarrow S^n$ is equivalently given by a null line subbundle $\Lambda \leqslant \underline{\mathbb{R}}^{n+1,1} := \Sigma \times \mathbb{R}^{n+1,1}$ and the map is an immersion if and only if Λ satisfies a Cartan condition with respect to the flat derivative on $\underline{\mathbb{R}}^{n+1,1}$.

A congruence is a reduction of the structure group of $\underline{\mathbb{R}}^{n+1,1}$ to H which is compatible with the reduction to P given by Λ. Such a congruence is given simply by an orthogonal direct sum decomposition $\underline{\mathbb{R}}^{n+1,1} = V \oplus V^\perp$, where V is a $(m+1,1)$-signature bundle and $\Lambda \subset V$. Then $d = (\mathcal{D}^V + \nabla^V) + \mathcal{N}^V$, where the H-connection $\mathcal{D}^V + \nabla^V$ is a sum of metric connections on V and $V^\perp$, and $\mathcal{N}^V$ is a 1-form valued in the bundle associated to $\mathfrak{m}$.

The enveloping condition reduces to $\mathcal{N}^V_X \sigma = 0$ $(\forall \sigma \in \Lambda, X \in T\Sigma)$.

Enveloped sphere congruences always exist: they are given by complements $I_V : N\Sigma \otimes \Lambda \cong V^\perp \hookrightarrow \Lambda^\perp$ to Λ such that $I_V \mod \Lambda$ is the natural inclusion $N\Sigma \otimes \Lambda \hookrightarrow \Lambda^\perp/\Lambda = (T\Sigma \oplus N\Sigma) \otimes \Lambda$. Enveloped sphere congruences form an affine space modelled on $\mathrm{C}^\infty(N^*\Sigma)$: for $\gamma \in \mathrm{C}^\infty(N^*\Sigma)$, the enveloped sphere congruence $V + \gamma$ is defined by $I_{V+\gamma}(\theta) = I_V(\theta) - \gamma(\theta)$. In fact this is the action of the gauge transformation $\exp(\gamma)$ on reductions.

We must now compute the condition $\partial \mathcal{N} = 0$ for an enveloped sphere congruence to be normal. For this it suffices to compute $\partial \mathcal{N}(v) = -\sum_i \mathcal{N}^V_{e_i}(\varepsilon_i \cdot v)$ for $v \in V$. Observe that $\varepsilon_i \cdot v$ lies in $\Lambda^\perp$. Now, since $\mathcal{N}^V$ vanishes on Λ, its restriction to $\Lambda^\perp$ is determined by a map $T\Sigma \otimes \Lambda^\perp/\Lambda \to V^\perp$, or equivalently, using the solder form, by a map $\mathrm{II}^V : T\Sigma \otimes T\Sigma \to N\Sigma$. This is the 'second fundamental form' of V. It is easy to see that II^V is symmetric and also that

- $\partial \mathcal{N}^V(v) = tr_c\, \mathrm{II}^V \pi^2(v)$,
- $\mathrm{II}^{V+\gamma} = \mathrm{II}^V + c \otimes \gamma^\sharp$.

As a consequence, there is a unique enveloped sphere congruence with tracefree second fundamental form. This is our 'normal' sphere congruence. In fact it turns out to be the 'central' sphere congruence of Blaschke and Thomsen [2] (the unique sphere congruence such that the sphere at $x \in \Sigma$ has the same mean curvature as Σ at x), and the conformal Gauss map of Bryant [3].

We shall denote the normal sphere congruence by V_Σ and write $\mathcal{N}$ for $\mathcal{N}^{V_\Sigma}$, ∇ for ∇^{V_Σ}. The induced Cartan connection $\mathcal{D}^{V_\Sigma}$ need not be a normal Cartan connection. However, we know that it differs from a normal Cartan connection by a unique $\mathcal{E} \in im\, \partial \leqslant \mathrm{C}^\infty(T^*\Sigma \otimes T^*\Sigma)$, so we can write $\mathcal{D}^{V_\Sigma} = \mathcal{D} + \mathcal{E}$ with $\mathcal{D}$ normal.

Thus the flat derivative on $\underline{\mathbb{R}}^{n+1,1}$ decomposes as $d = (\mathcal{D} + \nabla) + \mathcal{E} + \mathcal{N}$, and its flatness gives the following equations.

Gauss-Ricci: $$0 = R^{\mathcal{D},\nabla} + d^{\mathcal{D}}\mathcal{E} + \mathcal{N} \wedge \mathcal{N}$$

Codazzi: $$0 = d^{\mathcal{D},\nabla}\mathcal{N} + \mathcal{E} \wedge \mathcal{N} - \mathcal{N} \wedge \mathcal{E}.$$

The general theory tells us that these equations are equivalent to the simpler 'homological' equations

Gauss-Ricci: $0 = [R^{\mathcal{D},\nabla}] + [\mathcal{N}] \sqcup [\mathcal{N}]$

Codazzi: $0 = d^{\mathcal{D}}_{BGG}[\mathcal{N}]$.

(Note that the A_∞ series terminates very quickly in this case.) In fact it is not hard to check the general result directly. Let us introduce a Weyl structure $V = \Lambda \oplus V_0 \oplus \hat{\Lambda}$, to compute these equations more explicitly. Then, with respect to this decomposition, we have

$$\begin{aligned} \mathcal{D} &= r^D + D + \mathrm{id} \\ \mathcal{N} &= A^D + \mathrm{I\!I}^0 \\ \mathcal{E} &= s. \end{aligned}$$

In terms of these quantities the GCR equations become

Ricci: $0 = R^\nabla + \frac{1}{2}[\mathrm{I\!I}^0 \wedge \mathrm{I\!I}^0]|_{N\Sigma}$

Gauss 1: $0 = W^\Sigma + [s \wedge \mathrm{id}] + \frac{1}{2}[\mathrm{I\!I}^0 \wedge \mathrm{I\!I}^0]|_{T\Sigma}$

Gauss 2: $0 = C^{\Sigma,D} + d^D s + A^D \wedge \mathrm{I\!I}^0$

Codazzi 1: $0 = d^{\nabla,D}\mathrm{I\!I}^0 + A^D \wedge \mathrm{id} + c \wedge (A^D)^\sharp$

Codazzi 2: $0 = d^{\nabla,D}A^D + (r^D + s) \wedge \mathrm{I\!I}^0$

It follows from these equations that $(1-m)A^D = \mathrm{div}^{\nabla,D}\,\mathrm{I\!I}^0$ and that s is algebraic (quadratic) in $\mathrm{I\!I}^0$: in terms of the general theory $\mathrm{I\!I}^0$ is the homology class of $\mathcal{N}$ (unless $m = 1$, when A^D is the homology class and is independent of D) and we see explicitly that $\mathcal{N}$ is the differential lift of its homology class.

Only one of the Gauss and one of the Codazzi is needed in each dimension, so we finally arrive at the homological GCR equations

Gauss:	$0 = W^\Sigma + B_0(\mathrm{I\!I}^0)$	$m \geqslant 4$
	$0 = C^\Sigma + B_1^\nabla(\mathrm{I\!I}^0)$	$m = 2, 3$
Codazzi:	$0 = \mathrm{Coda}^\nabla\,\mathrm{I\!I}^0$	$m \geqslant 3$
	$0 = (\mathcal{H}^\nabla)^*(J\mathrm{I\!I}^0)$	$m = 2$
Ricci:	$0 = R^\nabla + \frac{1}{2}[\mathrm{I\!I}^0 \wedge \mathrm{I\!I}^0]\|_{N\Sigma}$	

where B_0 is a quadratic algebraic operator, B_1^∇ is a quadratic first order operator, Coda is a linear first order differential operator and $\mathcal{H}^\nabla$ is a linear second order operator.

Thus, to summarize, on a conformal m-submanifold of S^n there is induced the following primitive data:

- A conformal metric c and, for $m = 2$ a tracefree conformal Hessian $\mathcal{H}$ or for $m = 1$ a real projective structure $\mathcal{S}$.
- A weightless normal bundle $N\Sigma \otimes \Lambda$ with a metric and metric connection ∇.
- A tracefree second fundamental form $\mathrm{I\!I}^0$ (for $m \geqslant 2$) or a conformal acceleration A (for $m = 1$).

These data satisfy the homological GCR equations.

Conversely given these data we construct a normal Cartan connection $(\Lambda, V, \mathcal{D})$ from $(c, \mathcal{H}$ or $\mathcal{S})$, a shape operator $\mathcal{N} = \Pi(\mathrm{II}^0$ or $A^D)$, and a section $\mathcal{E} = s$ of $T^*\Sigma \otimes T^*\Sigma$.

Then $\mathcal{D} + \nabla + \mathcal{E} + \mathcal{N}$ is a flat connection on $V \oplus (N\Sigma \otimes \Lambda)$ and the inclusion $\Lambda \hookrightarrow V$, viewed as a map from Σ to the space of parallel null lines in $V \oplus (N\Sigma \otimes \Lambda)$ gives a local conformal immersion into S^n with these induced data, unique up to Möbius transformation.

The conformal theory of submanifolds is most interesting when $m = 2$, when the conformal metric, tracefree second fundamental form and normal connection do not suffice to determine the immersion: we also need to know the conformal Hessian operator

$$\mathcal{H} \colon \mathrm{C}^\infty(\Sigma, L) \to \mathrm{C}^\infty(\Sigma, S^2_0 T^*\Sigma \otimes L).$$

If we add a quadratic differential $q \in S^2_0 T^*\Sigma$ to $\mathcal{H}$ then the Cotton-York curvature C^Σ changes by $\operatorname{div} Jq$, which is essentially $\bar\partial q^{(2,0)}$.

9.1. **Theorem.** *$(c, \mathcal{H} + tq, \nabla, \mathrm{II}^0)$ satisfy the GCR equations for all $t \in \mathbb{R}$ if they do for $t = 0$, and q is a holomorphic quadratic differential commuting with II^0.*

Proof. The Gauss equation is $0 = C^\Sigma + t \operatorname{div} Jq + B^\nabla_1(\mathrm{II}^0)$, the Codazzi equation is $0 = \mathcal{H}^{\nabla *}(J\mathrm{II}^0) - t\langle Jq, \mathrm{II}^0\rangle$ and the Ricci equation is independent of t. Thus these hold for all t if and only if they hold for $t = 0$, $\operatorname{div} Jq = 0$ (i.e., q is holomorphic) and $\langle Jq, \mathrm{II}^0\rangle = 0$ (so, viewed as symmetric tracefree endomorphisms of $T\Sigma$, q and each normal component of II^0 are linearly dependent, hence commute). □

9.2. *Remarks.* The existence of a holomorphic quadratic differential commuting with II^0 is a modern way to say that Σ admits conformal curvature line coordinates: since q is holomorphic, there is locally a holomorphic coordinate $z = x + iy$ (so x, y are conformal coordinates) such that $q^{2,0} = dz^2$, and since q commutes with II^0 the x, y coordinates diagonalize the second fundamental form. For this reason these surfaces are said to be *isothermic*

Classical interest in isothermic surfaces arose because they admit deformations (given by the above theorem) with the same induced conformal metric and tracefree second fundamental form—the T transforms of Calapso—demonstrating that these data do not determined the immersion.

A more modern point of view is that these data induce a family of flat connections (for each t we have a flat connection on $V_\Sigma \oplus V_\Sigma^\perp$), which means they give rise to an integrable system. The rich transformation theory of isothermic surfaces (e.g., the so-called Darboux and Christoffel transformations) may be viewed as a manifestation of this integrability.

10. Sphere congruences with two envelopes

If one looks at a picture of a sphere congruence (see Figure 3), one is inclined to believe that they should admit not just one enveloping submanifold, but two, and indeed this is generically true for hypersphere congruences (codimension one). Sphere congruences with two envelopes then give rise to transformations of submanifolds: starting with an immersion of Σ in S^n, one can introduce an enveloped sphere congruence and look for the second envelope, which (if it exists) is generically another immersion of Σ in S^n.

Recall that an immersion $\Lambda \subset \underline{\mathbb{R}}^{n+1,1}$ envelopes a sphere congruence $V \subset \underline{\mathbb{R}}^{n+1,1}$ if $\mathcal{N}^V|_\Lambda = 0$. Thus a sphere congruence has two envelopes if and only if there are null line

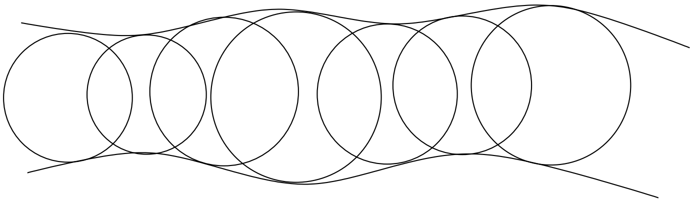

FIGURE 3. A sphere congruence with two envelopes

subbundles Λ and $\hat{\Lambda}$ in $V \subset \underline{\mathbb{R}}^{n+1,1}$ with $\mathcal{N}^V|_{\Lambda+\hat{\Lambda}} = 0$—of course for these envelopes to be immersions, Λ and $\hat{\Lambda}$ must both satisfy a Cartan condition. Note that the point of view of one of the immersions, the other immersion is just a special kind of Weyl structure (one with $A^D = 0$). Thus the theory presented here makes it easy to study sphere congruences with two envelopes.

Assuming the envelopes are disjoint, we have a direct sum decomposition $\underline{\mathbb{R}}^{n+1,1} = \Lambda \oplus V_0 \oplus \hat{\Lambda} \oplus V^\perp$, with respect to which we can decompose d as $(r^{D,V} + D + \mathrm{id}) + \nabla + \mathrm{II}^V$. Its flatness then amounts to the following equations:

$$0 = d^D id; \qquad 0 = d^D r^{D,V};$$
$$0 = W^V_{X,Y} Z + [\mathrm{II}^V_X, \mathrm{II}^V_Y]Z; \qquad 0 = R^\nabla_{X,Y}\xi + [\mathrm{II}^V_X, \mathrm{II}^V_Y]\xi;$$
$$0 = id \wedge \mathrm{II}^V; \qquad 0 = r^{D,V} \wedge \mathrm{II}^V;$$
$$0 = d^{\nabla,D}\mathrm{II}^V.$$

Using these equations, one can study special classes of sphere congruences. The sphere congruences most classical interest are the following.

- *Ribaucour sphere congruences* for which the two envelopes have the same curvature lines at corresponding points—this condition turns out to be essentially equivalent to the flatness of D on $\Lambda \oplus \hat{\Lambda}$.
- *conformal sphere congruences* for which the two envelopes have the same conformal metric at corresponding points—since $r^{D,V}$ is essentially the solder form of the second envelope, this condition may be written $r^{D,V}(r^{D,V})^T = \lambda^2\,\mathrm{id}$.

There are sphere congruences that are both conformal and Ribaucour: they exist on isothermic surfaces and give rise to the so-called Darboux transform.

There are beautiful theorems that one can prove about special classes of sphere congruence. One such theorem (of Blaschke and Thomsen, at least in codimension one) classifies 2-dimensional conformal sphere congruences with two envelopes. The two envelopes are either isothermic surfaces (forming a Darboux pair), Willmore surfaces (forming a dual pair in the sense of Bryant), or congruent surfaces in a spaceform.

11. Hypersurfaces in projective space

The model for the study of hypersurface in projective space is the immersion of a quadric into $\mathbb{R}P^{m+1}$ given by a choice of quadratic form on $\mathbb{R}^{m+2}$. For simplicity, we suppose that this quadratic form as signature $(m+1, 1)$ so that we are dealing with the conformal sphere $S^m \hookrightarrow \mathbb{R}P^{m+1}$. We then have $\mathfrak{g} = \mathfrak{sl}(m+2)$, $\mathfrak{h} = \mathfrak{so}(m+1,1)$ and $\mathfrak{m} = S^2_0\mathbb{R}^{m+1,1}$. The parabolic subalgebras are $\mathfrak{p} = \mathfrak{gl}(m+1) \ltimes (\mathbb{R}^{m+1})^*$ and $\mathfrak{q} = \mathfrak{co}(m) \ltimes \mathbb{R}^{m*}$.

Before wheeling out the Cartan geometric machinery, let us see how far we can get with more naive considerations.

If we have an immersion $\Sigma^m \hookrightarrow M = \mathbb{R}P^{m+1}$ of a hypersurface in projective space (or any projective manifold), then we have a short exact sequence of bundles along Σ

$$0 \to T\Sigma \to TM|_\Sigma \xrightarrow{p} N\Sigma \to 0, \tag{11.1}$$

where the quotient $N\Sigma$ is a line bundle, which we shall refer to as the normal bundle. Since M is a projective manifold it has an equivalence class of torsion-free connections where D and $\tilde{D}$ are equivalent if there is a 1-form γ on M such that $\tilde{D}_X Y = D_X Y + \gamma(X)Y + \gamma(Y)X$. These connections can be pulled back to give a family of connections on $TM|_\Sigma$.

For $X, Y \in T\Sigma$ let us define $\mathrm{II}(X,Y) = p(D_X Y) \in N\Sigma$. This is independent of the choice of D in the projective equivalence class since $p(X) = p(Y) = 0$. It is also symmetric, since D is torsion-free. Hence we have a symmetric bilinear form with values in a line bundle. Generically this is nondegenerate so that we have a conformal metric on Σ. Thus the conformal subgeometry (of some signature) is the generic geometry induced on a hypersurface in projective space. We shall assume for simplicity that this metric is definite (in line with our choice of model).

For $X \in T\Sigma$ and $Y \in TM|_\Sigma$, we define $\Gamma^D(X,Y) = p(D_X Y) - D_X(p(Y)) = -(D_X p)(Y)$, where we have used the fact that D induces a connection on $N\Sigma$ (more precisely it induces a connection on the pullback of $\mathcal{O}(1)$ to Σ, but the short exact sequence (11.1) and the volume form of the conformal metric on Σ identify $N\Sigma$ with the square of this line bundle). For X, Y in $T\Sigma$, we have $\Gamma^D(X,Y) = \mathrm{II}(X,Y)$, and so, by the nondegeneracy of II, the kernel of the map $Y \mapsto \Gamma^D(\cdot, Y)$ is a complement to $T\Sigma$ in $TM|_\Sigma$. In other words, the choice of D gives a splitting of (11.1) and hence a direct sum decomposition $TM|_\Sigma = T\Sigma \oplus N\Sigma$. By restriction and projection we obtain a connection $\hat{D}$ on $T\Sigma$, which is torsion-free, but not necessarily conformal, i.e., $\hat{D}\mathrm{II}$ need not be zero. We claim, however, that (with $M = \mathbb{R}P^{m+1}$)

- $\hat{D}\mathrm{II} \in T^*\Sigma \otimes S^2_0 T^*\Sigma \otimes N\Sigma$ is independent of the choice of D and is totally symmetric, i.e., a section of $S^3_0 T^*\Sigma \otimes N\Sigma$.

We therefore have a symmetric traceless cubic form $\mathcal{C}$ on Σ, called the Darboux cubic form. It is the analogue in projective geometry of the second fundamental form in euclidean geometry, being the extrinsic curvature of the hypersurface. (Note that the usual second fundamental form II is actually used to define the intrinsic geometry of the hypersurface, namely its conformal metric.)

As in conformal submanifold geometry, it is natural to ask whether the induced conformal metric and the Darboux cubic form are sufficient to recover the immersion up to projective transformation. This turns out to be true for $m \geqslant 3$, but is not true for $m = 2$, where there are surfaces in $\mathbb{R}P^3$ with the same conformal metric and Darboux cubic form which are not projectively equivalent. These are the so-called *projectively applicable surfaces.*

To see all this, let us now introduce some Cartan geometry. As in the conformal case, we prefer to take a linear point of view on Cartan connections. A projective Cartan connection on an $m+1$-manifold M is

- A rank $m+2$ bundle V with a volume form and a unimodular connection $\mathcal{D}^M$
- A line subbundle $\Lambda \subset V$ satisfying the Cartan condition that $\mathcal{D}^M|_\Lambda \mod \Lambda$ is an isomorphism $TM \otimes \Lambda \to V/\Lambda$.

We note that $\mathbb{R}P^{m+1}$ has a canonical flat projective Cartan connection: the trivial $\mathbb{R}^{m+1}$ bundle has a tautological line subbundle Λ satisfying the Cartan condition with respect the flat derivative d.

A smooth map $\Sigma^m \hookrightarrow \mathbb{R}P^{m+1}$ is therefore equivalently given by a line subbundle $\Lambda \leqslant \underline{\mathbb{R}}^{m+1} := \Sigma \times \mathbb{R}^{m+1}$ and the map is an immersion if and only if Λ satisfies a Cartan condition with respect to the flat derivative on $\underline{\mathbb{R}}^{m+1}$.

A congruence is a reduction of the structure group of $\underline{\mathbb{R}}^{m+1}$ to the orthogonal group H which is compatible with the reduction to P given by Λ. Such a congruence is given simply by a unimodular metric g on $\underline{\mathbb{R}}^{m+1}$ for which Λ is null, i.e., $g(\sigma,\sigma) = 0$ for $\sigma \in \Lambda$. Then $d = \mathcal{D}^g + \mathcal{N}^g$, where $\mathcal{D}^g$ is a metric connection and $\mathcal{N}^g$ a 1-form valued in the bundle $\mathrm{Sym}_0(\underline{\mathbb{R}}^{m+1})$ associated to $\mathfrak{m}$. In geometric terms, a congruence associates to each point of Σ a quadric meeting Σ at that point.

The enveloping condition states that $\mathcal{N}^g_X\sigma \in \Lambda$ $(\forall \sigma \in \Lambda, X \in T\Sigma)$. To unravel this definition, we first observe that it implies that $\mathcal{N}^g_X\sigma \in \Lambda^\perp$, i.e., $g(\sigma, \mathcal{N}^g_X\sigma) = 0$. Since $g(\sigma, \mathcal{D}^g\sigma) = 0$, this is equivalent to $g(\sigma, d_X\sigma) = 0$, which is precisely the condition that the quadric is tangent to Σ.

Under this tangency condition, the requirement that $\mathcal{N}^g_X\sigma$ is in Λ means (equivalently) that $g(\mathcal{N}^g_X\sigma, d_Y\sigma) = 0$ for all $Y \in T\Sigma$. Since $\mathcal{N}^g$ is symmetric and $\mathcal{D}^g$ is a metric connection, this is equivalent to the condition

$$g(d_X\sigma, d_Y\sigma) = -g(\sigma, d_X d_Y\sigma).$$

On dividing by σ^2, the left hand side is the conformal metric induced by g on $T\Sigma$, while the right hand side is $\mathrm{II}(X,Y)$. This means in fact that at any point of Σ the quadric given by g has the same second fundamental form II as Σ—in other words the quadric and Σ have second order contact at that point.

Enveloped congruences always exist. This is clear geometrically from the fact that (at a given point) a quadric can have any second fundamental form. Alternatively, one can choose a Weyl structure $\hat{\Lambda}$ and use this to define explicitly an enveloped congruence g.

Enveloped congruences of quadrics form an affine space modelled on the smooth sections of $T^*\mathbb{R}P^{m+1}|_\Sigma$. To see this, suppose g and $\hat{g}$ are both enveloped congruences of quadrics and write $\hat{g}(v,w) = g(Av,w)$ for a g-symmetric endomorphism V with determinant one. The enveloping condition for g and $\hat{g}$ implies that A preserves Λ and $\Lambda^\perp$, and that $A - \mathrm{id}$ maps $\Lambda^\perp$ into Λ. It follows that we can write $A = \exp 2\gamma = \mathrm{id} + 2\gamma + 2\gamma^2$ where γ is a g-symmetric endomorphism that annihilates Λ and maps $\Lambda^\perp$ into Λ. These nilpotent endomorphisms γ therefore act freely and transitively on enveloped congruences by gauge transformations: $\hat{g}(v,w) = g(\exp(\gamma)v, \exp(\gamma)w)$. However, one easily sees that they form an abelian Lie algebra isomorphic to $T^*\mathbb{R}P^{m+1}|_\Sigma$, so the action is actually affine.

We must now show that we can gauge an enveloped congruence g in a unique way so that they new congruence $\exp\gamma \cdot g$ has $\partial\mathcal{N}^{\exp\gamma\cdot g} = 0$. To compute the change in $\mathcal{N}^g$, we gauge the flat connection d rather than the congruence, and we find that

$$\partial\mathcal{N}^{\exp\gamma\cdot g} = \partial\mathcal{N}^g + \partial\mathcal{D}^g\gamma$$

and so $\gamma = -(\square^{\mathcal{D}^g})\partial\mathcal{N}^g$ gives the gauge transformation we want, and this is unique, as the Lie algebra we have defined is in the image of ∂.

Using the Codazzi equation $d^{\mathcal{D}^g}\mathcal{N}^g = 0$, the condition $\partial\mathcal{N}^g \in \Lambda^2$ reduces to $\mathcal{N}^g|_\Lambda = 0$, and gives rise to a family of congruences (called the Darboux family) forming an affine space modelled on sections of $N^*\Sigma \subset T^*\mathbb{R}P^{m+1}|_\Sigma$. The unique congruence with $\partial\mathcal{N}^g = 0$ is called the congruence of *Lie quadrics* and its homology class $[\mathcal{N}^g]$ belongs to $\Lambda^{-2} \otimes S^3_0 T^*\Sigma$ and can be identified with the Darboux cubic form $\mathcal{C}$.

Following the general theory, we can normalize the induced Cartan connection, and reduce the Gauss–Codazzi equations to homological form. The Gauss equation reads

$$0 = K^{\mathcal{D}} + \mathcal{C} \sqcup \mathcal{C},$$

where for $M \geqslant 4$, $K^{\mathcal{D}}$ is the Weyl curvature tensor and $\mathcal{C} \sqcup \mathcal{C}$ is an algebraic pairing, while for $m = 2, 3$, $K^{\mathcal{D}}$ is the Cotton-York tensor and $\mathcal{C} \sqcup \mathcal{C}$ is first order.

The Codazzi equation turns out to be linear:

$$0 = d^{\mathcal{D}}_{BGG}\mathcal{C}.$$

For $m \geqslant 3$ this operator is first order, but for $m = 2$ it is in fact third order. As in conformal submanifold geometry, the case $m = 2$ is also interesting because the conformal metric does not suffice to determine the normal Cartan connection: we must also deal with the Möbius operator $\mathcal{H}^{\mathcal{D}}$, too. As we indicated already, this leads to a class of surfaces which admit deformations with the same conformal metric and Darboux cubic form (but different Möbius structures), called the projectively applicable surfaces, which may be regarded as an analogue of isothermic surfaces in the projective theory. Like the isothermic surfaces, the Möbius structures of the family have the form $\mathcal{H}^{\mathcal{D}} + tq$ for a holomorphic quadratic differential q, but the condition that q commutes with II^0 in the conformal case is replaced by a first order differential relation between q and the Darboux cubic form. The fact that this relation is first order means that it is possible for there to be even a three dimensional family of deformations with the same conformal metric and cubic form. We refer the reader to Ferapontov [13], who studies surfaces in projective space in a more explicit way, for a discussion of these special surfaces.

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