TORIC SELF-DUAL EINSTEIN METRICS AS QUOTIENTS

CHARLES P. BOYER, DAVID M. J. CALDERBANK, KRZYSZTOF GALICKI, AND PAOLO PICCINNI

Abstract. We use the quaternion Kähler reduction technique to study old and new self-dual Einstein metrics of negative scalar curvature with at least a two-dimensional isometry group, and relate the quotient construction to the hyperbolic eigenfunction Ansatz. We focus in particular on the (semi-)quaternion Kähler quotients of (semi-)quaternion Kähler hyperboloids, analysing the completeness and topology, and relating them to the self-dual Einstein Hermitian metrics of Apostolov–Gauduchon and Bryant.

Introduction

There has been quite a lot of interest in self-dual Einstein (SDE) metrics in dimension four. In the negative scalar curvature case, such metrics naturally generalize the symmetric metrics on the real 4-ball $\mathcal{H}^4 \simeq \mathbb{H}\mathcal{H}^1$ (the real or quaternionic hyperbolic metric) and on the complex 2-disc $\mathbb{C}\mathcal{H}^2$ (the complex hyperbolic or Bergman metric).

A rather general construction of negative SDE metrics was offered by C. LeBrun in 1982 [LeB82]. LeBrun observed that for any real-analytic conformal structure $[h]$ on S^3, there is a Riemannian metric g_0 defined on some open neighborhood of $S^3 \subset \mathbb{R}^4$ such that g_0 is self-dual, the restriction of it to S^3 is in the conformal class $[h]$, and moreover $g = f^{-2}g_0$ is Einstein for some defining function f for S^3 in this open neighbourhood. However, this result is purely local: the Einstein metric it defines typically cannot be extended to a complete metric everywhere inside the ball.

Nevertheless, in later work [LeB91], LeBrun showed that the moduli space of negative complete SDE metrics on a ball is infinite dimensional, which led him to formulate a conjecture. A conformal structure on S^3 is said to have positive frequency if it bounds a complete SDE metric on the ball and negative frequency if bounds a complete anti-self-dual Einstein (ASDE) metric on the ball. The conjecture then asserts that near the standard conformal structure on S^3 (in an appropriate sense) the moduli spaces of positive and negative frequency subspaces are transverse (i.e., their tangent spaces at the standard conformal structure give a direct sum decomposition). The positive frequency conjecture is now proven, thanks to the remarkable work of O. Biquard [Biq02] (see also [Biq00, Biq99]). However, this still provides very little information about which conformal structures on S^3 bound complete SDE metrics on the ball.

The known examples are rather few. Apart from the hyperbolic metric, the first such metrics were obtained by H. Pedersen in [Ped86]: the conformal class $[h]$ on S^3 is represented by a Berger sphere metric $\sigma_1^2+\sigma_2^2+\lambda^2\sigma_3^2$ (where σ_1, σ_2 and σ_3 are the standard left-invariant 1-forms on $S^3 \simeq \mathrm{Sp}(1)$ and λ is a nonzero constant), and the corresponding complete SDE metric on the 4-ball is equally explicit. Later, N. Hitchin [Hit95] generalized this result by showing that any left-invariant conformal structure on S^3 determines a complete SDE metric on the ball, although now explicitness requires elliptic rather than elementary functions.

Date: November 2005.

During the preparation of this work the first and third authors were supported by NSF grant DMS-0203219. The second author was supported by the Leverhulme Trust, the William Gordon Seggie Brown Trust and an EPSRC Advanced Fellowship. The fourth author was supported by the MIUR Project "Proprietà Geometriche delle Varietà Reali e Complesse". The authors are also grateful for support from EDGE, Research Training Network HPRN-CT-2000-00101, funded by the European Human Potential Programme.

The reason that these metrics are tractible is the presence of symmetry. The real and complex hyperbolic metrics have isometry groups $SO(4,1)$ and $U(2,1)$ respectively, while the Pedersen metrics on the ball have isometry group $U(2)$. There are also (related) $U(2)$-invariant SDE metrics on complex line bundles $\mathcal{O}(n) \to \mathbb{C}P^2$, with $n \geqslant 3$, called the Pedersen–LeBrun metrics [LeB88, Hit95]. Hitchin [Hit95] actually classifies all SU(2)-invariant SDE metrics, and proves that the complete examples of negative scalar curvature consist only of the real and complex hyperbolic metrics, the Pedersen and Pedersen–LeBrun metrics, and SDE metrics on the ball associated to a left-invariant conformal or CR structure on $S^3 \simeq SU(2)$.

Recent progress on SDE metrics with symmetry concerns the much smaller symmetry group $T^2 \simeq S^1 \times S^1$ (and its non-compact forms): such SDE metrics are said to be *toric*. In the positive case, these metrics can be constructed using the Galicki–Lawson quaternion Kähler reduction [GL88] of the quaternionic projective space $\mathbb{H}P^n$ by the action of an $(n-1)$-dimensional subtorus of the maximal torus of $\mathrm{Sp}(n+1)$. Although the only positive SDE metrics on compact manifolds are the standard metrics on S^4 and $\mathbb{C}P^2$, these methods produce positive SDE metrics on compact orbifolds. The general such metrics were described by C. Boyer et al. in [BGMR98], following the construction by Galicki and Lawson of positive SDE metrics on weighted projective spaces [GL88]. It is natural to conjecture that all positive compact SDE orbifolds arise in this way: this would be similar to a related result of R. Bielawski [Bie99] stating that all toric 3-Sasakian manifolds (in any dimension) are the 3-Sasakian quotients considered in [BGMR98].

Another impetus to study toric SDE metrics comes from the recent work [CP02] of D. Calderbank and H. Pedersen, who proved that if a (positive or negative) SDE metric admits two commuting Killing vector fields, it can be expressed locally in an explicit form depending on a single function F on the upper-half plane, where F is an eigenfunction of the hyperbolic Laplacian with eigenvalue $3/4$. Conversely, any metric of this form is an SDE metric. Calderbank and Pedersen then showed explicitly how the positive SDE metrics of Galicki–Lawson and Boyer et al. arise from such an eigenfunction F, and tied together a number of examples of negative SDE metrics.

The (locally) toric SDE metrics of [CP02] also relate to a recent study by V. Apostolov and P. Gauduchon of SDE Hermitian metrics [AG02]. SDE metrics with symmetry are conformal to metrics which are Kähler with the opposite orientation (hence scalar-flat), but it is much rarer for an SDE metric to admit a Hermitian structure inducing the given orientation. Nevertheless, many of the examples of SDE metrics discussed so far are Hermitian in this sense. Other non-locally symmetric examples of SDE Hermitian metrics include cohomogeneity one metrics under the action of $\mathbb{R} \times \mathrm{Isom}(\mathbb{R}^2)$, $U(1,1)$, and $U(2)$ constructed by A. Derdziński [Der81] (the $U(2)$ case being the Pedersen–LeBrun metrics mentioned above). Apostolov and Gauduchon show, quite generally, that SDE Hermitian metrics always admit two distinguished commuting Killing vector fields, and that if the induced local $\mathbb{R}^2$ action does not have two dimensional generic orbits, then the isometry group necessarily acts transitively or with cohomogeneity one. In either case, they show that SDE Hermitian metrics are toric, hence given locally by the metrics of Calderbank and Pedersen.

The emergence of non-trivial isometries for SDE Hermitian metrics is perhaps less surprising in view of a link with recent work of R. Bryant on Bochner-flat Kähler metrics [Bry01]. In four dimensions, the Bochner tensor coincides with the anti-self-dual Weyl tensor and so Kähler metrics with vanishing Bochner tensor are just self-dual Kähler metrics. Apostolov and Gauduchon show that SDE Hermitian metrics are necessarily conformal to self-dual Kähler metrics, hence they belong to the class of metrics studied by Bryant. In his impressive paper, Bryant obtains an explicit local classification of Bochner-flat Kähler metrics and

studies in detail their global geometry. The symmetries here arise naturally from a differential system, which amounts to the realisation of Bochner-flat Kähler $2n$-manifolds as local quotients of the flat CR structure on S^{2n+1}. Bryant's work not only provides an alternative way of classifying SDE Hermitian metrics locally, but it also gives insight into the question of completeness, and he discusses some examples in an appendix to his paper.

In spite of this work (and in contrast to the case of SU(2) symmetry, where Hitchin provides a classification) the issue of completeness for negative SDE Hermitian metrics is not yet fully explored, and for the toric SDE metrics in general, the complete examples are far from understood. In fact, there are very many examples. In [CS03], Calderbank and M. A. Singer constructed examples of complete SDE metrics on resolutions of complex cyclic singularities and showed that the moduli of such metrics is (continuously) infinite dimensional. In particular these metrics can have arbitrarily large second Betti number (cf. [BGMR98] in the positive case). Examples of infinite topological type are also known.

The simplest examples in [CS03] are quaternion Kähler quotients of $\mathbb{H}\mathcal{H}^m$ generalizing the Pedersen–LeBrun metrics on $\mathcal{O}(n)$ ($n \geqslant 3$), and may be viewed as negative analogues of the compact orbifold SDE metrics of Galicki–Lawson and Boyer et al.

In fact many of the metrics discussed in this introduction occur as quaternion Kähler quotients [Gal87a, GL88]. For positive toric SDE metrics, compact orbifold examples are well understood (as we have discussed). For negative toric SDE metrics, many examples have been introduced as quotients by Galicki [Gal87b, Gal91], but the quotient approach has not been thoroughly explored. Our purpose in this work is to develop systematically the quotient approach to toric SDE metrics, which has a number of advantages. In addition to producing an abundance of examples locally, the quotient approach provides more direct insight into the global behaviour of such metrics (completeness or topology), as well as a systematic way to organise these examples into families.

In this paper we set the initial stage for such a systematic study by considering the toric SDE metrics arising as (semi-)quaternion Kähler quotients of 8-dimensional quaternionic hyperbolic space $\mathbb{H}\mathcal{H}^2$ and its indefinite signature analogue $\mathbb{H}\mathcal{H}^{1,1}$ by a one dimensional group action. A given reduction may be encoded by the adjoint orbits in $\mathfrak{sp}(1,2)$ of the generator of the action, which in turn may be classified using work of Burgoyne and Cushman [BC77]. There are essentially four distinct possible types of generator:

(i) elements belonging to the Lie algebra a maximal torus;
(ii) elements in a Cartan subalgebra with exponential image $S^1 \times \mathbb{R}$;
(iii) non-semisimple elements with two step nilpotent part;
(iv) non-semisimple elements with three step nilpotent part.

The quaternion Kähler quotients by generators in the first two classes correspond to the 3-pole solutions discussed in [CP02], but we present a detailed and self-contained analysis of the completeness and topology of the quotient. The other two classes may be regarded as limiting cases, but the geometry of the quotient is less well studied.

According to Apostolov and Gauduchon [AG02], quaternion Kähler quotients of $\mathbb{H}\mathcal{H}^2$ (and $\mathbb{H}P^2$) by one dimensional group actions are SDE Hermitian, and their argument applies also to quotients of $\mathbb{H}\mathcal{H}^{1,1}$. Therefore all of the quotients we discuss in this paper are SDE Hermitian manifolds. Furthermore, by comparing our examples with the classification of self-dual Kähler metrics by Bryant [Bry01], we see that in fact all SDE Hermitian metrics with nonzero scalar curvature are (at least locally) quaternion Kähler quotients of $\mathbb{H}P^2$, $\mathbb{H}\mathcal{H}^2$ or $\mathbb{H}\mathcal{H}^{1,1}$.

In addition to studying the quotients of $\mathbb{H}\mathcal{H}^2$ and $\mathbb{H}\mathcal{H}^{1,1}$ in detail, we develop some aspects of the general theory of quotients of $\mathbb{H}\mathcal{H}^{k,l}$ by $(k+l-1)$-dimensional Abelian semi-quaternion Kähler group actions. In particular we show how the quotient metrics are related to the hyperbolic eigenfunction Ansatz, simplifying and extending a result of [CP02].

Acknowledgments. The first author thanks the Ecole Polytechnique, Palaiseau and the Università di Roma "La Sapienza" for hospitality and support. The third author would like to thank the Università di Roma "La Sapienza", I.N.d.A.M, M.P.I-Bonn, and IHES as parts of this paper were written during his visits there. The fourth named author would like to thank University of New Mexico for hospitality and support. The authors are grateful to Paul Gauduchon, Michael Singer and Pavel Winternitz for invaluable discussions.

1. Semi-Quaternionic Projective Spaces

Definition 1.1: *Let (M^{4n}, g) be a semi-Riemannian manifold of signature $(4\nu, 4n-4\nu)$. We say that (M^{4n}, g) is semi-quaternion Kähler if the holonomy group of the metric connection is a subgroup of $\mathrm{Sp}(\nu, n-\nu)\cdot\mathrm{Sp}(1)$ when $n > 1$. As usual, when $n = 1$ we extend our definition and require that (M, g) be self-dual and Einstein. We will always suppose that the scalar curvature of (M, g) is nonzero. We refer to ν as the quaternionic index of M.*

Exactly as in the Riemannian case, the above definition implies the existence of the quaternion Kähler 4-form Ω which is parallel with respect to the Levi-Civita connection and gives rise to the quaternionic rank 3 bundle $\mathcal{V}$ over M.

The simplest example of semi-quaternion Kähler manifolds are obtained as follows. Let $\mathbb{H}^{k,l} = \{\mathbf{u} = (\mathbf{a}, \mathbf{b}) \ \mid \ \mathbf{a} = (u_0, \ldots, u_{k-1}), \ \ \mathbf{b} = (u_k, \ldots, u_{k+l})\}$ be the set of all quaternionic $(n+1)$-vectors together with the symmetric form

$$F_{k,l}(\mathbf{u}^1, \mathbf{u}^2) = -\sum_{\alpha=0}^{k-1} \bar{u}^1_\alpha u^2_\alpha + \sum_{\alpha=k}^{k+l} \bar{u}^1_\alpha u^2_\alpha = -\langle \mathbf{a}^1, \mathbf{a}^2\rangle + \langle \mathbf{b}^1, \mathbf{b}^2\rangle \tag{1.1}$$

Here $\langle \mathbf{a}^1, \mathbf{a}^2\rangle$ denotes the standard quaternionic-Hermitian inner product on $\mathbb{H}^k$ and we shall denote the associated norm by $\|\mathbf{a}\|^2 = \langle \mathbf{a}, \mathbf{a}\rangle$. The form $F_{k,l}$ defines the flat semi-Riemannian metric of signature $(4k, 4l)$ on $\mathbb{H}^{k,l}$.

Definition 1.2: *Let $\mathcal{H}_{k,l}(\epsilon) = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{H}^{k,l} \ \mid \ -\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \epsilon\}$.*

(i) *$\mathcal{H}_{k,l}(-1) \simeq S^{4k-1} \times \mathbb{H}^l$, where $k > 0$, is a semi-Riemannian submanifold of signature $(4k-1, 4l)$ called the pseudosphere.*
(ii) *$\mathcal{H}_{k,l}(+1) \simeq \mathbb{H}^k \times S^{4l-1}$, where $l > 0$, is a semi-Riemannian submanifold of signature $(4k, 4l-1)$ called the pseudohyperboloid.*
(iii) *$\mathcal{H}_{k,l}(0) \simeq S^{4k-1} \times S^{4l-1} \times \mathbb{R}/\sim$, where $k, l > 0$ and $\sim$ identifies $S^{4k-1} \times S^{4l-1} \times \{0\}$ with a point, is called the null cone.*

Let $k + l = n + 1$ and $\mathrm{Sp}(k, l) \subset \mathrm{GL}(n+1, \mathbb{H})$ which preserves the form $F_{k,l}$. It is well-known that $\mathcal{H}_{k,l}(\pm 1)$ are spaces of constant curvature and as homogeneous spaces of the semi-symplectic group $\mathrm{Sp}(k, l)$ they are

$$\mathcal{H}_{k,l}(\epsilon) = \begin{cases} \mathrm{Sp}(k,l)/\mathrm{Sp}(k,l-1) & \text{when } l > 0 \text{ and } \epsilon = -1, \\ \mathrm{Sp}(k,l)/\mathrm{Sp}(k-1,l) & \text{when } k > 0 \text{ and } \epsilon = +1. \end{cases}$$

Consider $\mathbb{H}^{k,l}_- = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{H}^{k,l} \ \mid \ \|\mathbf{a}\|^2 < \|\mathbf{b}\|^2\}$, and $\mathbb{H}^{k,l}_+ = \{(\mathbf{a}, \mathbf{b}) \in \mathbb{H}^{k,l} \ \mid \ \|\mathbf{a}\|^2 > \|\mathbf{b}\|^2\}$. Also, let us write $\mathbb{H}^{k,l}_0$ for $\mathcal{H}_{k,l}(0)$ as an alternative notation. We can then write

$$\mathbb{H}^{k,l} = \mathbb{H}^{k,l}_- \cup \mathbb{H}^{k,l}_0 \cup \mathbb{H}^{k,l}_+. \tag{1.2}$$

After removing $\mathbf{0} \in \mathbb{H}^{k,l}$ we consider the action of $\mathbb{H}^*$ on (1.2) by right multiplication.

Definition 1.3: *Let $\mathbb{H}^{k,l}$ be the quaternionic vector space with semi-hyperkähler metric of signature $(4k, 4l)$. We define the following projective spaces.*

(i) $\mathbb{H}\mathcal{H}^{k,l-1} := P_{\mathbb{H}}(\mathbb{H}^{k,l}_-) = \mathbb{H}^{k,l}_-/\mathbb{H}^* = \mathcal{H}_{k,l}(-1)/\mathrm{Sp}(1)$,
(ii) $\mathbb{H}\mathcal{H}^{k-1,l} := P_{\mathbb{H}}(\mathbb{H}^{k,l}_+) = \mathbb{H}^{k,l}_+/\mathbb{H}^* = \mathcal{H}_{k,l}(+1)/\mathrm{Sp}(1)$,

(iii) $P_{\mathbb{H}}(\mathbb{H}^{k,l}_0) = (\mathbb{H}^{k,l}_0 \setminus \{\mathbf{0}\})/\mathbb{H}^* = S^{4k-1} \times_{\mathrm{Sp}(1)} S^{4l-1}$.

If we make a choice of $\mathbb{C}^* \subset \mathbb{H}^*$ we also have complex 'projective' spaces.

Definition 1.4: *Let $\mathbb{H}^{k,l}$ be the quaternionic vector space with semi-hyperkähler metric of signature $(4k, 4l)$. Let $\mathbb{C}^* \subset \mathbb{H}^*$. We define*

(i) $P_{\mathbb{C}}(\mathbb{H}^{k,l}_-) = \mathbb{H}^{k,l}_-/\mathbb{C}^* = \mathcal{H}_{k,l}(-1)/U(1)$,
(ii) $P_{\mathbb{C}}(\mathbb{H}^{k,l}_+) = \mathbb{H}^{k,l}_+/\mathbb{C}^* = \mathcal{H}_{k,l}(+1)/U(1)$,
(iii) $P_{\mathbb{C}}(\mathbb{H}^{k,l}_0) = (\mathbb{H}^{k,l}_0 \setminus \{\mathbf{0}\})/\mathbb{C}^* = S^{4k-1} \times_{U(1)} S^{4l-1}$.

Proposition 1.5: *As homogeneous spaces of the semi-symplectic group*

$$\begin{aligned} P_{\mathbb{H}}(\mathbb{H}^{k,l}_-) &= \frac{\mathrm{Sp}(k,l)}{\mathrm{Sp}(1)\times\mathrm{Sp}(k-1,l)}, \qquad & P_{\mathbb{C}}(\mathbb{H}^{k,l}_-) &= \frac{\mathrm{Sp}(k,l)}{U(1)\times\mathrm{Sp}(k-1,l)}, \quad k>0 \\ P_{\mathbb{H}}(\mathbb{H}^{k,l}_+) &= \frac{\mathrm{Sp}(k,l)}{\mathrm{Sp}(k,l-1)\times\mathrm{Sp}(1)}, \qquad & P_{\mathbb{C}}(\mathbb{H}^{k,l}_+) &= \frac{\mathrm{Sp}(k,l)}{\mathrm{Sp}(k,l-1)\times U(1)}, \quad l>0. \end{aligned} \tag{1.3}$$

Furthermore, we have the natural fibrations

$$\begin{tikzcd} & \mathbb{H}^{k,l}_- \arrow[dl] \arrow[d] & \mathbb{H}^{k,l}_+ \arrow[d] \arrow[dr] & \\ P_{\mathbb{C}}(\mathbb{H}^{k,l}_-) \arrow[dr] & P_{\mathbb{H}}(\mathbb{H}^{k,l}_-) & P_{\mathbb{H}}(\mathbb{H}^{k,l}_+) \arrow[dl] & P_{\mathbb{C}}(\mathbb{H}^{k,l}_+) \end{tikzcd} \tag{1.4}$$

which can be glued together along the common boundary

$$\begin{array}{ccll} \mathbb{C}^* & \to & \mathbb{H}^{k,l}_0 \setminus \{\mathbf{0}\} & \\ & & \downarrow & \\ S^2 & \to & P_{\mathbb{C}}(\mathbb{H}^{k,l}_0) & \simeq S^{4k-1} \times_{S^1} S^{4l-1} \\ & & \downarrow & \\ & & P_{\mathbb{H}}(\mathbb{H}^{k,l}_0) & \simeq S^{4k-1} \times_{S^3} S^{4l-1} \end{array} \tag{1.5}$$

to give $\mathbb{H}P^{k+l-1}$, its twistor space $\mathbb{C}P^{2k+2l-1}$ and the vector space $\mathbb{H}^{k+l} \setminus \{\mathbf{0}\}$. Note that $P_{\mathbb{H}}(\mathbb{H}^{k,l}_0) \simeq S^{4k-1} \times_{S^3} S^{4l-1}$ is both S^{4k-1}-bundle over $\mathbb{H}P^{l-1}$ and S^{4l-1}-bundle over $\mathbb{H}P^{k-1}$. The following proposition is straightforward.

Proposition 1.6: *The manifolds $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$, $k>0$, are semi-quaternion Kähler with holonomy group $\mathrm{Sp}(k-1,l)\cdot\mathrm{Sp}(1)$, index $\nu = k-1$, negative scalar curvature, twistor space $P_{\mathbb{C}}(\mathbb{H}^{k,l}_-)$, and Swann bundle $\mathbb{H}^{k,l}_-$; furthermore, $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$ is the quaternionic $\mathbb{H}^l$-bundle over the standard quaternionic projective space $\mathbb{H}P^{k-1}$ associated to the quaternionic Hopf fibration. The manifolds $P(\mathbb{H}^{k,l}_+)$, $l>0$, are semi-quaternion Kähler with holonomy group $\mathrm{Sp}(k,l-1)\cdot\mathrm{Sp}(1)$, index $\nu = k$, positive scalar curvature, twistor space $P_{\mathbb{C}}(\mathbb{H}^{k,l}_+)$, and Swann bundle $\mathbb{H}^{k,l}_+$; furthermore, $P_{\mathbb{H}}(\mathbb{H}^{k,l}_+)$ is the quaternionic $\mathbb{H}^k$-bundle over the standard quaternionic projective space $\mathbb{H}P^{l-1}$ associated to the quaternionic Hopf fibration. Topologically, $\mathbb{H}\mathcal{H}^{k,l-1} = P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$ and $\mathbb{H}\mathcal{H}^{k-1,l} = P_{\mathbb{H}}(\mathbb{H}^{k,l}_+)$ are the components of $\mathbb{H}P^{k+l-1} \setminus P_{\mathbb{H}}(\mathbb{H}^{k,l}_0)$.*

The bundle structure of $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$ is the one associated to the right quaternionic multiplication of the quaternionic vector space $\mathbb{H}^k$ by the unit quaternions. Explicitly, let $(\mathbf{a},\mathbf{b}) \in \mathcal{H}_{k,l}(-1) \subset \mathbb{H}^{k,l}_-$. Let us identify $\mathcal{H}_{k,l}(-1) \simeq S^{4k-1}(1) \times \mathbb{H}^l$ via a map

$$f(\mathbf{a},\mathbf{b}) = (\mathbf{v},\mathbf{b}) = \left(\frac{\mathbf{a}}{\|\mathbf{b}\|^2+1}, \mathbf{b}\right).$$

Then $\sigma \in \mathrm{Sp}(1)$ acting on $S^{4k-1}(1) \times \mathbb{H}^l$ by $(\mathbf{v}, \mathbf{b}) \to (\mathbf{v}\sigma, \mathbf{b}\sigma)$ gives the quotient which can be identified with $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$. Hence, $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$ is an $\mathbb{H}^k$-bundle (quaternionic vector bundle) over $\mathbb{H}P^{l-1}$ associated to the quaternionic Hopf bundle $S^3 \to S^{4l-1} \to \mathbb{H}P^{l-1}$.

Example 1.7: Let $(k, l) = (1, 2)$. Then $P_{\mathbb{H}}(\mathbb{H}^{1,2}_-)$ is simply the unit open 8-ball in $\mathbb{H}^2$. The boundary of this cell $P_{\mathbb{H}}(\mathbb{H}^{1,2}_0) = S^3 \times_{S^3} S^7 \simeq S^7$ is the unit sphere. The space $P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ is the $\mathbb{H} \simeq \mathbb{R}^4$ bundle over $\mathbb{H}P^1 \simeq S^4$ associated to the quaternionic Hopf bundle $S^3 \to S^7 \to S^4$. Viewed another way $P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ is a complement of the unit 8-ball in $\mathbb{H}^2$ with $\mathbb{H}P^1 \simeq S^4$ added in at infinity.

Remark 1.1: Note that the map $\psi\colon \mathbb{H}^{k,l} \to \mathbb{H}^{l,k}$ defined by

$$\psi(u_0, u_1, \ldots, u_{k-1}, u_k, \ldots, u_n) = (u_n, \ldots, u_k, u_{k-1}, \ldots, u_0) \tag{1.6}$$

is the anti-isometry (or metric reversal) which induces anti-isometries

$$\psi\colon \mathcal{H}_{k,l}(\epsilon) \to \mathcal{H}_{l,k}(-\epsilon), \text{ i.e., } \psi\colon \mathbb{H}\mathcal{H}^{k,l} \to \mathbb{H}\mathcal{H}^{l,k}.$$

For example, $P_{\mathbb{H}}(\mathbb{H}^{n+1,0}_-)$ is diffeomorphic to $\mathbb{H}P^n$ but has negative-definite metric. It can be identified with $P_{\mathbb{H}}(\mathbb{H}^{0,n+1}_+)$ which is obviously the usual definition of $\mathbb{H}P^n$ by changing the sign of the metric. As a result we can restrict our discussion only to the negative scalar curvature spaces $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$, $k > 0$. This is not natural if one talks about the projective space $P_{\mathbb{H}}(\mathbb{H}^{n+1,0}_-)$ but in this paper we will mostly deal with the case $k < n + 1$.

We now describe the spaces $(P_{\mathbb{H}}(\mathbb{H}^{k,l}_-), g^-_{k,l})$ in inhomogeneous quaternionic coordinates. One needs k quaternionic charts to cover $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$, namely

$$\mathcal{U}_\beta = \{\mathbf{u} \in P_{\mathbb{H}}(\mathbb{H}^{k,l}_-) \ \mid \ u_\beta \neq 0\}, \quad \beta = 0, \ldots, k-1. \tag{1.7}$$

On $\mathcal{U}_\beta$ we write

$$\mathbf{x}^\beta = (x^\beta_1, \ldots, x^\beta_n) = (u_0 u_\beta^{-1}, \ldots, u_{\beta-1} u_\beta^{-1}, u_{\beta+1} u_\beta^{-1} \ldots, u_n u_\beta^{-1}) \in \mathbb{H}^n. \tag{1.8}$$

Note that (1.1) implies that on $\mathcal{U}_\beta$ we have

$$1 - F_{k-1,l}(\mathbf{x}^\beta, \mathbf{x}^\beta) = 1 + \sum_{\alpha=1}^{k-1} |x^\beta_\alpha|^2 - \sum_{\alpha=k}^{n} |x^\beta_\alpha|^2 = 1/|u_\beta|^2 > 0. \tag{1.9}$$

Let us denote $F_{k-1,l}$ simply by $\langle *, * \rangle_{k-1,l}$ with the associated semi-norm $\| * \|_{k-1,l}$. Then, on $\mathcal{U}_\beta$, $\|\mathbf{x}^\beta\|_{k-1,l} < 1$ and the semi-quaternion Kähler metric $g^-_{k,l}$ reads

$$g^-_{k,l} = \frac{1}{1 - \|\mathbf{x}^\beta\|^2_{k-1,l}} \left(\|d\mathbf{x}^\beta\|^2_{k-1,l} + \frac{1}{1 - \|\mathbf{x}^\beta\|^2_{k-1,l}} |\langle d\mathbf{x}^\beta, \mathbf{x}^\beta \rangle_{k-1,l}|^2 \right). \tag{1.10}$$

We will often refer to $\mathbf{u} = (u_0, u_1, \ldots u_n)$ as homogeneous coordinates on $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$.

Example 1.8: It is clear that $P_{\mathbb{H}}(\mathbb{H}^{1,n}_-) = \mathbb{H}\mathcal{H}^n$ is simply the unit ball in $\mathbb{H}^n$ with the quaternionic hyperbolic metric. In this case $\mathcal{U}_0$ is the only chart so we have global inhomogeneous coordinates

$$\mathbf{x} = (x_1, \ldots, x_n) = (u_1 u_0^{-1}, \ldots, u_n u_0^{-1}) \in \mathbb{H}^n. \tag{1.11}$$

with the positive definite hyperbolic metric

$$g = \frac{1}{1 - |\mathbf{x}|^2} \left(|d\mathbf{x}|^2 + \frac{1}{1 - |\mathbf{x}|^2} |\langle d\mathbf{x}, \mathbf{x} \rangle|^2 \right). \tag{1.12}$$

These are not the only examples of semi-quaternion Kähler manifolds as we shall see. However, many other examples can be obtained by taking semi-quaternionic Kähler quotients of $P_{\mathbb{H}}(\mathbb{H}^{k,l}_{-})$ by subgroups of $\mathrm{Sp}(k,l)$. The quotient construction in the semi-Riemannian case works in the similar way as in the Riemannian case. However, the zero-level set for the moment map need not be a semi-Riemannian submanifold. For example, when G is a 1-parameter subgroup acting on a semi-quaternion Kähler manifold (M^{4n}, g) of index ν then $N = \mu^{-1}(0) \subset M$ can have regions of signature $(4\nu - 3, 4n - 4\nu)$ and $(4\nu, 4n - 4\nu - 3)$ separated by all points in M with $g(V, V) = 0$, where V is the vector field of the G-action on M. Let us call these two regions by $N_- = \mu_-^{-1}(0)$ and $N_+ = \mu_+^{-1}(0)$. We have

Theorem 1.9: *Let (M^{4n}, g) be a semi-Riemannian manifold with quaternionic index ν and $G \subset \mathrm{Isom}_\Omega(M, g)$ be a one-parameter subgroup of isometries of M preserving the quaternion Kähler 4-form Ω. Let $\mu : M \to \mathcal{V}$ be the quaternion Kähler moment map for this action and let $\mu_-^{-1}(0) \subset M$, $\mu_+^{-1}(0) \subset M$ be a semi-Riemannian submanifolds of signature $(4\nu - 3, 4\nu)$ and $(4\nu, 4\nu - 3)$. If G acts freely and properly on $\mu_\pm^{-1}(0)$ the quotients $M_- = \mu_-^{-1}(0)/G$ and $M_+ = \mu_+^{-1}(0)/G$ are semi-quaternion Kähler manifolds of dimension $4n-4$ and quaternionic index $\nu - 1$ and ν, respectively.*

The situation is even more complex when we choose an arbitrary $G \subset \mathrm{Isom}_\Omega(M, g)$. In general, depending on how G acts on M one should separate $\mu^{-1}(0)$ into submanifolds of signature $(4\nu - 3c, 4n - 4\nu - 3d)$, where $c + d = \dim(G)$ and one could expect quotients of various quaternionic indices ranging from 0 to $\min(\dim(G), \nu)$.

However, in this paper we shall focus our interest on the special case when $M = P_{\mathbb{H}}(\mathbb{H}^{k,l}_{-})$ or $M = P_{\mathbb{H}}(\mathbb{H}^{k,l}_{+})$ with $k + l = 3$ and $\dim(G) = k + l - 2 = 1$. When $(k, l) = \{(0, 3), (3, 0)\}$ we are in the realm of the S^1 reductions of $\mathbb{H}P^2$, which have been already studied in [GL88]: the quotients are orbifold complex weighted projective planes. In the case of $(k, l) = (1, 2)$ we have two projective spaces one can consider: $P_{\mathbb{H}}(\mathbb{H}^{1,2}_{-}) = \mathbb{H}\mathcal{H}^2$ and $P_{\mathbb{H}}(\mathbb{H}^{1,2}_{+}) = \mathbb{H}\mathcal{H}^{1,1}$. However, as described in Example 1.7 these are two pieces of $\mathbb{H}P^2$ cut along a 7-sphere. The choice of $G \subset \mathrm{Sp}(1, 2)$, simultaneously determines the quaternion Kähler reduction of both $\mathbb{H}\mathcal{H}^2$ and $\mathbb{H}\mathcal{H}^{1,1}$ by G. In fact, the reduction depends only on the conjugacy classes of such 1-parameter subgroups in $\mathrm{Sp}(1, 2)$. These, on the other hand, are given by adjoint orbits in the Lie algebra $\mathfrak{sp}(1, 2)$. For each such adjoint orbit $[\Delta]$ ($\Delta \in \mathfrak{sp}(1, 2)$) one can consider 1-parameter group

$$G(\Delta) = \{A \subset \mathrm{Sp}(1, 2) \ \mid \ A = e^{\Delta t}, \quad t \in \mathbb{R}\} \tag{1.13}$$

acting on $\mathbb{H}^{1,2}$ as a subgroup of $\mathrm{Sp}(1, 2) \subset \mathrm{GL}(3, \mathbb{H})$. This action descends to an action on

$$\mathbb{H}P^2 = \mathbb{H}\mathcal{H}^2 \cup S^7 \cup \mathbb{H}\mathcal{H}^{1,1} \tag{1.14}$$

preserving the above decomposition and defining the semi-quaternion Kähler moment maps. Following Swann [Swa91], it is convenient to consider the semi-hyperkähler moment map $\mu\colon \mathbb{H}^{1,2} \to \mathrm{Im}(\mathbb{H})$ and the corresponding decomposition of the Swann bundle. We then write

$$\mu_\Delta^{-1}(\mathbf{0}) = N_-(\Delta) \cup N_0(\Delta) \cup N_+(\Delta), \tag{1.15}$$

where $N_\epsilon(\Delta)$ are restriction of $\mu_\Delta^{-1}(\mathbf{0})$ to $\mathbb{H}^{1,2}_\epsilon$. As we shall see $N_-(\Delta)$ can be empty, $N_+(\Delta)$ is never empty. Let $N_-(\Delta)$ be nonempty and suppose

$$P_{\mathbb{H}}(N_-(\Delta)) = N_-(\Delta)/\mathbb{H}^* \subset \mathbb{H}\mathcal{H}^2$$

is a submanifold in the 8-ball $\mathbb{H}\mathcal{H}^2$. Further assuming that $G(\Delta)$ acts freely and properly on $P_{\mathbb{H}}(N_-(\Delta))$ we define the quotient

$$G(\Delta) \to P_{\mathbb{H}}(N_-(\Delta)) \to M_-(\Delta) = G(\Delta)\backslash N_\epsilon(\Delta)/\mathbb{H}^*. \tag{1.16}$$

It follows that the metric $g(\Delta)$ on $M_-(\Delta)$ obtained by inclusion and submersion in the quotient construction is a complete SDE metric of negative scalar curvature. Its Swann bundle $\mathcal{U}(M_-(\Delta)) = G(\Delta)\backslash N_-(\Delta)$ is a semi-hyperkähler manifold of index 1. Hence, for every $\Delta \in \mathfrak{sp}(1,2)$ such that $P_{\mathbb{H}}(N_-(\Delta)) \subset \mathbb{H}\mathcal{H}^2$ and $G(\Delta)$ acts freely and properly on it we get a negative SDE manifold $(M_-(\Delta), g(\Delta))$. What remains is to enumerate all possible adjoint orbits (this will be done in the next section) and examine all the possible quotients (the following four sections).

The projectivisation $P_{\mathbb{H}}(N_+(\Delta)) \subset P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$, in general, does not need to be a semi-Riemannian submanifold. Let $V(\mathbf{u}) = \Delta \cdot \mathbf{u}$ be the vector field for the $G(\Delta)$-action on $P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$. Then the norm square of V in the semi-Riemannian metric $g^+_{1,2}$ can be negative, positive, or it can vanish. Let $P^+_{\mathbb{H}}(N_+(\Delta)) \subset P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ be the subset on which $g^+_{1,2}(V,V) > 0$ while $P^-_{\mathbb{H}}(N_+(\Delta)) \subset P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ the subset on which $g^+_{1,2}(V,V) < 0$. If $P^+_{\mathbb{H}}(N_+(\Delta))$ is a submanifold in $P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ then it is a semi-Riemannian submanifold of signature $(4,1)$. On the other hand, if $P^-_{\mathbb{H}}(N_+(\Delta))$ is a submanifold in $P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$ than it is a semi-Riemannian submanifold of signature $(1,4)$. At least locally we can define two different quotient metrics: (1) if $P^+_{\mathbb{H}}(N_+(\Delta))$ is not empty we have positive scalar curvature metric $g^+(\Delta)$ on $P^+_{\mathbb{H}}(N_+(\Delta))/G(\Delta)$ of signature $(4,0)$ (anti-Riemannian); (2) if $P^-_{\mathbb{H}}(N_+(\Delta))$ is not empty we have positive scalar curvature metric $g^-(\Delta)$ on $M^-_+(\Delta) = P^-_{\mathbb{H}}(N_+(\Delta))/G(\Delta)$ of signature $(0,4)$. The metric $g^-(\Delta)$, is a Riemannian metric of positive scalar curvature. Typically this metric is not complete, unless the quotient can be globally extended to the symmetric metric on S^4 or $\mathbb{C}P^2$. On the other hand $g^+(\Delta)$ is anti-Riemannian metric of positive scalar curvature so that $-g^+(\Delta)$ is a Riemannian metric of negative scalar curvature on $M^+_+(\Delta) = P^+_{\mathbb{H}}(N_+(\Delta))/G(\Delta)$. Generally this metric is not complete. However, as we shall see in section 7 complete metrics of this type can occur.

Hence, a priori, for each $\Delta \in \mathfrak{sp}(1,2)$ we have locally three different metrics: $g(\Delta)$, $-g^+(\Delta)$ and $g^-(\Delta)$. The two metrics $g(\Delta)$, $-g^+(\Delta)$ are negative SDE while $g^-(\Delta)$ is positive SDE.

Remark 1.2: Similarly, we can consider any orbits $[\Delta]$ under $\mathrm{Sp}(1,n)$ of the $(n-1)$-dimensional subalgebras $\Delta \subset \mathfrak{sp}(1,n)$. Our analysis carried out for $\mathrm{Sp}(1,2)$ applies without any changes and, a priori, for each Δ we obtain locally 3 different metrics: $g(\Delta)$, $-g^+(\Delta)$ and $g^-(\Delta)$. In addition, when $\Delta \subset \mathfrak{sp}(1,n)$ is Abelian these metrics have two commuting Killing vectors. Even more generally, we could consider any orbit $[\Delta]$ under $\mathrm{Sp}(k,l)$ of $(k+l-2)$-dimensional subalgebras $\mathfrak{g} \subset \mathfrak{sp}(k,l)$. If both k,l are greater than one $P_{\mathbb{H}}(\mathbb{H}^{k,l}_\pm)$ are both semi-quaternion Kähler. Our analysis carried out for $(1,2)$ still applies and, a priori, for each Δ we get locally four different metrics: two from the reduction of $P_{\mathbb{H}}(\mathbb{H}^{k,l}_-)$ and the other two from the reduction of $P_{\mathbb{H}}(\mathbb{H}^{k,l}_+)$. Two of these metrics will have negative scalar curvature. The case of $k = l$ is of special interest as we shall see in section 7. Again, for Abelian subalgebras the metrics will have two commuting Killing vectors while the non-Abelian case is more general.

2. Adjoint orbits in $\mathfrak{sp}(1,2)$

Adjoint orbits of elements in the classical Lie algebras $\mathfrak{g}$ have been determined by Burgoyne and Cushman [BC77]. We shall use this work to find all the conjugacy classes of one-parameter subgroups of $\mathrm{Sp}(1,2)$. First let us review some basic definitions. The symmetric form on $\mathbb{H}^{1,2}$ is given by $\mathbf{u}^\dagger \mathbb{F}\mathbf{u}$, where

$$\mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} \qquad \mathbb{F} = \mathbb{F}_{1,2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

We can describe $\mathrm{Sp}(1,2)$ and its Lie algebra $\mathfrak{sp}(1,2)$ as 3×3 matrices preserving $\mathbb{F}$, i.e.,

$$\mathrm{Sp}(1,2) = \{g \in \mathcal{M}_{3\times 3}(\mathbb{H}) \ \mid \ g^\dagger \mathbb{F} g = \mathbb{F}\} \tag{2.2}$$

$$\mathfrak{sp}(1,2) = \{Y \in \mathcal{M}_{3\times 3}(\mathbb{H}) \ \mid \ \mathbb{F}Y + Y^\dagger \mathbb{F} = \mathbf{0}\} \tag{2.3}$$

Explicitly, an element of $\mathfrak{sp}(1,2)$ can be written as

$$Y = \begin{pmatrix} a & \alpha & \beta \\ \bar{\alpha} & b & \gamma \\ \bar{\beta} & -\bar{\gamma} & c \end{pmatrix}, \tag{2.4}$$

Setting $\alpha = \beta = 0$ gives the maximal compact subalgebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ while $\beta = \gamma = 0$ yields $\mathfrak{sp}(1,1) \oplus \mathfrak{sp}(1)$.

We say that $Y \in \mathfrak{sp}(1,2)$ is *decomposable* if $\mathbb{H}^{1,2}$ may be split non-trivially as a direct sum of mutually orthogonal Y-invariant quaternionic subspaces. Otherwise we say that Y is *indecomposable*. Choosing a particular unit quaternion i identifies $\mathbb{H}^{1,2} \cong \mathbb{H}^3$ with $\mathbb{C}^6$, which realizes $\mathfrak{sp}(1,2)$ as a subalgebra of $\mathfrak{gl}(6,\mathbb{C})$. We shall say an element $Y \in \mathfrak{sp}(1,2)$ is *semisimple* iff it is diagonalizable as an element of $\mathfrak{gl}(6,\mathbb{C})$. Any $Y \in \mathfrak{sp}(1,2)$ can be uniquely written as $Y = S + N$, where S is semisimple, and N is nilpotent with $[S,N] = 0$. If $N^{m+1} = 0, N^m \neq 0$ then the integer m is called the *height* of Y. Semisimple elements have height equal to zero.

Definition 2.1: *We define the following elements of* $\mathfrak{sp}(1,2)$:

$$T_0(ip_0, ip_1, ip_2) = \begin{pmatrix} ip_0 & 0 & 0 \\ 0 & ip_1 & 0 \\ 0 & 0 & ip_2 \end{pmatrix},$$

$$T_0(\lambda, ip, iq) = \begin{pmatrix} ip & \lambda & 0 \\ \lambda & ip & 0 \\ 0 & 0 & iq \end{pmatrix},$$

$$T_1(\lambda, ip, iq) = \begin{pmatrix} ip & 0 & 0 \\ 0 & ip & 0 \\ 0 & 0 & iq \end{pmatrix} + \lambda \begin{pmatrix} i & i & 0 \\ -i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_2(\lambda, ip) = ip\,\mathbb{I}_3 + \lambda \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ i & i & 0 \end{pmatrix},$$

where (throughout) $\lambda \neq 0$.

The first two 3-parameter families of elements are semisimple and they are in two different Cartan subalgebras of $\mathfrak{sp}(1,2)$. They are necessarily decomposable. The first one corresponds to the decomposition of $\mathbb{H}^{1,2}$ into $\mathbb{H}\oplus\mathbb{H}\oplus\mathbb{H}$ while the second decomposes $\mathbb{H}^{1,2}$ into $\mathbb{H}^{1,1}\oplus\mathbb{H}$. The 3-parameter family $T_1(\lambda, ip, iq)$ has height one (and $T_1(\lambda,0,0)$ is 2-step nilpotent). These are decomposable, splitting $\mathbb{H}^{1,2}$ into $\mathbb{H}^{1,1}\oplus\mathbb{H}$. Finally, the 2-parameter family $T_2(\lambda, ip)$ has height two (and $T_2(\lambda,0)$ is 3-step nilpotent). These are indecomposable. Note that all elements in the Definition 2.1 are inside the subalgebra $\mathfrak{u}(1,2)$. Furthermore, note that we chose $T_1 := T_1(1,0,0)$ and $T_2 := T_2(1,0)$, so that they commute. In fact $\{i\mathbb{I}_3, T_2, T_1 = iT_2^2\}$ span a maximal nilpotent Abelian subalgebra of $\mathfrak{sp}(1,2)$.

The following proposition follows from [BC77].

Proposition 2.2: *Let Y be an arbitrary non-zero element of* $\mathfrak{sp}(1,2)$. *Then Y is conjugate under the adjoint* $\mathrm{Sp}(1,2)$ *action to an element Δ of Definition 2.1. This element is unique, except in the height one case, where $T_1(\lambda,p,q)$ is conjugate to $T_1(1,p,q)$ or $T_1(-1,p,q)$ for $p \neq 0$, and to $T_1(1,0,q)$ for $p = 0$, and in the height two case, where $T_2(\lambda,p)$ is conjugate to $T_2(1,p)$.*

Furthermore, any one-parameter subgroup in $\mathrm{Sp}(1,2)$ *is conjugate to* $G(\Delta) = \{A \in \mathrm{Sp}(1,2) \mid A = e^{\Delta t}\}$*, where* Δ *is one of the types of the Definition 2.1.*

In other words, the list of Definition 2.1 enumerates all adjoint orbits in $\mathfrak{sp}(1,2)$. The corresponding conjugacy classes of one parameter subgroups of $\mathrm{Sp}(1,2)$ are enumerated by these elements up to scale: Δ and $c\Delta$ define the same subgroup for any $c \neq 0$.

In the following, it will sometimes be more convenient to work with a different basis of $\mathbb{H}^{1,2}$ in which the symmetric form may be written $\mathbf{v}^\dagger \tilde{\mathbb{F}} \mathbf{v}$ with

$$\mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} \qquad \tilde{\mathbb{F}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

The advantage of this basis is that the last three matrices in Definition 2.1 are conjugated to the following simpler forms.

$$\tilde{T}_0(\lambda, ip, iq) = \begin{pmatrix} ip+\lambda & 0 & 0 \\ 0 & ip-\lambda & 0 \\ 0 & 0 & iq \end{pmatrix},$$

$$\tilde{T}_1(\lambda, ip, iq) = \begin{pmatrix} ip & 0 & 0 \\ -i\lambda & ip & 0 \\ 0 & 0 & iq \end{pmatrix},$$

$$\tilde{T}_2(\lambda, ip) = ip\,\mathbb{I}_3 + \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}.$$

We end our discussion by noting that it is straightforward to compute the momentum map in homogeneous coordinates associated to a generator T or $\tilde{T}$ using the general formulae

$$\mu_T(\mathbf{u}) = \mathbf{u}^\dagger \mathbb{F} T \mathbf{u} = (-\bar{u}_0, \bar{u}_1, \bar{u}_2) T \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

$$\mu_{\tilde{T}}(\mathbf{v}) = \mathbf{v}^\dagger \tilde{\mathbb{F}} \tilde{T} \mathbf{v} = (\bar{v}_1, \bar{v}_0, \bar{v}_2) \tilde{T} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}.$$

3. The Pedersen–LeBrun Metrics on Line Bundles over $\mathbb{C}P^1$

In this section we will examine the case of $\Delta = \Delta_0(\mathbf{p}) = T_0(ip_0, ip_1, ip_2)$. We shall assume that this generates a circle action, which means, after rescaling Δ, that we may assume that the p_i's are integers with $\gcd(p_0, p_1, p_2) = 1$. (We can assume that the weights do not vanish as the cases when one or two of the weights vanish are degenerate.) We then have a circle action on the quaternionic hyperbolic 2-ball $\mathbb{H}\mathcal{H}^2$ given in homogeneous coordinates by

$$\varphi_t(u_0, u_1, u_2) = (e^{2\pi i p_0 t} u_0,\ e^{2\pi i p_1 t} u_1,\ e^{2\pi i p_2 t} u_2) \tag{3.1}$$

where $t \in [0,1)$. We note that this action is effective unless the weights p_0, p_1, p_2 are all odd, in which case we obtain an effective action of a quotient circle by taking $t \in [0, 1/2)$. In inhomogeneous coordinates (x_1, x_2) we have

$$\varphi_t^{\mathbf{p}}(x_1, x_2) = (e^{2\pi i p_1 t} x_1 e^{-2\pi i p_0 t}, e^{2\pi i p_2 t} x_2 e^{-2\pi i p_0 t}), \tag{3.2}$$

and the moment map is given as

$$\mu_{\mathbf{p}}(\mathbf{u}) = -p_0 \bar{u}_0 i u_0 + p_1 \bar{u}_1 i u_1 + p_2 \bar{u}_2 i u_2, \tag{3.3}$$

$$f_{\mathbf{p}}(\mathbf{x}) = u_0 \mu_{\mathbf{p}}(\mathbf{u}) u_0^{-1} = -ip_0 + p_1 \bar{x}_1 i x_1 + p_2 \bar{x}_2 i x_2 \tag{3.4}$$

in homogeneous or inhomogeneous coordinates. We now write

$$\mathbf{x} = \mathbf{z} + \mathbf{w}j = \mathbf{z} + j\bar{\mathbf{w}} \tag{3.5}$$

where $\mathbf{z}, \mathbf{w} \in \mathbb{C}^2$ and observe that

$$\varphi_t^{\mathbf{p}} \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \end{pmatrix} = \begin{pmatrix} e^{2\pi i(p_1-p_0)t} z_1 & e^{2\pi i(p_1+p_0)t} w_1 \\ e^{2\pi i(p_2-p_0)t} z_2 & e^{2\pi i(p_2+p_0)t} w_2 \end{pmatrix}.$$

$$\mu_{\mathbf{p}}^{-1}(0) = \{(\mathbf{z}, \mathbf{w}) \in \mathbb{H}\mathcal{H}^2 : \sum_{\alpha=1,2} p_\alpha(|z_\alpha|^2 - |w_\alpha|^2) = p_0, \sum_{\alpha=1,2} p_\alpha \bar{w}_\alpha \cdot z_\alpha = 0\}. \tag{3.6}$$

Proposition 3.1: *Let $q_\alpha = p_\alpha/p_0$. Then the subset $\mu_{\mathbf{p}}^{-1}(0) \subset \mathbb{H}\mathcal{H}^2$ is empty unless $|q_\alpha| > 1$ for at least one α. Otherwise, $\mu_{\mathbf{p}}^{-1}(0)$ is an open smooth submanifold of codimension 3.*

Proof. We can assume that both q_1, q_2 are positive (otherwise we simply reverse the role of z_α and w_α in the argument below). On the one hand, the momentum constraint gives

$$q_1(|z_1|^2 - |w_1|^2) + q_2(|z_2|^2 - |w_2|^2) = 1,$$

so that

$$-q_1|z_1|^2 - q_2|z_2|^2 \leqslant -1.$$

On the other hand, we have

$$|z_1|^2 + |z_2|^2 \leqslant |z_1|^2 + |w_1|^2 + |z_2|^2 + |w_2|^2 < 1$$

by the unit ball condition. Adding the two inequalities we get

$$(1 - q_1)|z_1|^2 + (1 - q_2)|z_2|^2 < 0.$$

This has no solutions when $1 \geqslant q_1$ and $1 \geqslant q_2$. Otherwise, if (say) $|q_1| > 1$ then by taking $z_2, w_2 = 0$, it is easy to see that $\mu_{\mathbf{p}}^{-1}(0)$ is nonempty. The last statement follows because straightforward computation reveals that 0 is a regular value of the Jacobian of $\mu_{\mathbf{p}}$. □

Without loss of generality we will further assume that all weights are positive. We will also choose $q_1 > 1$ that is that $p_1 > p_0$. Then we have the following

Proposition 3.2: *The $\varphi_t^{\mathbf{p}}$-action on the level set $\mu_{\mathbf{p}}^{-1}(0) \subset \mathbb{H}\mathcal{H}^2$ is free if and only if $p_1 = p_0 + 1$ and $0 < p_2 \leqslant p_0 + 1$ when one of the weights is even, or $p_1 = p_0 + 2$ and $0 < p_2 \leqslant p_0 + 2$ when all the weights are odd.*

Proof. Consider the set described by $(z_1, 0, 0, 0)$. This meets $\mu_{\mathbf{p}}^{-1}(0)$ in a circle, but any point on this circle is fixed by $\mathbb{Z}_{p_1-p_0}$. Hence we must have $p_1 = p_0 + 1$, unless all weights are odd when we have $p_1 = p_0 + 2$. Next, suppose that $p_2 > p_0$. Then the set described by $(0, z_2, 0, 0)$ also meets $\mu_{\mathbf{p}}^{-1}(0)$ in a circle and any point on this circle is fixed by $\mathbb{Z}_{p_2-p_0}$. Thus if $p_2 > p_0$, we must have $p_2 = p_0 + 1$ (or $p_0 + 2$ if all weights are odd). It is easy to see that p_2 can be any integer with $0 < p_2 \leqslant p_0 + 1$ (or $p_0 + 2$ if all weights are odd). □

We now have:

Theorem 3.3: *For $\mathbf{p} \in \mathbb{Z}_+^3$ as in Proposition 3.2, the quotient $M(\mathbf{p}) = \mu_{\mathbf{p}}^{-1}(0)/S^1(\mathbf{p})$ is a complete self-dual Einstein manifold with negative scalar curvature and at least two commuting Killing vectors. When $p_2 = p_1$ (which is $p_0 + 1$ or $p_0 + 2$) the metric is $U(2)$-invariant while when $p_2 = p_0$ the metric is $U(1,1)$-invariant.*

Proof. Only completeness of the induced metric on $M(\mathbf{p})$ remains to be proven, and this follows from the fact that the induced metric on the closed embedded submanifold $\mu_{\mathbf{p}}^{-1}(0) \hookrightarrow \mathbb{H}\mathcal{H}^2$ is complete and the action 3.1 is proper. □

We continue with describing the total space of these metrics. When $p_2 = p_1$, we expect that the metric is complete and, hence, it has to be one of the possibilities listed by Hitchin in Theorem 13 of [Hit95]. We will show that our quotient metrics are the Pedersen–LeBrun metrics on complex line bundles $\mathcal{O}(n) \to \mathbb{C}P^1$, $n \geqslant 3$ (Theorem 13:3(d) of [Hit95]). Before we analyze $M(p, p+1, p+1)$ and $M(p, p+2, p+2)$ let us recall a standard description of a complex line bundle over $\mathbb{C}P^1$ with first Chern class s. Let $S^3 = \{\mathbf{v} \in \mathbb{C}^2 : \quad \|\mathbf{v}\| = 1\}$ and let $s \in \mathbb{Z}^+$. Then we set

$$\mathcal{L}_s \equiv S^3 \times \mathbb{C}/\Phi^s, \tag{3.7}$$

where Φ^s is the action of S^1 on $S^3 \times \mathbb{C}$ given by

$$\Phi^s_\tau(\mathbf{v}, \alpha) = (\tau\mathbf{v},\ \tau^s\alpha). \tag{3.8}$$

The natural projection $\mathcal{L}_s \longrightarrow S^2 \cong S^3/S^1$ makes $\mathcal{L}_s$ a complex line bundle over S^2 with $c_1(\mathcal{L}_s) = s$.

Note that we get the same conclusion when we replace $\mathbb{C}$ by $D^1_{\mathbb{C}}(1) = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$. Then $\mathcal{L}_{r,s}$ is a complex unit disk bundle with first Chern class s. Now we are ready for

Theorem 3.4: *Let $p \in \mathbb{Z}$ and $\mathbf{p} = (p-1, p, p)$, $p > 1$. Then the quotient metric $g(\mathbf{p})$ is complete, $U(2)$-invariant and the total space $M(\mathbf{p})$ can be identified with the complex unit disk bundle $\mathcal{L}_{2p} \to \mathbb{C}P^1$ with first Chern class equal to $2p$.*

Proof. By Theorem 3.3 it suffices only to identify the quotient in this special case. Let $(\mathbf{z}, \mathbf{w}) \in \mathbb{H}\mathcal{H}^2$. We make a slight change of these coordinates by setting

$$\mathbf{x} = \frac{1}{\sqrt{p_0/p + \|\mathbf{w}\|^2}}\mathbf{z}, \qquad \mathbf{y} = \sqrt{2p}\mathbf{w}. \tag{3.9}$$

In these coordinates the moment map equations can be written

$$\mu_{\mathbf{p}}^{-1}(0) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^2 \times \mathbb{C}^2 : \quad \|\mathbf{x}\|^2 = 1, \quad \bar{\mathbf{y}} \cdot \mathbf{x} = 0, \quad ||\mathbf{y}|| < 1\}. \tag{3.10}$$

Then the circle action is given by

$$\varphi_\tau(\mathbf{x}, \mathbf{y}) = (\tau\mathbf{x},\ \tau^{2p-1}\mathbf{y}) \tag{3.11}$$

for $\tau = e^{2\pi i t} \in S^1$. Consequently, we have that $M(\mathbf{p})$ is equivalent to the quotient of the set

$$\mu_{\mathbf{p}}^{-1}(0) = \{(\mathbf{x}, \mathbf{y}) \in S^3 \times D^2_{\mathbb{C}}(1) : \quad \mathbf{x} \perp \mathbf{y}\} \simeq S^3 \times D^1_{\mathbb{C}}(1)$$

by the action (3.11). We define a map $f : S^3 \times D^1_{\mathbb{C}}(1) \longrightarrow M(\mathbf{p})$ by setting $f(\mathbf{v}, \alpha) = (\mathbf{v}, \alpha\mathbf{v}^\dagger)$ where if $\mathbf{v} = (v_o, v_1)$ then $\mathbf{v}^\dagger = (-\bar{v}_1, \bar{v}_o)$. Note that for any $\tau \in S^1$ we have the commutative diagram

$$\begin{array}{ccc} (\mathbf{v}, \alpha) & \longrightarrow & (\mathbf{v}, \alpha\mathbf{v}^\dagger) \\ \downarrow & & \downarrow \\ (\tau\mathbf{v}, \tau^{2p}\alpha) & \longrightarrow & (\tau\mathbf{v}, \tau^{2p-1}\alpha\mathbf{v}^\dagger) \end{array};$$

i.e., we have that $f \circ \Phi^{2p}_\tau = \varphi_\tau \circ f$. Thus f is an S^1-equivariant diffeomorphism and therefore f induces a smooth equivalence of the quotient spaces. Hence $M(\mathbf{p}) \simeq \mathcal{L}_{2p}$.

□

When $\mathbf{p} = (p-2, p, p)$ we immediately get the other half of the line bundles with odd Chern classes:

Theorem 3.5: *Let $p \in \mathbb{Z}$ and $\mathbf{p} = (p-2, p, p)$, $p = 2k+1 > 2$. Then the quotient metric $g(\mathbf{p})$ is complete, $U(2)$-invariant and the total space $M(\mathbf{p})$ can be identified with the complex unit disk bundle $\mathcal{L}_p \to \mathbb{C}P^1$ with first Chern class equal to p.*

Note that this construction does not give the line bundles over $\mathbb{C}P^1$ with first Chern classes $c_1 = 1, 2$. The metrics on $\mathcal{L}_p$ with $p \geqslant 3$ have a curious history. The quotient construction presented here was written in [Gal87b]. The Pedersen metric on the 4-ball [Ped86] depends on a single parameter $m^2 \in (-1, \infty)$. It was realized later (see [Hit95]) that setting this parameter to $(2-n)/n$ (with $n \in \mathbb{Z}, n > 2$) allows for the analytic continuation of this metric to a complete metric on $\mathcal{O}(n) \to \mathbb{C}P^1$. The reason these metrics are called Pedersen–LeBrun in [Hit95] is that they are conformal to the scalar flat Kähler metrics on $\mathcal{O}(-n) \to \mathbb{C}P^1$ constructed by LeBrun [LeB88].

When, $p_0 + 1 = p_1 > p_2 > 0$ we take a different approach. Observe that one can still easily solve the complex equation of the moment map by setting

$$(w_1, w_2) = \alpha(-p_2\bar{z}_2, p_1\bar{z}_1), \tag{3.12}$$

where $\alpha \in \mathbb{C}$. The unit ball condition in terms of (z_1, z_2, α) reads:

$$|z_1|^2 + |z_2|^2 + |\alpha|^2 p_1^2 |z_1|^2 + |\alpha|^2 p_2^2 |z_2|^2 = |z_1|^2(1 + p_1^2|\alpha|^2) + |z_2|^2(1 + p_2^2|\alpha|^2) < 1, \tag{3.13}$$

while the remaining moment map equation is

$$|z_1|^2(p_1 - p_2 p_1^2|\alpha|^2) + |z_2|^2(p_2 - p_1 p_2^2|\alpha|^2) = p_0.$$

Let us solve this equation with respect to $|z_1|^2$:

$$|z_1|^2 = \frac{p_0}{p_1}\frac{1}{1 - p_1 p_2|\alpha|^2} - \frac{p_2}{p_1}|z_2|^2. \tag{3.14}$$

One can immediately see that z_1 cannot vanish as then

$$|z_2|^2 = \frac{p_0}{p_2}\frac{1}{1 - p_1 p_2|\alpha|^2} \geqslant \frac{p_0}{p_2} \geqslant 1. \tag{3.15}$$

Let $\rho = \frac{z_1}{|z_1|}$. It is easy to see that

$$\phi_\tau(\rho, z_2, \alpha) = (\tau\rho, \tau^{p_2 - p_0} z_2, \tau^{p_2 + p_1}\alpha).$$

Proposition 3.6: *The level set $\mu_{\mathbf{p}}^{-1}(0) \simeq D \times S^1$, where $D \subset \mathbb{C}^2$ is an open 4-ball.*

Proof. It is clear that $(\rho, z_2, \alpha) \in S^1 \times D$ are coordinates on $\mu_{\mathbf{p}}^{-1}(0)$. We have to check that D is diffeomorphic to a 4-ball. To do that let us consider

$$\left(\frac{p_0}{p_1}\frac{1}{1 - p_1 p_2|\alpha|^2} - \frac{p_2}{p_1}|z_2|^2\right)(1 + p_1^2|\alpha|^2) + |z_2|^2(1 + p_2^2|\alpha|^2) < 1$$

which can be written as

$$f_{\mathbf{p}}(z_2, \alpha) = (p_1 - p_2)|z_2|^2[1 - p_1 p_2|\alpha|^2]^2 + p_1^2 p_2|\alpha|^2 - 1 < 0.$$

One can easily see that $|\alpha|^2 < \frac{1}{p_1^2 p_2}$

$$D(\mathbf{p}) = \{(z_2, \alpha) \in \mathbb{C} \times \mathbb{C} \ \mid \ f_{\mathbf{p}}(z_2, \alpha) < 0\}$$

is an open 4-ball. □

Theorem 3.7: *The quotient $M(\mathbf{p}) \simeq D(\mathbf{p})$ is diffeomorphic to a 4-ball. The self-dual Einstein metric $g(\mathbf{p})$ obtained from the quaternion Kähler quotient is complete and it has two commuting Killing vectors. Furthermore, $M(p, p+1, p)$ is of cohomogeneity one with respect to $U(1,1)$.*

The cohomogeneity one $U(1,1)$ action on $M(p, p+1, p)$ can be explicitly described as follows. Let

$$\mathbb{A} = \begin{pmatrix} a & \tau b \\ \bar{b} & \tau\bar{a} \end{pmatrix} \in U(1,1),$$

where $a, b, \tau \in \mathbb{C}$ with $|a|^2 - |b|^2 = 1$ and $|\tau|^2 = 1$. This group acts on the quaternionic ball as

$$\varphi_{\mathbb{A}}(\mathbf{u}) = \begin{pmatrix} a & 0 & \tau b \\ 0 & 1 & 0 \\ \bar{b} & 0 & \tau\bar{a} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix}$$

and, it commutes with the circle action given by $\varphi_t^{\mathbf{p}}$. In the inhomogeneous chart we get

$$\varphi_{\mathbb{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1(a + \tau b x_2)^{-1} \\ (\bar{b} + \tau\bar{a}x_2)(a + \tau b x_2)^{-1} \end{pmatrix}.$$

The above action preserves the zero level set of the moment map and it descends to a cohomogeneity one isometric action on the quotient space $M(p, p+1, p)$. Cohomogeneity one SDE metrics with an isometric action of a four-dimensional Lie group have been studied by Derdziński. Hence, up to isometries $M(p, p+1, p)$ must be the cohomogeneity one self-dual Kähler metric introduced in [Der81] and more recently studied by Apostolov and Gauduchon in [AG02].

4. Generalized Pedersen Metrics on the Ball

In this section and the following two, we will consider the $\mathbb{R}$-actions on $\mathbb{H}\mathcal{H}^2$ whose generators do not belong to the Lie algebra of a maximal torus. To do this we shall work in the $\mathbf{v} = (v_0, v_1, v_2)$ coordinates introduced in section 2. In these coordinates $\mathbb{H}\mathcal{H}^2$ is the open subset of $\mathbb{H}P^2$ defined by the equation

$$\bar{v}_0 v_1 + \bar{v}_1 v_0 + |v_2|^2 < 0. \tag{4.1}$$

It follows that v_0 does not vanish on $\mathbb{H}\mathcal{H}^2$ and so the inhomogeneous coordinates $y_1 = v_1 v_0^{-1}$, $y_2 = v_2 v_0^{-1}$ provide a global chart identifying $\mathbb{H}\mathcal{H}^2$ with the domain

$$y_1 + \bar{y}_1 + |y_2|^2 < 0 \tag{4.2}$$

in $\mathbb{H}^2$. We remark (for later use) that the real part of y_1 is strictly negative on this domain.

We begin by considering the case of $\Delta_0(p, q) = \tilde{T}_0(1, ip, iq)$, where we have taken $\lambda = 1$ by rescaling. The $\mathbb{R}$-action on the quaternionic hyperbolic 2-ball $(\mathbb{H}\mathcal{H}^2, g)$ is given explicitly by

$$\varphi_t^{p,q}(\mathbf{v}) = \begin{pmatrix} e^{(ip+1)t} & 0 & 0 \\ 0 & e^{(ip-1)t} & 0 \\ 0 & 0 & e^{iqt} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{ipt}e^{t}v_0 \\ e^{ipt}e^{-t}v_1 \\ e^{iqt}v_2 \end{pmatrix}, \tag{4.3}$$

which reduces, in inhomogeneous coordinates $\mathbf{y} = (y_1, y_2)$, to

$$\varphi_t^{p,q} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{ipt}e^{-2t}y_1 e^{-ipt} \\ e^{iqt}y_2 e^{-ipt} \end{pmatrix}. \tag{4.4}$$

This action is a quaternionic isometry of the hyperbolic metric g and it defines a bundle valued momentum map $\mu_{p,q} \colon \mathbb{H}\mathcal{H}^2 \to \mathcal{V}$ given in homogeneous and inhomogeneous coordinates by the function

$$\begin{aligned} \mu_{p,q}(\mathbf{v}) &= \bar{v}_1 v_0 - \bar{v}_0 v_1 + p(\bar{v}_1 i v_0 + \bar{v}_0 i v_1) + q\bar{v}_2 i v_2, \\ f_{p,q}(\mathbf{y}) &= \bar{y}_1 - y_1 + p(\bar{y}_1 i + i y_1) + q\bar{y}_2 i y_2. \end{aligned}$$

Although this function is not invariant under the action (4.4), its zero set is, and the quaternion Kähler reduction of $\mathbb{H}\mathcal{H}^2$ by the one parameter group $e^{\Delta_0(p,q)t}$ is the quotient of this zero set by the group action. The resulting SDE metrics were first introduced in [Gal87b] and may be regarded as a deformation of the Pedersen metrics on the ball to metrics with fewer symmetries.

Theorem 4.1: *Let $\Delta = \Delta_0(p,q) = \tilde{T}_0(1, ip, iq)$ and consider the one parameter group $\varphi_t^{p,q} = e^{\Delta t}$ acting on the quaternionic hyperbolic space $\mathbb{H}\mathcal{H}^2$. The quaternion Kähler reduction $M(p,q) = \mu_{p,q}^{-1}(0)/\varphi^{p,q}$ is diffeomorphic to an open 4-ball for all $(p,q) \in \mathbb{R}^2$. The quotient metric is complete, self-dual, and Einstein of negative scalar curvature whose isometry group contains a 2-torus. Furthermore, the quotient metrics on $M(0,q)$ are isometric to the Pedersen metrics, and their isometry group contains $U(2)$.*

Proof. Consider the following set

$$\mathcal{S}_{p,q} = \{\mathbf{y} \mid f_{p,q}(y_1, y_2) = 0 \text{ and } y_1 + \bar{y}_1 = 2\,\mathrm{Re}(y_1) = -1\}. \tag{4.5}$$

For any $y_2 \in \mathbb{H}$ there is a unique y_1 such that $\mathbf{y} \in \mathcal{S}_{p,q}$. It follows that $\mathcal{S}_{p,q} \cap \mathbb{H}\mathcal{H}^2$ is diffeomorphic to the open 4-ball $|y_2|^2 < 1$ in $\mathbb{H} \cong \mathbb{R}^4$. Furthermore, $\mathcal{S}_{p,q} \cap \mathbb{H}\mathcal{H}^2$ provides a global slice for the action of $e^{\Delta t}$ in the zero set of the momentum map: to see this, we only have to note that $e^{\Delta t}$ sends $y_1 + \bar{y}_1$ to $e^{-2t}(y_1 + \bar{y}_1)$ and therefore, since $y_1 + \bar{y}_1 < 0$, there is a unique (y_1, y_2) in any orbit with $y_1 + \bar{y}_1 = -1$.

Therefore $M(p,q)$ is diffeomorphic to the open 4-ball equipped with the metric obtained by restriction to $\mu_{p,q}^{-1}(0)$ and submersion. The fact that the quotient is a complete Riemannian manifold follows as in the proof of Theorem 3.3. Moreover, it must be an SDE metric of negative scalar curvature since it is obtained as quaternion Kähler quotient of $\mathbb{H}\mathcal{H}^2$.

The isometry group contains a 2-torus since $\mathcal{S}_{p,q}$ is invariant under the transformation

$$(y_1, y_2) \mapsto (\sigma y_1 \sigma^{-1}, \tau y_2 \sigma^{-1})$$

by $(\tau, \sigma) \in U(1) \times U(1)$. As this action is by quaternionic isometries on $\mathbb{H}\mathcal{H}^2$ and commutes with $e^{\Delta t}$, it descends to give an action by isometries on $M(p,q)$. If $p = 0$ this action may be extended, by taking $\sigma \in Sp(1)$, to yield an action of $U(1) \cdot \mathrm{Sp}(1) \simeq U(2)$.

To identify $M(0,q)$ as the Pedersen family, one can compute the metric explicitly. Alternatively, we can use the classification of SDE metrics with SU(2) symmetry by Hitchin [Hit95]. This classification provides very few possible candidates with $U(2)$ symmetry: apart from the real and complex hyperbolic metrics, the Pedersen metrics are the only examples. In fact, one can see that $M(0,0)$ is real hyperbolic space but for other values of q the metric is not symmetric. □

5. The Height One Quotients

In this section we will examine the family of quotients of $\mathbb{H}\mathcal{H}^2$, obtained from $\tilde{T}_1(\lambda, ip, iq)$. By rescaling we can assume that p is 0 or 1, and if $p = 0$ we can scale q to 1 or 0. Since we assume λ is nonzero, we can then conjugate so that $\lambda = \pm 1$ (or $\lambda = 1$ if $p = 0$) and rescale by the sign. Hence we only need to consider the quotients $\Delta_1(p,q) = \tilde{T}_1(1, ip, iq)$ with $p \in \{-1, 0, 1\}$, and if $p = 0$ we can suppose $q \in \{0, 1\}$. Nevertheless, for convenience we shall carry out our analysis for arbitrary p, q. We have

$$\varphi_t^{p,q}(\mathbf{v}) = \begin{pmatrix} e^{ipt} & 0 & 0 \\ -it & e^{ipt} & 0 \\ 0 & 0 & e^{iqt} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{ipt} v_0 \\ e^{ipt} v_1 - it v_0 \\ e^{iqt} v_2 \end{pmatrix}, \tag{5.1}$$

which reduces, in inhomogeneous coordinates $\mathbf{y} = (y_1, y_2)$, to

$$\varphi_t^{p,q} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{ipt}(y_1 - it)e^{-ipt} \\ e^{iqt} y_2 e^{-ipt} \end{pmatrix}. \tag{5.2}$$

The moment map for this action is given in homogeneous or inhomogeneous coordinates by

$$\mu_{p,q}(\mathbf{v}) = -\bar{v}_0 i v_0 + p(\bar{v}_0 i v_1 + \bar{v}_1 i v_0) + q \bar{v}_2 i v_2,$$
$$f_{p,q}(\mathbf{y}) = -i + p(i y_1 + \bar{y}_1 i) + q \bar{y}_2 i y_2.$$

Theorem 5.1: *Let $\Delta = \Delta_1(p,q) = \tilde{T}_1(1, ip, iq)$ and consider the one parameter group $\varphi_t^{p,q} = e^{\Delta t}$ acting on the quaternionic hyperbolic space $\mathbb{H}\mathcal{H}^2$. Then*

(i) *the quaternion Kähler reduction $M(p,q) = \mu_{p,q}^{-1}(0)/\varphi^{p,q}$ is diffeomorphic to $\mathbb{R}^4$ for all (p,q) with $p < 0$.*
(ii) *the quaternion Kähler reduction $M(p,q) = \mu_{p,q}^{-1}(0)/\varphi^{p,q}$ is diffeomorphic to $S^1 \times \mathbb{R}^3$ for all (p,q) with $0 \leqslant p < |q|$.*

In these cases $M(p,q)$ has a complete self-dual Einstein metric of negative scalar curvature and its isometry group contains a 2-torus. In all other cases (i.e., if $p \geqslant |q|$) the zero set of the momentum map is empty.

Proof. We begin by defining the set

$$\mathcal{S}_{p,q} = \{(y_1, y_2) \ \mid \ f_{p,q}(y_1, y_2) = 0 \text{ and } iy_1 - \bar{y}_1 i = 2\,\mathrm{Re}(iy_1) = 0\} \tag{5.3}$$

and claim that $\mathcal{S}_{p,q} \cap \mathbb{H}\mathcal{H}^2$ can be identified with the quotient space $M(p,q)$ as a global slice for the $\varphi^{p,q}$ action on the momentum zero set. Indeed, it is clear that as the action of $e^{\Delta t}$ sends $\mathrm{Re}(iy_1)$ to $\mathrm{Re}(iy_1) + 2t$, so there is a unique point of $\mathcal{S}_{p,q}$ on each orbit of $e^{\Delta t}$ in $\mu_{p,q}^{-1}(0)$. It remains to describe the set $\mathcal{S}_{p,q} \cap \mathbb{H}\mathcal{H}^2$.

For $p \neq 0$, there is a unique $(y_1, y_2) \in \mathcal{S}_{p,q}$ for any $y_2 \in \mathbb{H}$. We now note that $\mathbb{H}\mathcal{H}^2$ is the domain

$$\begin{aligned} p(y_1 + \bar{y}_1) + p\bar{y}_2 y_2 &< 0, \qquad p > 0 \\ p(y_1 + \bar{y}_1) + p\bar{y}_2 y_2 &> 0, \qquad p < 0. \end{aligned}$$

On the other hand

$$0 = \mathrm{Re}(if_{p,q}) = 1 - p(y_1 + \bar{y}_1) - q\,\mathrm{Re}(i\bar{y}_2 i y_2)$$

so that $\mathcal{S}_{p,q} \cap \mathbb{H}\mathcal{H}^2$ may be identified with the set of $y_2 \in \mathbb{H}$ satisfying

$$\begin{aligned} -q\,\mathrm{Re}(i\bar{y}_2 i y_2) + p\bar{y}_2 y_2 &< -1, \qquad p > 0 \\ -q\,\mathrm{Re}(i\bar{y}_2 i y_2) + p\bar{y}_2 y_2 &> -1, \qquad p < 0. \end{aligned}$$

Writing $y_2 = z_2 + jw_2$ for $w_2, z_2 \in \mathbb{C}$, this is the domain in $\mathbb{C}^2$ given by

$$\begin{aligned} (p+q)|z_2|^2 + (p-q)|w_2|^2 &< -1, \qquad p > 0 \\ (p+q)|z_2|^2 + (p-q)|w_2|^2 &> -1, \qquad p < 0. \end{aligned}$$

For $p > 0$, this domain is empty unless $p < |q|$, in which case it is the exterior of a hyperboloid, which is diffeomorphic to $S^1 \times \mathbb{R}^3$. For $p < 0$, this domain is the interior of a hyperboloid for $-|q| < p < 0$, the interior of a cylinder for $p = -|q|$, and the interior of an ellipsoid for $p < -|q|$: all these domains are diffeomorphic to $\mathbb{R}^4$.

We now consider the case $p = 0$, when y_1 is not uniquely determined by y_2. For $q = 0$, the momentum zero set is empty. Otherwise, for $q > 0$, we have $y_2 = e^{is}/\sqrt{q}$ for $s \in \mathbb{R}$, while for $q < 0$, we have $y_2 = e^{is} j/\sqrt{-q}$ for $s \in \mathbb{R}$. In either case, $\mathcal{S}_{0,q} \cap \mathbb{H}\mathcal{H}^2$ is identified with the set of $(iy_1, e^{is}) \in \mathrm{Im}\,\mathbb{H} \times S^1$ with $y_1 + \bar{y}_1 < -p/q$. This is diffeomorphic to $S^1 \times \mathbb{R}^3$.

It is now clear that when $\mathcal{S}(p,q) \cap \mathbb{H}\mathcal{H}^2$ is non-empty, as in the previous cases, it carries a complete SDE metric of negative scalar curvature. The isometry group contains the 2-torus

$$(y_1, y_2) \mapsto (\sigma y_1 \sigma^{-1}, \tau y_2 \sigma^{-1})$$

with $(\tau, \sigma) \in U(1) \times U(1)$. $\square$

6. The Height Two Quotients

To complete our analysis of the quotients of $\mathbb{H}\mathcal{H}^2$ we consider the height two case $\tilde{T}_2(\lambda, ip)$. As in the height one case, by scaling and conjugation, we can suppose $\Delta_2(p) = \tilde{T}_2(1, ip)$ with $p \in \{0, 1\}$, so there are only two distinct quotients up to scale, but we shall carry out our computations for arbitrary p. We then have

$$\varphi_t^p(\mathbf{v}) = \begin{pmatrix} e^{ipt} & 0 & 0 \\ -t^2/2 & e^{ipt} & it \\ it & 0 & e^{ipt} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^{ipt} v_0 \\ e^{ipt} v_1 + itv_2 - t^2 v_0/2 \\ e^{ipt} v_2 + itv_0 \end{pmatrix}, \tag{6.1}$$

which reduces, in inhomogeneous coordinates $\mathbf{y} = (y_1, y_2)$, to

$$\varphi_t^p \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{ipt}(y_1 + ity_2 - t^2/2)e^{-ipt} \\ e^{ipt}(y_2 + it)e^{-ipt} \end{pmatrix}. \tag{6.2}$$

The moment map for this action is given in homogeneous or inhomogeneous coordinates by

$$\begin{aligned} \mu_p(\mathbf{v}) &= \bar{v}_0 i v_2 + \bar{v}_2 i v_0 + p(\bar{v}_0 i v_1 + \bar{v}_1 i v_0) + p\bar{v}_2 i v_2, \\ f_p(\mathbf{y}) &= iy_2 + \bar{y}_2 i + p(iy_1 + \bar{y}_1 i) + p\bar{y}_2 i y_2. \end{aligned}$$

Theorem 6.1: *Let $\Delta = \Delta_2(p) = \tilde{T}_2(1, ip)$ and consider the one parameter group $\varphi_t^p = e^{\Delta t}$ acting on the quaternionic hyperbolic space $\mathbb{H}\mathcal{H}^2$. Then the quaternion Kähler reduction $M(p) = \mu_p^{-1}(0)/\varphi^p$ is diffeomorphic to $\mathbb{R}^4$ for all p, and carries a complete self-dual Einstein metric of negative scalar curvature whose isometry group contains $S^1 \times \mathbb{R}$. Furthermore, $M(0)$ is quaternionic hyperbolic space.*

Proof. Consider the following set

$$\mathcal{S}_p = \{\mathbf{y} \ \mid \ f_p(y_1, y_2) = 0 \text{ and } iy_2 - \bar{y}_2 i = 2\,\mathrm{Re}(iy_2) = 0\}. \tag{6.3}$$

It is clear that this is a global slice for the action of $e^{\Delta t}$ on the zero set of the momentum map. For $p = 0$, we obtain $y_2 = 0$, and hence $\mathcal{S}_p \cap \mathbb{H}\mathcal{H}^2$ is diffeomorphic to $\{y_1 \in \mathbb{H} : \mathrm{Re}\, y_1 < 0\}$, so let us suppose that $p \neq 0$. We write $y_2 = s_2 + jw_2$ with $s_2 \in \mathbb{R}$ and $w_2 \in \mathbb{C}$. The momentum constraint determines the imaginary part of iy_1 in terms of y_2. In particular, it implies that

$$2s_2/p + y_1 + \bar{y}_1 + s_2^2 - |w_2|^2 = 0.$$

We find that y_2 is constrained to lie in the paraboloid $s_2/p > |w_2|^2$. $M(p)$ is diffeomorphic to the product of this paraboloid with the real line, which is diffeomorphic to $\mathbb{R}^4$, and as before has a complete SDE metric of negative scalar curvature.

The isometry group of the quotient metric contains the group generated by $\tilde{T}_1(\lambda, ip, ip)$, which is isomorphic to $S^1 \times \mathbb{R}$. The last statement follows from a direct computation. □

7. The Bergman Metric on the 4-Ball

In this section we turn our attention to the quotients of $\mathbb{H}\mathcal{H}^{1,1} = P_{\mathbb{H}}(\mathbb{H}^{1,2}_+)$. One could consider all the cases studied in the previous four sections. Locally we will get families of metrics of both positive and negative scalar curvature. However, because $\mathbb{H}\mathcal{H}^{1,1}$ is not Riemannian, singularities can arise when the vector field generating the $\varphi_\Delta(t) = e^{\Delta t}$ action is null somewhere on the zero-set of the momentum map. For this reason, we shall restrict our attention to the special case $\Delta = \Delta_0(\mathbf{p})$ (cf. [Gal87a]). Furthermore, it will be convenient to switch signature and take quotients of $P_{\mathbb{H}}(\mathbb{H}^{2,1}_-)$: this means we don't have to reverse the sign of the quotient metric to get a positive definite metric of negative scalar curvature.

We begin by placing the case $\mathbf{p} = (1, 1, 1)$ in a more general context. Recall the following construction of the Wolf space $X(2, k) = U(2, k)/U(2) \times U(k)$. We start with the space

$P_{\mathbb{H}}(\mathbb{H}^{2,k}_-)$ and the diagonal circle action on $P_{\mathbb{H}}(\mathbb{H}^{2,k}_-)$, described in quaternionic coordinates $\mathbf{u} = (u_0, u_1, u_2, \ldots, u_{k+1})$ as

$$\varphi_t(\mathbf{u}) = e^{2\pi i t}\mathbf{u}, \tag{7.1}$$

where $t \in [0, 1/2)$. The moment map for this action reads

$$\mu(\mathbf{u}) = -\bar{u}_0 i u_0 - \bar{u}_1 i u_1 + \sum_{\alpha=2}^{k+1} \bar{u}_\alpha i u_\alpha. \tag{7.2}$$

By introducing the complex coordinates $u_\alpha = z_\alpha + j\bar{w}_\alpha$ and the matrices

$$\mathbb{Z} = \begin{pmatrix} \mathbb{Z}_0 \\ \mathbb{Z}_1 \end{pmatrix}, \quad \mathbb{Z}_0 = \begin{pmatrix} z_0 & w_0 \\ z_1 & w_1 \end{pmatrix}, \quad \mathbb{Z}_1 = \begin{pmatrix} z_2 & w_2 \\ \vdots & \vdots \\ z_{k+1} & w_{k+1} \end{pmatrix}, \tag{7.3}$$

we can describe the set $\mu^{-1}(0) \cap \mathcal{H}_{2,k}(-1)$ by a matrix equation

$$-\mathbb{Z}_0^\dagger \mathbb{Z}_0 + \mathbb{Z}_1^\dagger \mathbb{Z}_1 = -\mathbb{I}_{2\times 2}. \tag{7.4}$$

Now, one observes that the $U(1)\cdot \mathrm{Sp}(1) \simeq U(2)$ which takes us from $\mu^{-1}(0) \cap \mathcal{H}_{2,k}(-1)$ to the quotient is nothing but $U(2)$ matrix multiplication of $\mathbb{Z}$ from the right. This action is free and the quotient is simple a bounded domain in $\mathbb{C}^{2k}$. As homogeneous (symmetric) spaces

$$\mu^{-1}(0) \cap \mathcal{H}_{2,k}(-1) \simeq U(2,k)/U(k), \tag{7.5}$$

and

$$M = \frac{\mu^{-1}(0) \cap \mathcal{H}_{2,k}(-1)}{U(2)} = \frac{U(2,k)}{U(2) \times U(k)}. \tag{7.6}$$

In particular, when $k = 1$ we get the complex hyperbolic (or Bergman) metric on the until ball in $\mathbb{C}^2$.

Below, we will show that this construction is rigid in a sense that an introduction of weights automatically leads to orbifold singularities. As we are interested in 4-dimensional quotients we will set $k = 1$. In the previous sections we have seen that all of the complete $U(2)$-symmetric SDE metrics of negative scalar curvature can be obtained as quaternion Kähler quotients of the ball $P_{\mathbb{H}}(\mathbb{H}^{1,2}_-)$. The only exception is the complex hyperbolic Bergman metric. The above calculation now shows that this metric can be constructed as a quotient of the pseudo-Riemannian quaternion Kähler manifold $P_{\mathbb{H}}(\mathbb{H}^{2,1}_-)$. More generally, take $\Delta = \Delta(\mathbf{p})$ and examine the following circle action

$$\varphi_t^{\mathbf{p}}(u_0, u_1, u_2) = (e^{2\pi i p_0 t}u_0,\ e^{2\pi i p_1 t}u_1,\ e^{2\pi i p_2 t}u_2) \tag{7.7}$$

where $\mathbf{p} = (p_0, p_1, p_2) \in \mathbb{Z}^3$, $\gcd(p_0, p_1, p_2) = 1$, $t \in [0, 1)$ when all the weights are odd, and $t \in [0, 1/2)$ otherwise. Now, the moment map $\mu_{\mathbf{p}} : P_{\mathbb{H}}(\mathbb{H}^{2,1}_-) \to \mathcal{V}$ is given as

$$\mu_{\mathbf{p}}(\mathbf{u}) = -p_0 \bar{u}_0 i u_0 - p_1 \bar{u}_1 i u_1 + p_2 \bar{u}_2 i u_2. \tag{7.8}$$

Theorem 7.1: *Let $\mathbf{p} \in (\mathbb{Z}^+)^3$ and let $M(\mathbf{p})$ be the quaternion Kähler quotient of $P_{\mathbb{H}}(\mathbb{H}^{2,1}_-)$ by the above circle action. Then $M(\mathbf{p})$ has orbifold singularities unless $\mathbf{p} = (1,1,1)$ in which case $M(1,1,1) \simeq U(2,1)/U(2) \times U(1)$ is the symmetric complex hyperbolic metric on the unit ball in $\mathbb{C}^2$.*

Proof. Unlike in the case of $P_{\mathbb{H}}(\mathbb{H}^{2,1}_-)$ we no longer have the advantage of global coordinates. We need to consider two cases

$$P_{\mathbb{H}}(\mathbb{H}^{2,1}_-) = \mathcal{U}_0 \cup \mathcal{U}_1, \tag{7.9}$$

where $\mathcal{U}_i$ are defined as a submanifold on which $u_i \neq 0$. We first consider $\mathcal{U}_0$, where we can switch to inhomogeneous local chart $(x_1^0, x_2^0) = (u_1 u_0^{-1}, u_2 u_0^{-1})$. On $\mathcal{U}_0$ we have

$$-|x_1^0|^2 + |x_2^0|^2 < 1. \tag{7.10}$$

As before, the action and the zero level of the moment map become

$$\varphi_t^{\mathbf{p}}(x_1^0, x_2^0) = (e^{2\pi i p_1 t} x_1^0 e^{-2\pi i p_0 t}, e^{2\pi i p_2 t} x_2^0 e^{-2\pi i p_0 t}), \tag{7.11}$$

$$0 = -ip_0 - p_1 \bar{x}_1^0 i x_1^0 + p_2 \bar{x}_2^0 i x_2^0. \tag{7.12}$$

We then write

$$\mathbf{x}^0 = \mathbf{z}^0 + j\bar{\mathbf{w}}^0, \tag{7.13}$$

where $(\mathbf{z}^0, \mathbf{w}^0) \in \mathcal{U}_0$ and observe that on $\mathcal{U}_0$

$$\varphi_t^{\mathbf{p}} \begin{pmatrix} z_1^0 & w_1^0 \\ z_2^0 & w_2^0 \end{pmatrix} = \begin{pmatrix} e^{2\pi i(p_1-p_0)t} z_1^0 & e^{2\pi i(p_1+p_0)t} w_1^0 \\ e^{2\pi i(p_2-p_0)t} z_2^0 & e^{2\pi i(p_2+p_0)t} w_2^0 \end{pmatrix}$$

while the moment map equations (7.12) become

$$-p_1(|z_1^0|^2 - |w_1^0|^2) + p_2(|z_2^0|^2 - |w_2^0|^2) = p_0, \quad -p_1 \bar{w}_1^0 z_1^0 + p_2 \bar{w}_2^0 z_2^0 = 0. \tag{7.14}$$

In this case we no longer have any analogue of Proposition 3.1 as $\mu_{\mathbf{p}}^{-1}(0)$ always intersects the open set defined by (7.10).

Without loss of generality we will further assume that all weights are non-negative. Furthermore, neither p_0 nor p_1 can equal 0 if we want the quotient to be non-singular. If, say, $p_0 = 0$ then take $(u_0, 0, 0) \in P_{\mathbb{H}}(\mathbb{H}_-^{2,1})$. This point is also on the level set of the moment map and it is fixed by every element of $S^1(\mathbf{p})$. The third weight p_2 can be zero. On $\mathcal{U}_0 \cap \mu_{\mathbf{p}}^{-1}(0)$ we can choose $z_1^0 = z_2^0 = w_2^0 = 0$ and $|w_1^0|^2 = p_0/p_1$ which is a circle of points fixed by $\mathbb{Z}_{p_0+p_1}$. Hence, in order to get smooth quotient we must assume all $p_0 = p_1 = 1$ and p_2 is odd. But then, taking $z_1^0 = w_1^0 = w_2^0 = 0$ and $|z_2^0|^2 = p_0/p_2 = 1/p_2$ one gets a circle of points where the isotropy group equals $\mathbb{Z}_{(p_2+p_0)/2}$. This forces $p_0 = p_1 = p_2 = 1$. From our previous example we know that $M(1,1,1)$ is the complex hyperbolic Bergman metric on $\mathbb{C}^2$. □

Remark 7.1: Let us observe that all quaternion Kähler reductions of the symmetric space $X(2,2) \simeq U(2,2)/U(2) \times U(2)$ can now be obtained using our construction in a very simple manner. As $X(2,2)$ is by itself reduction of $P_{\mathbb{H}}(\mathbb{H}_-^{2,2})$ by the circle action corresponding to the generator $T_1 = ip\mathbb{I}_4$ we can consider all possible quotients of $P_{\mathbb{H}}(\mathbb{H}_-^{2,2})$ by 2-dimensional Lie algebras $\mathfrak{g} = \{T_1, T_2\}$, where $T_2 \in \mathfrak{sp}(2,2)$. Since T_1 is fixed to be a multiple of the identity these are classified by the adjoint orbit $[T_2]$ in $\mathfrak{sp}(2,2)$. Hence, one could begin by enumerating all such classes. Here, there are many more cases. To begin with $\mathfrak{sp}(2,2)$ has 3 different Cartan subalgebras. In addition, we have elements of height 0,1,2,3. In fact, there are six distinct families of 'purely' nilpotent classes [BC77]. All of these quotients can be examined and they should lead to many new metrics.

Example 7.2: The simplest example when one gets non-trivial negative SDE Hermitian metric is deformation of the Bergman metric on the 4-ball. We choose the second generator as

$$T_2(p,q,r) = \begin{pmatrix} ip & 0 & 0 & 0 \\ 0 & iq & 1 & 0 \\ 0 & 1 & iq & 0 \\ 0 & 0 & 0 & ir \end{pmatrix} \in \mathfrak{sp}(2,2), \tag{7.15}$$

One can easily see that $p = q = r = 0$ gives the Bergman metric which should correspond to a 4-parameter family of deformations of this metric. Detailed analysis of this and other quotients will be carried out elsewhere.

8. Quotients, hyperbolic eigenfunctions and Bochner-flat metrics

The SDE metrics that we have constructed have in common that they possess (at least) two commuting Killing vector fields, and therefore belong to the class of metrics classified locally by Calderbank and Pedersen [CP02]. Furthermore, according to Apostolov–Gauduchon [AG02], quaternion Kähler quotients of $\mathbb{H}\mathcal{H}^2$ are not just SDE, but Hermitian, and are therefore conformal to the self-dual (and therefore Bochner-flat) Kähler metrics classified by Bryant [Bry01]. In this section we relate our metrics to the hyperbolic eigenfunction Ansatz of Calderbank–Pedersen (which gives the explicit local form of the metrics), and to the SDE Hermitian metrics of Apostolov–Gauduchon and Bryant.

We recall that the work of Calderbank and Pedersen shows that an SDE metric of nonzero scalar curvature with two commuting Killing vector fields is determined explicitly (on the open set where the vector fields are linearly independent) by an eigenfunction F of the Laplacian on the hyperbolic plane with eigenvalue $3/4$, so it suffices to find the eigenfunction F corresponding to our quotients. According to [CP02], the hyperbolic eigenfunctions F arising as quotients of $\mathbb{H}\mathcal{H}^2$ or $\mathbb{H}\mathcal{H}^{1,1}$ should be either '3-pole' solutions, or limits in which one or more of the 'centers' of the 3-pole coincide. We shall justify this claim here.

8.1. **Quotients and hyperbolic eigenfunctions.** We first consider SDE manifolds arising as semi-quaternion Kähler quotients of $\mathbb{H}\mathcal{H}^{k-1,l}$ or $\mathbb{H}\mathcal{H}^{k,l-1}$ by $n-1$ dimensional Abelian subgroups G of $\mathrm{Sp}(k,l)$ (with $k+l=n+1$) in full generality. Following [CP02], we study such a quotient (M,g) using the Swann bundle $(\tilde{M},\tilde{g})$, which is the principal CO(3) bundle over (M,g) arising as the corresponding semi-hyperkähler quotient of $\mathbb{H}^{k,l}$. More precisely, we take the semi-hyperkähler quotient by G of (a connected component of) $\mathbb{H}^{k,l}_* = (\mathbb{H}^{k,l}\smallsetminus\mathbb{H}^{k,l}_0)/\{\pm 1\}$, which is a principal CO(3)-bundle over $\mathbb{H}\mathcal{H}^{k-1,l}\cup\mathbb{H}\mathcal{H}^{k,l-1}$. $\tilde{M}$ is thus the quotient by G of the zero-set of the momentum map of G in $\mathbb{H}^{k,l}_*$ and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{H}^{k,l}_* & \longrightarrow & \mathbb{H}\mathcal{H}^{k-1,l}\cup\mathbb{H}\mathcal{H}^{k,l-1} \\ \downarrow & & \downarrow \\ M & \longrightarrow & M, \end{array}$$

where the vertical arrows denote semi-hyperkähler and quaternion Kähler quotients, and the horizontal arrows are principal CO(3) bundles: SO(3) acts by isometries, and $\mathbb{R}^+$ by homotheties, so that if q is an $\mathbb{H}^\times$ valued function on the double cover of $\tilde{M}$ coming from a local trivialization, we have

$$\tilde{g} = s|q|^2 g + |dq + q\omega|^2,$$

where ω is the principal SO(3) connection on $\tilde{M}$ and s is a positive multiple of the scalar curvature of g, so that sg is a (possibly negative definite) SDE metric of positive scalar curvature. We can then arrange our conventions so that $|q|^2$ pulls back to the zero-set of the momentum map in $\mathbb{H}^{k,l}$ to give the absolute value $|F_{k,l}(\mathbf{u},\mathbf{u})|$ of the quadratic form.

Any $(n-1)$-dimensional Abelian subgroup G of $\mathrm{Sp}(k,l)$ lies in a maximal Abelian subgroup H, which has dimension $n+1$. For generic G this maximal Abelian subgroup will be unique, but in general we must choose such a group H so that we have a quotient group H/G acting on $\tilde{M}$ and M. The Lie algebra of this quotient group may be identified with $\mathbb{R}^2$.

Now, according to [CP02], there is also a commutative diagram

$$\begin{array}{ccc} \tilde{M} & \longrightarrow & M \\ \downarrow & & \downarrow \\ \mathrm{Im}\,\mathbb{H}\otimes_0\mathbb{R}^2 & \longrightarrow & \mathcal{H}^2, \end{array}$$

where the vertical arrows are (possibly only locally defined) isometric quotients by H/G, $\operatorname{Im}\mathbb{H}\otimes_0\mathbb{R}^2$ is the open subset of indecomposable elements of the flat vector space $\operatorname{Im}\mathbb{H}\otimes\mathbb{R}^2 \cong \operatorname{Im}\mathbb{H}\oplus\operatorname{Im}\mathbb{H}$, and $\mathcal{H}^2$ is the hyperboloid of positive definite elements of determinant one in $S^2\mathbb{R}^2$, equipped with the metric induced by the determinant on $S^2\mathbb{R}^2$ (which is the hyperbolic metric). The lower horizontal map, like the upper map, is a principal CO(3)-bundle, and is given explicitly by the Grammian map

$$\mathbf{x} = (x_1, x_2) \in \operatorname{Im}\mathbb{H}\otimes_0\mathbb{R}^2 \to \frac{1}{|x_1\wedge x_2|}\begin{pmatrix} |x_1|^2 & \langle x_1, x_2\rangle \\ \langle x_1, x_2\rangle & |x_2|^2 \end{pmatrix}.$$

Given a hyperbolic eigenfunction F on (an open subset of) $\mathcal{H}^2$, we can lift F to a homogeneity $1/2$ function $\tilde{F}$ on the corresponding union of rays in the space of positive definite elements of $S^2\mathbb{R}^2$. Now we have the following result, which was proven in the definite case (i.e., $k=0$ or $l=0$) in [CP02]. The more general result also has a more direct proof, and we correct a minor error in [CP02].

Theorem 8.1: *Let (M^4, g) be an SDE metric with two commuting Killing vector fields obtained as a semi-quaternion Kähler quotient of $\mathbb{H}\mathcal{H}^{k-1,l}$ or $\mathbb{H}\mathcal{H}^{k,l-1}$ by an $n-1$ dimensional Abelian subgroup G of $\operatorname{Sp}(k,l)$ (where $k+l=n+1$), and let $\tilde{F}$ be the homogeneity $1/2$ lift of the hyperbolic eigenfunction F generating g, locally, with respect to a 2-dimensional Abelian quotient group acting by isometries. Then the pullback of $\tilde{F}$ to the zero-set of the momentum map in $\mathbb{H}^{k,l}$ is a nonzero constant multiple of the restriction of the quadratic form $F_{k,l}(\mathbf{u},\mathbf{u})$.*

Proof. Let A be a positive definite element of $S^2\mathbb{R}^2$, and write $A = \sqrt{\det A}A_1$ with $A_1 \in \mathcal{H}^2$ having determinant one. Then by definition $\tilde{F}(A) = (\det A)^{1/4}F(A_1)$ and so $\tilde{F} = (\det A)^{1/4}F$, where F now denotes the (homogeneity 0) pullback to $S^2\mathbb{R}^2$. Now it was shown in [CP02] that the pullback of the function $A \mapsto \det A$ to the Swann bundle $\tilde{M}$ is $|q|^8/|F|^4$ (although the result is incorrectly stated there). It follows that $\tilde{F}$ pulls back to the Swann bundle to give $|q|^2F/|F|$, which pulls back to the momentum zero set in $\mathbb{H}^{k,l}$ to give a nonzero constant multiple of the absolute value of the quadratic form times a (possibly nonconstant) sign. However, $\tilde{F}$ is smooth, even through its zero-set, so the result follows. □

Note that the pullback of $\tilde{F}$ is independent of the choice of quotient torus (in the case that such a choice exists).

We are going to use this result to calculate the hyperbolic eigenfunction corresponding to the metrics we have studied in detail here. In order to do this we just need to write the quadratic form $F_{k,l}(\mathbf{u},\mathbf{u})$ in momentum coordinates and restrict it to the zero-set of the momentum map, as we now explain.

Having chosen (if there is a choice) the maximal Abelian subgroup H of $\operatorname{Sp}(k,l)$ containing G (and a basis for the Lie algebra of H so that we can identify it with $\mathbb{R}^{n+1}$), we have momentum coordinates $y_0, \dots y_n \in \operatorname{Im}\mathbb{H}$ which are independent on the open subset U of $\mathbb{H}^{k,l}$ where the H action is free. Since $F_{k,l}(\mathbf{u},\mathbf{u})$ is H-invariant it will be a function of $\mathbf{y} = (y_0, \dots y_n)$ on U, and our first task is to compute this function. Then, secondly, we must restrict to the zero-set of the momentum map of the G action. For this second step, following Bielawski–Dancer [BD00], we introduce an explicit parameterization of the momentum zero-set of G in terms of the momentum coordinates of the quotient torus. To do this, we write the Lie algebra $\mathfrak{g}$ of G as the kernel of a $2\times(n+1)$ matrix $S\colon \mathbb{R}^{n+1}\to\mathbb{R}^2$. Then the transpose matrix $S^t\colon \mathbb{R}^{2*}\to\mathbb{R}^{(n+1)*}$ parameterizes the kernel of the projection $\mathbb{R}^{(n+1)*}\to\mathfrak{g}^*$. Since the momentum map of H is injective on U, the momentum zero-set of G in U is the subset where the momentum map of H takes values in the image of S^t, so we can parameterize it by writing $\mathbf{w} = S^t\mathbf{x}$, with $\mathbf{x} = (x_1, x_2)$.

The hyperbolic eigenfunction is now obtained by substituting this into the quadratic form $F_{k,l}$, writing the result in terms of the SO(3) invariants $\langle x_i, x_j\rangle$ and restricting to the

hyperboloid $\det\langle x_i, x_j\rangle = 1$, where we can write

$$\begin{pmatrix} |x_1|^2 & \langle x_1, x_2\rangle \\ \langle x_1, x_2\rangle & |x_2|^2 \end{pmatrix} = \begin{pmatrix} 1/\rho & \eta/\rho \\ \eta/\rho & (\rho^2+\eta^2)/\rho \end{pmatrix}$$

for half-space coordinates (ρ, η) on $\mathcal{H}^2$. We now carry out this procedure for the examples we have studied.

8.2. **Subgroups of a maximal torus and the generalized Pedersen–LeBrun metrics.** Let $H \cong (S^1)^{n+1}$ be the standard maximal torus in $\mathrm{Sp}(k,l)$ acting diagonally on $\mathbb{H}^{k,l}$ with respect to the coordinates $(u_0, \dots u_n)$, i.e., the jth circle acts by scalar multiplication by e^{it} on the jth coordinate u_j, and has momentum map $y_j = \bar{u}_j i u_j$. We therefore have

$$F_{k,l}(\mathbf{u}, \mathbf{u}) = -\sum_{j=0}^{k-1} |y_j| + \sum_{j=k}^{k+l} |y_j|.$$

On the zero-set of the momentum map of G we then get

$$F_{k,l}(\mathbf{u}, \mathbf{u}) = -\sum_{j=0}^{k-1} |a_j x_2 - b_j x_1| + \sum_{j=k}^{k+l} |a_j x_2 - b_j x_1|,$$

where the matrix S_{ij} defining $\mathfrak{g}$ has columns $(-b_j, a_j)$. We now observe that

$$\begin{aligned} |ax_2 - bx_1| &= \sqrt{a^2|x_2|^2 - 2ab\langle x_1, x_2\rangle + b^2|x_1|^2} \\ &= \frac{\sqrt{a^2(\rho^2+\eta^2) - 2ab\eta + b^2}}{\sqrt{\rho}} = \frac{\sqrt{a^2\rho^2 + (a\eta - b)^2}}{\sqrt{\rho}} \end{aligned} \tag{8.1}$$

and thus the corresponding hyperbolic eigenfunction is

$$F(\rho, \eta) = -\sum_{j=0}^{k-1} \frac{\sqrt{a_j^2\rho^2 + (a_j\eta - b_j)^2}}{\sqrt{\rho}} + \sum_{j=k}^{k+l} \frac{\sqrt{a_j^2\rho^2 + (a_j\eta - b_j)^2}}{\sqrt{\rho}}$$

in accordance with the discussion in [CP02]—see also [CS03].

These hyperbolic eigenfunctions may be interpreted as 'multipole' solutions, in the sense that they are a linear combination of solutions of the form (8.1) which we regard as the eigenfunction generated by a monopole source at the point $\eta = b/a$ on the boundary $\rho = 0$ of the hyperbolic plane (which is a circle $\mathbb{R} \cup \{\infty\}$).

In the case studied in this paper, $n = 3$, and the vectors (a_0, a_1, a_2) and (b_0, b_1, b_2) are any two linearly independent solutions to the equation $a_0p_0 + a_1p_1 + a_2p_2 = 0$, where p_0, p_1, p_2 are the weights of the torus action.

Note that $\mathrm{SL}(2,\mathbb{R})$ acts on the vectors (a_j, b_j) to give equivalent solutions so that the points b_j/a_j can be fixed (for instance at $1, -1$ and ∞, as in [CP02]—we remark that the points are distinct provided the weights p_0, p_1, p_2 are nonzero).

8.3. **The generalized Pedersen metrics.** For the generalized Pedersen metrics, the family of generators that we are using span a Cartan subalgebra of $\mathfrak{sp}(1,2)$ which is not the Lie algebra of a maximal torus. However, this can be understood as an analytic continuation of the generalized Pedersen–LeBrun metrics (replace λ by it, where i acting on the left is a complex scalar commuting with the right quaternionic action, and diagonalize). As discussed in [CP02] this implies that the hyperbolic eigenfunction can be assumed to take the form

$$F(\rho, \eta) = \frac{a}{\sqrt{\rho}} + \frac{b+ic}{2}\frac{\sqrt{\rho^2 + (\eta+i)^2}}{\sqrt{\rho}} + \frac{b-ic}{2}\frac{\sqrt{\rho^2 + (\eta-i)^2}}{\sqrt{\rho}}.$$

This is still a 3-pole solution, but two of the sources are complex conjugate rather than real.

8.4. **The height one quotients.** In the remaining cases, it is more convenient to begin with the coordinates $\mathbf{v} = (v_0, v_1, v_2)$ that we introduced already before, so that

$$F_{1,2}(\mathbf{u}, \mathbf{u}) = \bar{v}_0 v_1 + \bar{v}_1 v_0 + \bar{v}_2 v_2.$$

In the case of the height one quotients, the momentum coordinates (in terms of the v_0, v_1, v_2 coordinates) that we shall use are

$$y_0 = \bar{v}_1 i v_0 + \bar{v}_0 i v_1, \qquad y_1 = -\bar{v}_0 i v_0, \qquad y_2 = \bar{v}_2 i v_2$$

and we compute

$$F_{1,2}(\mathbf{u}, \mathbf{u}) = \bar{v}_0 v_1 + \bar{v}_1 v_0 + \bar{v}_2 v_2 = \frac{\langle y_0, y_1 \rangle}{|y_1|} + |y_2|.$$

After substituting for x_1, x_2, the second term is treated as before, so it suffices to compute

$$\frac{\langle a_0 x_2 - b_0 x_1, a_1 x_2 - b_1 x_1 \rangle}{|a_1 x_2 - b_1 x_1|} = \frac{a_0 a_1 \rho^2 + (a_0 \eta - b_0)(a_1 \eta - b_1)}{\sqrt{\rho}\sqrt{a_1^2 \rho^2 + (a_1 \eta - b_1)^2}}.$$

Under the action of $\mathrm{SL}(2, \mathbb{R})$ this is equivalent to

$$F(\rho, \eta) = \frac{\eta}{\sqrt{\rho}\sqrt{\rho^2 + \eta^2}} = \frac{\partial}{\partial \eta} \frac{\sqrt{\rho^2 + \eta^2}}{\sqrt{\rho}}$$

which may be interpreted as an 'infinitesimal dipole', i.e., a limit of oppositely charged monopoles at $\eta = \pm\varepsilon$ as $\varepsilon \to 0$.

Thus the hyperbolic eigenfunctions corresponding to height one quotients are combinations of a monopole and an infinitesimal dipole.

8.5. **The height two quotients.** In the height two case, the maximal Abelian subalgebra containing T_2 is unique, being spanned by $i\mathbb{I}_3$, T_2 and $T_1 = iT_2^2$. The momentum coordinates of these generators are

$$y_0 = \bar{v}_1 i v_0 + \bar{v}_0 i v_1 + \bar{v}_2 i v_2, \qquad y_1 = \bar{v}_2 i v_0 + \bar{v}_0 i v_2, \qquad y_2 = -\bar{v}_0 i v_0.$$

Writing the quadratic form in these coordinates is straightforward once one has computed all the inner products between them. The result is

$$F_{1,2}(\mathbf{u}, \mathbf{u}) = \bar{v}_0 v_1 + \bar{v}_1 v_0 + \bar{v}_2 v_2 = \frac{|y_1|^2 |y_2|^2 - \langle y_1, y_2 \rangle^2 + 2\langle y_0, y_2 \rangle |y_2|^2}{2|y_2|^3}.$$

The family of quotients we consider is the span of $i\mathbb{I}_3$ and T_2, so we can take $y_0 = a_0 x_1$, $y_1 = a_1 x_1$ and $y_2 = x_2$ as our parameterization in quotient coordinates to yield

$$\frac{a_1^2(|x_1|^2 |x_2|^2 - \langle x_1, x_2 \rangle^2) + 2a_0 \langle x_1, x_2 \rangle |x_2|^2}{2|x_2|^3} = a_0 \frac{\eta}{\sqrt{\rho}\sqrt{\rho^2 + \eta^2}} + \frac{a_1^2}{2} \frac{\rho^{3/2}}{(\rho^2 + \eta^2)^{3/2}}.$$

We recognise the first term as an infinitesimal dipole. Differentiating again with respect to η, we see that the second term may be regarded as an infinitesimal tripole. As the two terms have different homogeneities in (ρ, η), by scaling the coordinates and the eigenfunction, we have just three distinct quotients:

- $a_0 = 0$, the pure tripole, corresponds the quotient by $i\mathbb{I}_3$, which is complex hyperbolic space (under the non-semisimple $\mathbb{R}^2$ action induced by T_1 and T_2);
- $a_1 = 0$, the pure dipole, corresponds to the quotient by T_2, which is real hyperbolic space (under the non-semisimple $S^1 \times \mathbb{R}$ action induced by T_1 and $i\mathbb{I}_3$);
- the nontrivial case with a_0, a_1 both nonzero.

8.6. **Infinitesimal multipoles from the quotient point of view.** We have seen, as conjectured in [CP02], that the nilpotent cases (height one and height two quotients) can be regarded as limiting cases in which two or more monopoles come together to form an infinitesimal multipole. This can be seen from the group theory of the quotient construction by realizing a non-semisimple element as a limit of semisimple ones.

For example, consider the following generator in $\mathfrak{sp}(1,2)$:

$$\mathbb{T}_{\mathbf{p},\lambda} = \begin{pmatrix} ip_0 & \lambda & 0 \\ \lambda & ip_1 & 0 \\ 0 & 0 & ip_2 \end{pmatrix} \in \mathfrak{sp}(1,2). \tag{8.2}$$

Generically this generator is of height 0 but a special choice of the parameters $(\mathbf{p},\lambda)$ raises the height to 1. To see it let us consider the following one parameter group actions on the ball

$$\varphi_t^{\mathbf{p},\lambda}(\mathbf{u}) = \exp(\mathbb{T}_{\mathbf{p},\lambda}t)\cdot\mathbf{u} \equiv \mathbb{A}_{\mathbf{p},\lambda}(t)\cdot\mathbf{u},$$

where now all (λ, p_0, p_1, p_2) are real parameters and $\mathbb{A}_{\mathbf{p},\lambda}(t) \in U(1,2) \subset Sp(1,2)$ and we assume $\lambda \neq 0$. We set

$$\alpha = \frac{p_0 - p_1}{2}, \quad \beta = \frac{p_0 + p_1}{2}, \quad \gamma = \sqrt{|\alpha^2 - \lambda^2|}. \tag{8.3}$$

We can compute the matrix $\mathbb{A}_{\mathbf{p},\lambda}(t)$ explicitly. Depending on the sign of $\alpha^2 - \lambda^2$ we get three distinct cases. If we denote the corresponding $U(1,2)$ matrices by $\mathbb{A}^+_{\mathbf{p},\lambda}(t), \mathbb{A}^0_{\mathbf{p},\lambda}(t)$, and $\mathbb{A}^-_{\mathbf{p},\lambda}(t)$, we obtain

$$\mathbb{A}^+_{\mathbf{p},\lambda}(t) = \begin{pmatrix} e^{i\beta t}(\cosh\gamma t + \frac{i\alpha}{\gamma}\sinh\gamma t) & \frac{\lambda}{\gamma}e^{i\beta t}\sinh\gamma t & 0 \\ \frac{\lambda}{\gamma}e^{i\beta t}\sinh\gamma t & e^{i\beta t}(\cosh\gamma t - \frac{i\alpha}{\gamma}\sinh\gamma t) & 0 \\ 0 & 0 & e^{ip_2 t} \end{pmatrix},$$

$$\mathbb{A}^0_{\mathbf{p},\lambda}(t) = \begin{pmatrix} e^{i\beta t}(1 + i\alpha t) & e^{i\beta t}\lambda t & 0 \\ e^{i\beta t}\lambda t & e^{i\beta t}(1 - i\alpha t) & 0 \\ 0 & 0 & e^{ip_2 t} \end{pmatrix},$$

$$\mathbb{A}^-_{\mathbf{p},\lambda}(t) = \begin{pmatrix} e^{i\beta t}(\cos\gamma t + \frac{i\alpha}{\gamma}\sin\gamma t) & \frac{\lambda}{\gamma}e^{i\beta t}\sin\gamma t & 0 \\ \frac{\lambda}{\gamma}e^{i\beta t}\sin\gamma t & e^{i\beta t}(\cos\gamma t - \frac{i\alpha}{\gamma}\sin\gamma t) & 0 \\ 0 & 0 & e^{ip_2 t} \end{pmatrix}.$$

Note that $\lim_{\gamma\to 0}\mathbb{A}^+_{\mathbf{p},\lambda}(t) = \lim_{\gamma\to 0}\mathbb{A}^-_{\mathbf{p},\lambda}(t) = \mathbb{A}^0_{\mathbf{p},\lambda}(t)$. Also, $\mathbb{A}^-_{\mathbf{p},\lambda}(t)$ is actually a circle provided the triple (γ, β, p_2) is commensurate (all ratios are in $\mathbb{Q}$).

Note that the above calculation has to do with writing

$$T_{\mathbf{p},\lambda} = L + N = \begin{pmatrix} i\beta & 0 & 0 \\ 0 & i\beta & 0 \\ 0 & 0 & ip_2 \end{pmatrix} + \begin{pmatrix} i\alpha & \lambda & 0 \\ \lambda & -i\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{8.4}$$

where $[L, N] = 0$ and $L = T_0(i\beta, i\beta, ip_2)$. Now, $N^2 = 0$ when $\lambda^2 = \alpha^2$. This shows that when $\lambda^2 = \alpha^2$, $T_{\mathbf{p},\lambda}$ must be conjugated to some $T_1(\mu, p, q)$ of Definition 2.1. When $\lambda^2 \neq \alpha^2$ the generator $T_{\mathbf{p},\lambda}$ has height 0 and, depending on the sign of $\lambda^2 - \alpha^2$, is conjugated either to some $T_0(\mu, ip, iq)$ or $T_0(iq_0, iq_1, iq_2)$. In either case, one can think of height 1 metrics as certain limits of height 0 metrics.

8.7. **Quotients and Bochner-flat Kähler metrics.** We finally discuss the relationship between quaternion Kähler quotients of $\mathbb{H}\mathcal{H}^2$ or $\mathbb{H}\mathcal{H}^{1,1}$ and Bochner-flat (i.e., self-dual) Kähler surfaces. On a self-dual Kähler surface (M, h, J) the conformal metric $g = s_h^{-2}h$, defined wherever the scalar curvature s_h of h is nonzero is an SDE metric. Conversely, h can be recovered from g using the fact that the Weyl tensor $W = W^+$ of a self-dual Kähler

surface is a constant multiple of $s_h\omega\otimes_0\omega$, where ω is the Kähler form and the subscript zero denotes the tracefree part in $S^2_0(\Lambda^2T^*_+M)$: thus, up to a constant $s_h = |W|_h = |W|_g^{1/3}$ and $h = |W|^{2/3}g$. This sets up a one to one correspondence, at least locally, between self-dual Kähler metrics and SDE Hermitian metrics [Der83, AG02] which are not conformally flat. (h and g are equal up to homothety iff they are locally symmetric.)

Bochner-flat Kähler manifolds have been completely classified, locally and globally, by Bryant [Bry01]. The local classification is quite easy to understand: over a Bochner-flat Kähler $2n$-manifold M, the (locally defined) rank 1 bundle with connection, whose curvature is the Kähler form of M, has a flat CR structure (given by the horizontal lift of the Kähler structure on M) and is therefore locally CR isomorphic to S^{2n+1}. This realises the Kähler metric on M as local quotient of S^{2n+1} by a one parameter subgroup of $\mathrm{PSU}(1,n+1)$, the group of CR automorphisms of S^{2n+1} (which is naturally realised as the quadric of totally null complex lines in the projective space of $\mathbb{C}^{1,n+1}$). It then follows that Bochner-flat Kähler metrics are classified by adjoint orbits in $\mathfrak{su}(1,n+1)$.

Specialising to $n=2$, self-dual Kähler surfaces are classified, as local quotients of S^5, by adjoint orbits in $\mathfrak{su}(1,3)$, and it is natural to conjecture that the corresponding SDE Hermitian metrics are obtained as (perhaps only local) quaternion Kähler quotients of $\mathbb{H}P^2$, $\mathbb{H}\mathcal{H}^2$ and $\mathbb{H}\mathcal{H}^{1,1}$, classified by adjoint orbits in $\mathfrak{sp}(3)$ and $\mathfrak{sp}(1,2)$. This is essentially correct, as the work of Apostolov–Gauduchon [AG02] shows.

Proposition 8.2: *Let (M,g) be a self-dual Einstein manifold given as a (semi-)quaternion Kähler quotient of $\mathbb{H}P^2$, $\mathbb{H}\mathcal{H}^2$ or $\mathbb{H}\mathcal{H}^{1,1}$ by a (possibly local) S^1 or $\mathbb{R}$ action. Then (M,g) admits a compatible Hermitian structure and there is an invariant Sasakian structure on the momentum zero-set of the action, whose underlying CR structure is flat.*

Sketch proof. We outline the arguments, refering the reader to Apostolov–Gauduchon [AG02] for more details. Let be K a quaternionic Killing vector field on a (semi-)quaternion Kähler manifold Q of nonzero scalar curvature; this means that

$$\nabla K \in \mathrm{C}^\infty(Q, \mathcal{V}_Q \oplus \mathfrak{sp}(TQ)) \subset \mathrm{C}^\infty(Q, \mathfrak{so}(TQ)),$$

where $\mathcal{V}_Q$ is the bundle of $\mathfrak{sp}(1)$'s in $\mathfrak{so}(TQ)$ defining the quaternionic structure and $\mathfrak{sp}(TQ) \subset \mathfrak{so}(TQ)$ is the bundle of $\mathfrak{sp}(n)$'s in $\mathfrak{so}(TQ)$ consisting the skew endomorphisms which commute with $\mathcal{V}_Q$. Since the scalar curvature is nonzero, the momentum map of K is defined to be the $\mathcal{V}_Q$ component of ∇K. It follows that on the zero-set $\mathcal{S}$ of the momentum map, ∇K is a section of $\mathfrak{sp}(TQ)$. It is also K invariant, so its horizontal part descends to the (perhaps only locally defined) quotient $M = \mathcal{S}/K$, which is the quaternion Kähler quotient of Q by K, to give a section Ψ of $\mathfrak{sp}(TM)$. If Q is an 8-manifold, then M is a 4-manifold and $\mathfrak{so}(TM) = \mathcal{V}_M \oplus \mathfrak{sp}(TM)$ and with our conventions $\mathcal{V}_M = \mathfrak{so}_-(TM)$ and $\mathfrak{sp}(TM) \cong \mathfrak{so}_+(TM)$, the bundles of (anti-)self-dual endomorphisms associated to $\Lambda^2_-T^*M$ and $\Lambda^2_+T^*M$ using the metric. It follows that wherever Ψ is nonzero $\sqrt{2}\Psi/|\Psi|$ is a almost complex structure which is self-dual (i.e., orthogonal and commuting with the quaternionic structure), so that M is an almost Hermitian manifold. Apostolov and Gauduchon show that this complex structure is integrable if Q is $\mathbb{H}P^2$ or $\mathbb{H}\mathcal{H}^2$ and their argument applies unchanged to $\mathbb{H}\mathcal{H}^{1,1}$ (it is a straightforward consequence of the fact that these spaces are flat as quaternionic manifolds). Thus M is SDE Hermitian, as claimed, and one can check that the conformal Kähler metric h is $|K|^{-2}g$.

Now the curvature of the rank 1 bundle $\mathcal{S} \to M$ (i.e., the horizontal part of the 2-form associated to $|K|^{-2}\nabla K$) is then the Kähler form of M, so that the Kähler structure on M lifts to the horizontal distribution to give a K-invariant Sasakian structure on $\mathcal{S}$. This is the canonical Sasakian structure associated to (M,h), and the underlying CR structure is flat because (M,h) is self-dual. □

A flat CR manifold is locally isomorphic to S^5 with its standard flat CR structure (as the projective light cone in $\mathbb{C}^{1,3}$). Since K generates an action by CR automorphisms, such a local isomorphism determines an element of $\mathfrak{su}(1,3)$, the Lie algebra of CR automorphisms of S^5. However, the local isomorphism is only determined up to conjugation by $\mathrm{PSU}(1,3)$, so we do not obtain a Lie algebra homomorphism from $\mathfrak{sp}(3)$ or $\mathfrak{sp}(1,2)$ to $\mathfrak{su}(1,3)$—these Lie algebras are certainly not isomorphic.

Nevertheless, the classifications of self-dual Kähler manifolds (in terms of adjoint orbits in $\mathfrak{su}(1,3)$) and quotients of $\mathbb{H}P^2$, $\mathbb{H}\mathcal{H}^2$ and $\mathbb{H}\mathcal{H}^{1,1}$ (in terms of adjoint orbits in $\mathfrak{sp}(3)$ and $\mathfrak{sp}(1,2)$) do essentially coincide. This is slightly subtle, as in both quotient constructions the manifold (or orbifold) corresponding to a conjugacy class may not be connected: for the Kähler metric, these components correspond to Bryant's 'momentum cells', whereas for the Einstein metric, the conformal infinity (which in Kähler terms is the zero-set of s_h) separates the quotients of $\mathbb{H}\mathcal{H}^2$ from the quotients of $\mathbb{H}\mathcal{H}^{1,1}$. Also some of the self-dual Kähler quotients of S^5 will have associated Einstein metrics which are scalar-flat, while some of the SDE quotients of $\mathbb{H}P^2$, $\mathbb{H}\mathcal{H}^2$ and $\mathbb{H}\mathcal{H}^{1,1}$ will be conformally flat.

One way to relate the classifications is to observe that every element of $\mathfrak{su}(1,3)$ has a spacelike eigenvector, and some of them (the 'elliptic' elements) have a timelike eigenvector too. Since $\mathrm{PSU}(1,3)$ acts transitively on the spacelike or timelike lines, we can fix one of each and conjugate any element of $\mathfrak{su}(1,3)$ into $\mathfrak{u}(1,2)$, and the elliptic elements into $\mathfrak{u}(3)$. On the other hand all adjoint orbits in $\mathfrak{sp}(3)$ are represented by elements of $\mathfrak{u}(3)$, and the same is true for $\mathfrak{sp}(1,2)$, since we have given representatives in $\mathfrak{u}(1,2)$ in Definition 2.1.

Remark 8.1: There is a rather beautiful Hermitian/quaternionic real form of the classical Klein correspondence that allows us to make the identification of adjoint orbits more natural. Recall that there is a special isomorphism between $\mathfrak{so}(6,\mathbb{C})$ and $\mathfrak{sl}(4,\mathbb{C})$: $\mathbb{C}^4$ is the spin representation of $\mathfrak{so}(6,\mathbb{C})$, or, more straightforwardly, $\mathfrak{sl}(4,\mathbb{C})$ acts on $\Lambda^2\mathbb{C}^4$ (via $A\cdot u\wedge v = A(u)\wedge v + u\wedge A(v)$) preserving a complex bilinear form g_c given by the contraction of $(\alpha,\beta)\mapsto \alpha\wedge\beta$ with the volume element. This isomorphism underlies the Klein correspondence:

- lines in $P(\mathbb{C}^4)$ correspond bijectively to points on the quadric in $P(\Lambda^2\mathbb{C}^4)$ ($P(U)$ corresponds to null line $\Lambda^2 U$);
- points in $P(\mathbb{C}^4)$ correspond bijectively to α-planes in the quadric ($[u]$ corresponds to projectivization of the maximal totally null subspace $\{u\wedge v : v\in\mathbb{C}^4\}$);
- planes in $P(\mathbb{C}^4)$ correspond bijectively to β-planes in the quadric ($P(W)$ corresponds to the projectivization of the maximal totally null subspace $\Lambda^2 W$).

Now $\mathfrak{su}(1,3)$ is the real form of $\mathfrak{sl}(4,\mathbb{C})$ preserving a Hermitian metric $(.,.)$ of signature $(1,3)$. Consider now the Hodge star operator on $\Lambda^2\mathbb{C}^{1,3}$ defined by $(*\alpha)\wedge\beta = (\alpha,\beta)vol$. For this to make sense, we must take the Hermitian metric to be anti-linear in α and thus $*$ anti-commutes with i. The signature of the metric implies that $*^2=-1$, so $j:=*$ defines a quaternionic structure on $\Lambda^2\mathbb{C}^{1,3}$. It is convenient to make $\Lambda^2\mathbb{C}^{1,3}$ into a *right* quaternionic vector space in this way (thus $k=ij=j\circ i$).

We denote by $\mathfrak{so}^*(3,\mathbb{H})$ the subalgebra of $\mathfrak{so}(6,\mathbb{C})$ commuting with j: it is the real form isomorphic to $\mathfrak{su}(1,3)$. We can describe it in quaternionic terms as the Lie algebra of the group of $\mathbb{H}$-linear transformations of $\mathbb{H}^3$ preserving an (i,j,k)-invariant skew form ω, and hence also the triple of signature $(6,6)$ symmetric forms g_i, g_j, g_k defined by $g_i(a,b)=\omega(ai,b)$ and so on. Note that g_i is i-invariant, but is anti-invariant with respect to j and k, and similarly for g_j and g_k. Hence the quaternionic definition is related to the complex one by taking g_j to be the real part of g_c (since g_c is i-bilinear and j-invariant).

A spacelike or timelike line in $\mathbb{C}^{1,3}$ defines a maximal totally null (α) subspace of $\Lambda^2\mathbb{C}^{1,3}$ and its perpendicular hyperplane defines a complementary maximal totally null (β) subspace. Such a decomposition is equivalently given by a g_j-orthogonal complex structure I on $\Lambda^2\mathbb{C}^{1,3}$

commuting with the quaternionic structure: the null subspaces are the $\pm i$ eigenspaces. Note that $g(a,b) = \omega(Ia,b)$ is therefore an (i,j,k)-invariant inner product and it is easy to check that it is indefinite or definite according to whether the line is spacelike or timelike.

An element of $\mathfrak{su}(1,3)$ belongs to $\mathfrak{u}(1,2)$ or $\mathfrak{u}(3)$ (i.e., preserves the spacelike or timelike line) if and only if its action on $\Lambda^2\mathbb{C}^{1,3}$ commutes with I if and only if it is skew with respect to g. In fact this realizes $\mathfrak{u}(1,2)$ and $\mathfrak{u}(3)$ as $\mathfrak{sp}(1,2)\cap\mathfrak{so}^*(3,\mathbb{H})$ and $\mathfrak{sp}(3)\cap\mathfrak{so}^*(3,\mathbb{H})$ respectively.

For example, consider the diagonal element

$$\begin{pmatrix} ir_0 & 0 & 0 & 0 \\ 0 & ir_1 & 0 & 0 \\ 0 & 0 & ir_2 & 0 \\ 0 & 0 & 0 & ir_3 \end{pmatrix}$$

with $r_0 + r_1 + r_2 + r_3 = 0$, defined using the standard basis e_0, e_1, e_2, e_3 for $\mathbb{C}^{1,3}$ with e_0 timelike. Its action on $\Lambda^2\mathbb{C}^{1,3}$ with respect to the quaternionic basis $e_0 \wedge e_1$, $e_0 \wedge e_2$, $e_0 \wedge e_3$ is easily computed to be

$$\begin{pmatrix} i(r_0+r_1) & 0 & 0 \\ 0 & i(r_0+r_2) & 0 \\ 0 & 0 & i(r_0+r_3) \end{pmatrix},$$

where i acts by left multiplication (we have chosen our quaternionic basis so that the complex structure I determined by e_0 is left multiplication by i).

Adjoint orbits in $\mathfrak{su}(1,3)$ are essentially determined by their characteristic and minimal polynomials, and Bryant [Bry01] gives his classification in these terms—more precisely, in terms of the polynomials of the associated Hermitian matrices. If P_c is the characteristic polynomial and P_m is the minimal polynomial, then the degree d of P_c/P_m determines the local cohomogeneity of the self-dual Kähler metric as $2-d$. In the generic, local cohomogeneity two, case Bryant discusses the classification in detail, which he divides into Cases 1–4.

For reference, we shall give the correspondence between adjoint orbits in $\mathfrak{su}(1,3)$ and $\mathfrak{sp}(1,2)$ which relate Bryant's classification to ours. We do this by giving the characteristic and minimal polynomials P_c corresponding to the representatives in Definition 2.1. These correspondences are obtained by choosing an element of $\mathfrak{su}(1,3)$ with given P_c, P_m and spacelike eigenvector e_1, and computing its action on $\Lambda^2\mathbb{C}^{1,3}$ with quaternionic basis $e_1 \wedge e_0$, $e_1 \wedge e_2$, $e_1 \wedge e_3$.

We recall that there are exceptional adjoint orbits which do not give SDE and self-dual Kähler metrics which are conformal. For these exceptional orbits, the self-dual Kähler metric is conformal to a scalar-flat SDE metric, while the quotient SDE metric is the real hyperbolic metric (conformally flat).

We begin with the (local) cohomogeneity two Kähler metrics, where $P_m(t) = P_c(t)$.

(i) $P_c(t) = (t-r_0)(t-r_1)(t-r_2)(t-r_3)$, where r_0, r_1, r_2, r_3 are distinct with $r_0+r_1+r_2+r_3 = 0$. This is Bryant's Case 4, and corresponds to

$$T_0(ip_0, ip_1, ip_2) = \begin{pmatrix} ip_0 & 0 & 0 \\ 0 & ip_1 & 0 \\ 0 & 0 & ip_2 \end{pmatrix}.$$

with $p_0 = r_0 + r_1$, $p_1 = -(r_0 + r_2)$, $p_2 = -(r_0 + r_3)$ (and so $p_i \neq \pm p_j$ for i, j distinct). The exceptional orbits arise when one of the weights vanish.

(ii) $P_c(t) = (t-r_1)(t-r_2)(t-r-i\lambda)(t-r+i\lambda)$, where r_1, r_2 are distinct with $r_1+r_2+2r = 0$. This is Bryant's Case 1 and corresponds to

$$T_0(\lambda, ip, iq) = \begin{pmatrix} ip & \lambda & 0 \\ \lambda & ip & 0 \\ 0 & 0 & iq \end{pmatrix}$$

with $p = r + r_1$, $q = -2r = r_1 + r_2$ (and so $p \neq 0$). The exceptional orbits arise when q vanishes.

(iii) $P_c(t) = (t - r_1)(t - r_2)(t - r)^2$, where r_1, r_2 and r are distinct with $r_1 + r_2 + 2r = 0$. This is Bryant's Case 3 and corresponds to

$$T_1(1, ip, iq) = \begin{pmatrix} ip & 0 & 0 \\ 0 & ip & 0 \\ 0 & 0 & iq \end{pmatrix} + \begin{pmatrix} i & i & 0 \\ -i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $p = r + r_1$, $q = -2r = r_1 + r_2$ (and so $p \neq 0$ and $p \neq \pm q$). The exceptional orbits arise when $q = 0$.

(iv) $P_c(t) = (t - r_1)(t - r)^3$, where r_1 and r are distinct with $r_1 + 3r = 0$. This is Bryant's Case 2 and corresponds to

$$T_2(1, ip) = \quad ip\,\mathbb{I}_3 + \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ i & i & 0 \end{pmatrix}$$

with $p = r + r_1$ (so that $p \neq 0$).

We finally consider the cohomogeneity one and homogeneous Kähler metrics.

(i) $P_c(t) = (t - r_0)^2(t - r_1)(t - r_2)$ and $P_m(t) = (t - r_0)(t - r_1)(t - r_2)$, where r_0, r_1, r_2 are distinct with $2r_0 + r_1 + r_2 = 0$. These metrics have cohomogeneity one under U(2) or U(1, 1) according to the signature of the Hermitian metric on the repeated eigenspace, and correspond to $T_0(ip, \pm ip, iq)$ or $T_0(iq, ip, \pm iq)$ with $p \neq \pm q$. When $q = 0$ we have an exceptional orbit.

Further degenerations give homogeneous metrics:

- $P_c(t) = (t - r)^3(t + 3r)$ and $P_m(t) = (t - r)(t + 3r)$ with $r \neq 0$ corresponds to $T_0(ip, ip, ip)$ and the Bergman metric;
- $P_c(t) = (t - r)^2(t + r)^2$ and $P_m(t) = (t - r)(t + r)$ with $r \neq 0$ is exceptional: the Kähler metric is the product metric on $S^2 \times \mathcal{H}^2$ and the SDE quotient (by $T_0(0, 0, ip)$ or $T_0(ip, 0, 0)$) is $\mathcal{H}^4$.

(ii) $P_c(t) = (t+r)^2(t-r-i\lambda)(t-r+i\lambda)$ and $P_m(t) = (t+r)(t-r-i\lambda)(t-r+i\lambda)$. These have cohomogeneity one under U(2) and correspond to the Pedersen metrics $T_0(\lambda, 0, ir)$, apart from exceptional orbits when $r = 0$.

(iii) $P_c(t) = (t+r)^2(t-r)^2$ or $(t+r)(t-r)^3$ and $P_m(t) = (t+r)(t-r)^2$ with $r \neq 0$, correspond to the height one quotients by $T_1(1, 0, iq)$ and $T_1(1, ip, \pm ip)$, which are cohomogeneity one metrics.

A further degeneration gives an exceptional orbit: $P_c(t) = t^4$ and $P_m(t) = t^2$. The Kähler metric in this case is flat, while the SDE quotient by $T_1(1, 0, 0)$ is $\mathcal{H}^4$.

(iv) $P_c(t) = t^4$ and $P_m(t) = t^3$ corresponds to the height two quotient $T_2(1, 0)$. This is an exceptional orbit: the self-dual Kähler metric has cohomogeneity one, but the SDE quotient is $\mathcal{H}^4$.

References

[AG02] V. Apostolov and P. Gauduchon, *Self-dual Einstein Hermitian four manifolds*, Ann. Sc. Nrom. Sup. Pisa **I** (2002), 203–246.

[BC77] N. Burgoyne and R. Cushman, *Conjugacy classes in linear groups*, J. Algebra **44** (1977), no. 2, 339–362. MR 55 #5761

[BD00] R. Bielawski and A. S. Dancer, *The geometry and topology of toric hyperkähler manifolds*, Comm. Anal. Geom. **8** (2000), 727–760.

[BGMR98] C. P. Boyer, K. Galicki, B. M. Mann, and E. G. Rees, *Compact 3-Sasakian 7-manifolds with arbitrary second Betti number*, Invent. Math. **131** (1998), no. 2, 321–344. MR **99b:**53066

[Bie99] R. Bielawski, *Complete hyper-Kähler $4n$-manifolds with a local tri-Hamiltonian R^n-action*, Math. Ann. **314** (1999), no. 3, 505–528. MR **2001c:**53058

[Biq99] O. Biquard, *Einstein deformations of hyperbolic metrics*, Surveys in differential geometry: essays on Einstein manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, pp. 235–246. MR **2001k:**53080

[Biq00] ———, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque (2000), no. 265, vi+109. MR **2001k:**53079

[Biq02] ———, *Métriques autoduales sur la boule*, Invent. Math. **148** (2002), no. 3, 545–607. MR 1 908 060

[Bry01] R. L. Bryant, *Bochner-Kähler metrics*, J. Amer. Math. Soc. **14** (2001), no. 3, 623–715 (electronic). MR **2002i:**53096

[CP02] D. M. J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, J. Differential Geom. **60** (2002), no. 3, 485–521. MR 1 950 174

[CS03] D. M. J. Calderbank and M. A. Singer, *Einstein metrics and complex singularities*, Invent. Math. (2003), to appear, math.DG/0206229.

[Der81] A. Derdziński, *Exemples de métriques de Kähler et d'Einstein autoduales sur le plan complexe*, Géometrie riemannienne en dimension 4, CEDIC, 1981.

[Der83] ———, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), no. 3, 405–433. MR **84h:**53060

[Gal87a] K. Galicki, *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Comm. Math. Phys. **108** (1987), no. 1, 117–138. MR **88f:**53088

[Gal87b] ———, *New matter couplings in $N = 2$ supergravity*, Nuclear Phys. B **289** (1987), no. 2, 573–588. MR **88j:**53076

[Gal91] ———, *Multi-centre metrics with negative cosmological constant*, Classical Quantum Gravity **8** (1991), no. 8, 1529–1543. MR **92i:**53040

[GL88] K. Galicki and H. B. Lawson, Jr., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), no. 1, 1–21. MR **89m:**53075

[Hit95] N. J. Hitchin, *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), no. 1, 30–112. MR **96g:**53057

[LeB82] C. R. LeBrun, *$\mathcal{H}$-space with a cosmological constant*, Proc. Roy. Soc. London Ser. A **380** (1982), no. 1778, 171–185. MR **83d:**83019

[LeB88] ———, *Counter-examples to the generalized positive action conjecture*, Comm. Math. Phys. **118** (1988), no. 4, 591–596. MR **89f:**53107

[LeB91] ———, *On complete quaternionic-Kähler manifolds*, Duke Math. J. **63** (1991), no. 3, 723–743. MR **92i:**53042

[Ped86] H. Pedersen, *Einstein metrics, spinning top motions and monopoles*, Math. Ann. **274** (1986), no. 1, 35–59. MR **87i:**53070

[Swa91] A. Swann, *Hyper-Kähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991), no. 3, 421–450. MR **92c:**53030

CPB and KG: Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA.
E-mail address: cboyer@math.unm.edu
E-mail address: galicki@math.unm.edu

DMJC: School of Mathematics, University of Edinburgh, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland.
E-mail address: davidmjc@maths.ed.ac.uk

PP: Università degli Studi di Roma, "La Sapienza", Dipartimento di Matematica, Piazzale Aldo Moro, I-00185 Roma, Italia.
E-mail address: piccinni@mat.uniroma1.it