HYDRODYNAMIC INTEGRABILITY VIA GEOMETRY

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ABSTRACT. This paper develops a geometric approach to the theory of integrability by hydrodynamic reductions to establish an equivalence, for a large class of quasilinear systems, between hydrodynamic integrability and the existence of nets compatible with the geometry induced on the codomain of the system. This unifies and extends known results for three subclasses of such systems. The generalization is obtained by studying the algebraic geometry of the characteristic correspondence of the system, and by introducing a generalized notion of conjugate nets.

1. Introduction

The method of hydrodynamic reductions is a test for the integrability of dispersionless systems introduced by E. Ferapontov and K. Khusnutdinova [9, 11]. It applies to PDE systems which can be written in the following first order quasilinear form:

$$A_1(\boldsymbol{u})\partial_{t_1}\boldsymbol{u} + \cdots + A_n(\boldsymbol{u})\partial_{t_n}\boldsymbol{u} = 0, \tag{1.1}$$

where the unknown is a function $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$, $\boldsymbol{t} = (t_1, \dots t_n)$ are standard coordinates on the open domain $\mathscr{M} \subseteq \mathfrak{t} \cong \mathbb{R}^n$ (with $n \geqslant 3$), the codomain $\mathscr{U}$ is (for simplicity here) an open subset of $\mathbb{R}^m$, and each $A_j\colon \mathscr{U} \to M_{k\times m}(\mathbb{R})$ is a given matrix-valued function (so k is the number of equations in the system). Note that $A_1, \dots A_n$ do not depend explicitly on $\boldsymbol{t}$, so the system is invariant under local translations in $\mathscr{M}$.

Fundamental examples include N-component hydrodynamic systems of the form

$$\partial_{t_j} R^a = \kappa_{aj}(\boldsymbol{R})\partial_{t_1} R^a, \qquad j \in \{2, \dots n\}, \quad a \in \{1, \dots N\}, \tag{1.2}$$

for $\boldsymbol{R} = (R^1, \dots R^N)\colon \mathscr{M} \to \mathscr{V}$, with $\mathscr{V}$ open in $\mathbb{R}^N$, where κ_{aj} are given functions of $\boldsymbol{r} = (r_1, \dots r_N) \in \mathscr{V}$ which satisfy the compatibility conditions that

$$\text{for all} \quad a \neq b \in \{1, \dots N\} \quad \text{and} \quad j \in \{2, \dots n\}, \quad \partial_b \kappa_{aj} = \gamma_{ab}(\boldsymbol{r})\,(\kappa_{bj} - \kappa_{aj}),$$

with $\gamma_{ab}(\boldsymbol{r})$ independent of j and $\partial_b := \partial_{r_b}$. Integrability for such systems was shown by Tsarev [23], who called them semi-hamiltonian systems of hydrodynamic type.

An N-component *hydrodynamic reduction* of (1.1) consists of functions $U\colon \mathscr{V} \to \mathscr{U}$ and $\kappa_{aj}\colon \mathscr{V} \to \mathbb{R}$, for $a \in \{1, \dots N\}$ and $j \in \{2, \dots n\}$, such that (1.2) is compatible and $\boldsymbol{u} = U \circ \boldsymbol{R}$ satisfies (1.1) if $\boldsymbol{R} = (R^1, \dots R^N)$ satisfies (1.2). These requirements impose a PDE system on the functions U and κ_{aj}, and (1.1) is *hydrodynamically integrable* if this PDE system is compatible for all $N \geqslant 2$ (although it suffices to check $N = 3$ [9, 11]).

Hydrodynamic integrability is known to be equivalent to the existence of a dispersionless Lax pair in some cases [3, 6, 7, 8, 10], but it does not require one to find the Lax pair. Thus it has the benefit that it is an algorithmic test of integrability (for this class of systems). However, it is computationally intensive, and for all but the simplest cases, carrying out the test requires symbolic computer algebra. Other algorithmic tests are available when $n = 3$ or 4 using Einstein–Weyl or self-dual conformal geometry respectively (called "integrable background geometries" in [4]), but these only apply when the characteristic variety of the system (see below) is a quadric—see [2, 5] for the general relation to Lax pairs in this setting. In some overlapping cases, Einstein–Weyl/self-dual integrability and hydrodynamic integrability are known to be equivalent [6, 7, 12], but this is established by direct computation, and a conceptual explanation is lacking.

Example 1.1. To illustrate the process in a case amenable to hand computation, let $n = 3$, $(t_1, t_2, t_3) = (t, x, y)$ and consider an equation of the form

$$(u_x + \tau(u)u_t)_t = u_{yy}, \tag{1.3}$$

for a scalar function u, where τ is a given function of one variable, and t, x, y subscripts denote partial derivatives. Then (1.3) is equivalent to the first order quasilinear system

$$u_y - v_t = 0, \qquad u_x + \tau(u)u_t - v_y = 0, \tag{1.4}$$

for $\boldsymbol{u} = (u, v)$. A hydrodynamic reduction in this case consists of functions U, V, $\mu_a = \kappa_{a2}$ and $\lambda_a = \kappa_{a3}$ on an open subset $\mathscr{V}$ of $\mathbb{R}^N$, and the compatibility of (1.2) requires that

$$\frac{\partial_b \mu_a}{\mu_b - \mu_a} = \frac{\partial_b \lambda_a}{\lambda_b - \lambda_a} \qquad \text{for all} \qquad a \neq b. \tag{1.5}$$

Using the chain rule for $u = U \circ \boldsymbol{R}$ and $v = V \circ \boldsymbol{R}$, the system

$$R^a_x = \mu_a(\boldsymbol{R}) R^a_t, \qquad R^a_y = \lambda_a(\boldsymbol{R}) R^a_t, \tag{1.6}$$

yields

$$u_y - v_t = \sum_a (\lambda_a\, \partial_a U - \partial_a V) R^a_t, \quad u_x + \tau(u)u_t - v_y = \sum_a \big((\mu_a + \tau(U))\, \partial_a U - \lambda_a\, \partial_a V\big) R^a_t.$$

Hence to satisfy (1.4) for any solution $\boldsymbol{R}$ of (1.6), the ansatz requires that

$$\text{for all} \quad a \in \{1, \dots N\}, \quad \lambda_a\, \partial_a U = \partial_a V \quad \text{and} \quad (\mu_a + \tau(U))\, \partial_a U = \lambda_a\, \partial_a V.$$

In particular

$$\mu_a + \tau(U) = \lambda_a^2,$$

which is called the *dispersion relation* and can be used to eliminate μ_a. Then (1.5) becomes

$$\frac{2\lambda_a \partial_b \lambda_a - \tau'(U) \partial_b U}{\lambda_b^2 - \lambda_a^2} = \frac{\partial_b \lambda_a}{\lambda_b - \lambda_a}, \quad \text{i.e.,} \quad \partial_b \lambda_a = -\frac{\tau'(U)\, \partial_b U}{\lambda_b - \lambda_a},$$

for $a \neq b$, while the symmetry of $\partial_b \partial_a V = \partial_b(\lambda_a\, \partial_a U)$ in a, b implies

$$\partial_b \partial_a U = -2 \frac{\tau'(U)\, \partial_a U\, \partial_b U}{(\lambda_b - \lambda_a)^2}$$

(and then V may also be eliminated). Hydrodynamic integrability requires that no further equations for U, λ_a arise in this way. In particular, the symmetry of $\partial_c \partial_b \lambda_a$ in b, c (for a, b, c distinct) implies that $\tau'' = 0$, and the symmetry of $\partial_c \partial_b \partial_a U$ is then consistent with the system. Thus (1.3) is hydrodynamically integrable if and only if either τ is constant, so that (1.3) is linear, or τ has degree 1, so that (1.3) is equivalent by an affine transformation of u to the dispersionless Kadomtsev–Petviashvilli (dKP) equation $(u_x + uu_t)_t = u_{yy}$.

The computational intensiveness motivates the search for a geometric underpinning to hydrodynamic integrability, and there is much that is already understood. First the dispersion relation has a geometric interpretation. In the case of equation (1.3), it states that for each a, $[1, \mu_a(r), \lambda_a(r)]$ is a point on the (u-dependent) projective variety

$$\{[\xi_1, \xi_2, \xi_3] \in P(\mathbb{R}^3) : \xi_1 \xi_2 + \tau(u)\xi_1^2 = \xi_3^2\},$$

at $u = U(r)$. This is the *characteristic variety* of (1.3), and for hydrodynamic reductions of general quasilinear systems (1.1), the *characteristic momenta* $\theta_a = \sum_{j=1}^n \kappa_{aj}\, dt_j$, with $\kappa_{a1} = 1$, always define points $[\theta_a]$ on the characteristic variety of the equation.

In the important case $n = 3$, A. Odeskii and V. Sokolov [17, 18] have axiomatised and studied the systems of Gibbons–Tsarev type [14] arising from hydrodynamic reductions. Here the characteristic variety is a bundle of curves over $\mathscr{U}$ and so the "Gibbons–Tsarev structure" may be described using 1-parameter families of vector fields on $\mathscr{U}$.

For arbitrary n, there are now several works [3, 6, 8] showing that for particular classes of quasilinear systems (1.1), hydrodynamic reductions correspond to nice submanifolds with respect to some interesting geometric structure on the codomain $\mathscr{U}$ of $\boldsymbol{u}$.

The main result of this paper places these latter classes in common framework. One of the central ingredients is inspired by work of A. Smith [19, 20, 21, 22], who emphasises the role not only of the characteristic variety, but also the so-called rank one variety. For a quasilinear system (1.1), the latter is quite straightforward to define: the system imposes a linear constraint on the derivative $d\boldsymbol{u}_z$ at each $z \in \mathscr{M}$, depending only on $p = \boldsymbol{u}(z)$, hence defining a p-dependent linear subspace $\mathscr{E}$ of $m \times n$ matrices ($p \in \mathscr{U}$); the *rank one variety* is the projective variety $\mathcal{R}^{\mathscr{E}}$ in $P(\mathscr{E})$ of rank one matrices in $\mathscr{E}$ up to scale. Thus it is a p-dependent projective variety lying in the *Segre variety* of all $m \times n$ matrices of rank one, up to scale. The latter is canonically isomorphic to the product $P(\mathbb{R}^n) \times P(\mathbb{R}^m)$ of the projective spaces of $\mathfrak{t} \cong \mathbb{R}^n$ and $\mathbb{R}^m$, because any rank one matrix factorizes as a product $\theta \otimes S$ of a column vector and a row vector, each determined uniquely up to scale.

Thus the rank one variety has projections onto both $P(\mathbb{R}^n)$ and $P(\mathbb{R}^m)$; the former is the characteristic variety $\mathcal{X}^{\mathscr{E}}$ of the system. I am unaware of a standard name for the latter projection $\mathcal{C}^{\mathscr{E}}$, so I refer to it as the *cocharacteristic variety* of the system. Note that, geometrically speaking, the $\mathbb{R}^m$ appearing here is the tangent space of the codomain $\mathscr{U}$ at a given point $p \in \mathscr{U}$. In this more invariant language, the p-dependent linear subspace $\mathscr{E}$ defining the equation is really a vector subbundle of the vector bundle $\mathfrak{t} \otimes T\mathscr{U}$ with fibre $\mathfrak{t} \otimes T_p\mathscr{U}$ at $p \in \mathscr{U}$, and the cocharacteristic variety $\mathcal{C}^{\mathscr{E}}$ is a subbundle of $P(T\mathscr{U})$, i.e., it defines a family of cones in the tangent spaces of $\mathscr{U}$, equipping $\mathscr{U}$ with a potentially interesting geometric structure.

A key observation about the method of hydrodynamic reductions is that the dispersion relation arises because the characteristic momenta are projections of points on the rank one variety. Indeed if $\boldsymbol{u} = U \circ \boldsymbol{R}$ and $\boldsymbol{R}$ satisfies (1.2) then, by the chain rule,

$$d\boldsymbol{u} = \boldsymbol{R}^* dU \circ d\boldsymbol{R} = \sum_{a=1}^{N} dR^a \otimes \partial_a U(\boldsymbol{R}) = \sum_{a=1}^{N} f_a(\boldsymbol{R}) \theta_a(\boldsymbol{R}) \otimes \partial_a U(\boldsymbol{R})$$

where $\theta_a(\boldsymbol{R}) = \sum_{j=1}^{n} \kappa_{aj}(\boldsymbol{R}) dt_j$, and (assuming $\kappa_{a1} = 1$ as before) $f_a(\boldsymbol{R}) = \partial_{t_1} R^a$. For this to yield solutions of (1.1) for given U and any such $\boldsymbol{R}$, $\theta_a \otimes \partial_a U$ must must be a section of $\mathscr{E}$ for each $a \in \{1, \dots N\}$, and this is clearly also sufficient. Since $\theta_a \otimes \partial_a U$ is a rank one matrix, its span $[\theta_a \otimes \partial_a U]$ must lie in the rank one variety $\mathcal{R}^{\mathscr{E}}$; thus $[\theta_a]$ is characteristic and $[\partial_a U]$ is cocharacteristic. The latter condition means that U maps coordinate lines in $\mathscr{V} \subseteq \mathbb{R}^N$ onto curves in $\mathscr{U}$ whose tangent lines belong to the cocharacteristic variety $\mathcal{C}^{\mathscr{E}}$ at each point. This is a special kind of *net* (see e.g. [1]) with respect to the geometric structure $\mathcal{C}^{\mathscr{E}}$ defines on $\mathscr{U}$.

In the classes of examples studied in [3, 6, 8], such a net turns out to be sufficient to recover the hydrodynamic reduction, as the coordinate derivatives $\partial_a U$ implicitly contain information about the characteristic momenta. This can be understood using the correspondence between characteristic and cocharacteristic varieties given by the rank one variety. For well-behaved systems, this correspondence is a bijection, i.e., for each $[\theta]$ in the characteristic variety, there is a unique $[S]$ in the cocharacteristic variety such that $[\theta \otimes S]$ is in the rank one variety, and vice versa. This is the case in Example 1.1: for given U, the cocharacteristic variety consists of all $[u, v]$ in $P(\mathbb{R}^2)$ and the characteristic variety is the Veronese embedding of $P(\mathbb{R}^2)$ as a conic in $P(\mathbb{R}^3)$. In general such a bijective rank one correspondence means that the characteristic and cocharacteristic varieties are simply different projective embeddings of the same underlying abstract variety.

Unfortunately this is not sufficient in general to describe hydrodynamic reductions purely in terms of the cocharacteristic variety, because it is necessary also to encode the compatibility condition of (1.2), and this depends on the embedding of $\mathcal{X}^{\mathscr{E}}$. Motivated again by [3, 6, 8], I describe a class of *compliant* quasilinear systems for which the characteristic embedding can be recovered from the cocharacteristic one, and a class of *cocharacteristic nets* for which the following result holds.

Theorem 1.1. *Let $\mathscr{E} \leqslant \mathfrak{t}^*\otimes T\mathscr{U}$ be a compliant quasilinear system and $N \leqslant \dim \mathscr{U}$. Then up to natural equivalences, there is a bijection between generic N-component hydrodynamic reductions of $\mathscr{E}$ and generic N-dimensional cocharacteristic nets in $\mathscr{U}$.*

It follows that hydrodynamic integrability for compliant systems is equivalent to the existence sufficiently many cocharacteristic nets in $\mathscr{U}$, a condition which is amenable to analysis by differential geometric and/or representation theoretic techniques (see e.g. [3, 13, 19, 20]).

Note that Example 1.1 is not compliant. However, even when a quasilinear system is not compliant, there may be a reformulation of the system (for example by prolongation or differential covering) which is, and this can be done for the dKP equation (see Remark 2.4).

The structure of the paper is as follows. In Section 2, I explain in more detail the affine geometry of first order quasilinear systems and the characteristic correspondence. Section 3 reviews the algebraic geometry of projective embeddings in order to define the notion of a compliant quasilinear system. In Section 4, I turn to the differential geometry of nets, and explain what is a cocharacteristic net. This turns out to include conjugate nets as a special case. The proof of Theorem 1.1 is then given in Section 5.

Throughout the paper, all manifolds are assumed connected, and can be real or complex, and all maps between manifolds are smooth over the base field (hence holomorphic in the complex case). For a vector bundle $\mathscr{E}$ over a manifold $\mathscr{M}$, the space of (smooth) k-forms on $\mathscr{M}$ with values in $\mathscr{E}$ will be denoted $\Omega^k_{\mathscr{M}}(\mathscr{E})$, or $\Omega^k_{\mathscr{M}}(V)$ if $\mathscr{E} = \mathscr{M} \times V$ is a trivial bundle; when $k = 0$ this is just the space of (smooth) sections of $\mathscr{E}$ or functions $\mathscr{M} \to V$. The derivative of a map $F\colon \mathscr{M} \to \mathscr{N}$ between manifolds is denoted $dF \in \Omega^1_{\mathscr{M}}(F^*T\mathscr{N})$, but in the case that $\mathscr{N}$ is a vector space V, $F^*TV \cong \mathscr{M} \times V$ is trivial, so $dF \in \Omega^1_{\mathscr{M}}(V)$.

2. Quasilinear systems and their characteristic geometry

2.1. Affine geometry of quasilinear systems. First order quasilinear systems (1.1) are translation invariant, and so the natural context for their study is affine geometry. An open subset $\mathscr{M}$ of an affine space is an example of a flat affine manifold. More precisely, if the affine space is modelled on an n-dimensional vector space $\mathfrak{t}$, then there is an exact 1-form $d\boldsymbol{t} \in \Omega^1_{\mathscr{M}}(\mathfrak{t})$ which defines a bundle isomorphism $T\mathscr{M} \cong \mathscr{M} \times \mathfrak{t}$. Under this isomorphism, the constant $\mathfrak{t}$-valued functions are identified with the infinitesimal generators of the local free and transitive translation action on $\mathscr{M}$. The affine coordinate $\boldsymbol{t}\colon \mathscr{M} \to \mathfrak{t}$ is defined up to an additive constant in $\mathfrak{t}$, which is usually fixed by specifying an origin $z \in \mathscr{M}$ with $\boldsymbol{t}(z) = 0$. I will not do this herein, nor will I assume $\mathscr{M}$ is an open subset of an affine space: it suffices to have an exact 1-form $d\boldsymbol{t}$, the *tautological 1-form*, inducing $T\mathscr{M} \cong \mathscr{M} \times \mathfrak{t}$.

Definition 2.1. Let $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$ be a map between manifolds, with domain $\mathscr{M}$ a flat affine n-manifold as above, and derivative $d\boldsymbol{u} \in \Omega^1_{\mathscr{M}}(\boldsymbol{u}^*T\mathscr{U})$. Then the *affine derivative* of $\boldsymbol{u}$ is the unique $\psi \in \Omega^0_{\mathscr{M}}(\mathfrak{t}^* \otimes \boldsymbol{u}^*T\mathscr{U})$ with $d\boldsymbol{u} = \langle \psi, d\boldsymbol{t}\rangle$, where angle brackets denote contraction of $\mathfrak{t}$ with $\mathfrak{t}^*$.

In particular if f is a scalar-valued function on $\mathscr{M}$, then its affine derivative is in $\Omega^0_{\mathscr{M}}(\mathfrak{t}^*)$; if this function is constant, f is said to be *affine*, and $\mathfrak{h}$ will denote the space of affine functions on $\mathscr{M}$.

Remarks 2.2. In view of the definition, it is natural to denote the affine derivative by $d\boldsymbol{u}/d\boldsymbol{t}$; indeed if $\mathscr{M}$ is 1-dimensional, this is the ordinary derivative of a function of 1-variable, expressed as a ratio of 1-forms. More generally, in local affine coordinates,

$$d\boldsymbol{t} = (dt_1, \dots dt_n) \quad \text{and} \quad \frac{d\boldsymbol{u}}{d\boldsymbol{t}} = (\partial_{t_1}\boldsymbol{u}, \dots \partial_{t_n}\boldsymbol{u}).$$

(the partial derivatives of $\boldsymbol{u}$).

The affine derivative restricts to a linear map from $\mathfrak{h}$ to $\mathfrak{t}^*$, with kernel the constant functions; if this map surjects (as it does locally) then there is a natural affine coordinate $\boldsymbol{t}\colon \mathscr{M} \to \mathfrak{h}^*$ defined by $\langle \boldsymbol{t}(z), f\rangle = f(z)$. This identifies $\mathscr{M}$ locally with the affine hyperplane $\langle \boldsymbol{t}, 1\rangle = 1$ in $\mathfrak{h}^*$; in particular, $\langle d\boldsymbol{t}, c\rangle = 0$ for any constant function c, so $d\boldsymbol{t}$ takes values in $\mathfrak{t}$, and agrees with the tautological 1-form, justifying the notation.

Although the above invariant language is not essential, it provides a convenient and computationally concise way to formalize the theory of quasilinear systems.

Definition 2.3. A *quasilinear system* (*QLS*), on maps from a flat affine space $\mathscr{M}$ with tautological 1-form $d\boldsymbol{t} \in \Omega^1_{\mathscr{M}}(\mathfrak{t})$ to an m-manifold $\mathscr{U}$, is vector subbundle $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ over $\mathscr{U}$; a *solution* of the QLS $\mathscr{E}$ is a map $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$ with $d\boldsymbol{u}/d\boldsymbol{t} \in \Omega^0_{\mathscr{M}}(\boldsymbol{u}^*\mathscr{E})$.

If $\mathscr{E}_p \leqslant \mathfrak{t}^* \otimes T_p\mathscr{U}$ has codimension k for each $p \in \mathscr{U}$, then it is locally the kernel of k independent linear forms on $\mathfrak{t}^* \otimes T_p\mathscr{U}$ depending smoothly on p, and so the QLS imposes k linear constraints on the derivative of $\boldsymbol{u}$. Identifying $\mathfrak{t}$ with $\mathbb{R}^n$ or $\mathbb{C}^n$ (via affine coordinates t_i) and $T\mathscr{U}$ locally with $\mathscr{U} \times \mathbb{R}^m$ or $\mathscr{U} \times \mathbb{C}^m$, the t_i components of these linear forms combine, for each $j \in \{1, \dots n\}$, into $k \times m$ matrix valued functions $A_j(p)$ of $p \in \mathscr{U}$, yielding the coordinate expression (1.1) for the QLS.

Example 2.1. Consider the first order formulation of the generalized dKP equation (1.3) for $\boldsymbol{u} = (u, v)\colon \mathscr{M} = \mathbb{R}^3 \to \mathscr{U} = \mathbb{R}^2$. Thus the fibre of $\mathscr{E}$ at $(u, v) \in \mathscr{U} = \mathbb{R}^2$ is the set of $(u_t, u_x, u_y) \otimes (1, 0) + (v_t, v_x, v_y) \otimes (0, 1)$ satisfying (1.4), which may be solved for (v_t, v_y) to give

$$\mathscr{E}_{(u,v)} = \{(u_t, u_x, u_y) \otimes (1, 0) + (u_y, v_x, u_x + \tau(u)u_t) \otimes (0, 1)\}.$$

Example 2.2. A QLS in $n = 2$ dimensions is often called a *hydrodynamic system*; it takes the form $B(\boldsymbol{u})\partial_t\boldsymbol{u} = C(\boldsymbol{u})\partial_x\boldsymbol{u}$ for affine coordinates (x, t) and matrix valued functions B, C. Multiplying by $dx \wedge dt$, this system can be rewritten as $(B(u)dx + C(u)dt) \wedge d\boldsymbol{u} = 0$.

This has an analogue in any dimension n. For later use, the unknown map will be denoted $\boldsymbol{R}\colon \mathscr{M} \to \mathscr{V}$, where the codomain $\mathscr{V}$ is an N-manifold. The hydrodynamic system is then determined by a section $A \in \Omega^0_{\mathscr{V}}(\mathfrak{t}^* \otimes \mathrm{End}(T\mathscr{V}))$, and $\boldsymbol{R}$ solves this system if

$$\boldsymbol{R}^* A \wedge \frac{d\boldsymbol{R}}{d\boldsymbol{t}} = 0 \quad \text{in} \quad \Omega^0_{\mathscr{M}}(\wedge^2\mathfrak{t}^* \otimes \boldsymbol{R}^*T\mathscr{V}), \tag{2.1}$$

where the wedge product is on $\mathfrak{t}^*$ and the action of $End(T\mathscr{V})$ on $T\mathscr{V}$ is used to multiply coefficients. Thus $\mathscr{E}$ is the kernel of the bundle map $\mathfrak{t}^* \otimes T\mathscr{V} \to \wedge^2\mathfrak{t}^* \otimes T\mathscr{V}$ sending ψ to $A \wedge \psi$. The QLS can also be formulated as an exterior differential system (EDS) on $\mathscr{M} \times \mathscr{V}$ using the projections $\pi_{\mathscr{M}}$ and $\pi_{\mathscr{V}}$ to $\mathscr{M}$ and $\mathscr{V}$: for any 1-form α on $\mathscr{V}$, $\pi^*_{\mathscr{V}}\alpha(A) \in \Omega^1_{\mathscr{M}\times\mathscr{V}}(\mathfrak{t}^*)$ and $\pi^*_{\mathscr{M}}d\boldsymbol{t} \in \Omega^1_{\mathscr{M}\times\mathscr{V}}(\mathfrak{t})$, hence their contracted wedge product $\langle \pi^*_{\mathscr{V}}\alpha(A) \wedge \pi^*_{\mathscr{M}}d\boldsymbol{t}\rangle$ is a 2-form on $\mathscr{M} \times \mathscr{V}$. Then $\boldsymbol{R}$ solves (2.1) if and only if these 2-forms pullback to zero by $(id_{\mathscr{M}}, \boldsymbol{R})\colon \mathscr{M} \to \mathscr{M} \times \mathscr{V}$, i.e., the graph of $\boldsymbol{R}$ is an integral manifold of the EDS (differential ideal) generated by these 2-forms.

Evidently if $\boldsymbol{R} = \Psi \circ \tilde{\boldsymbol{R}}$ for some diffeomorphism Ψ of $\mathscr{V}$, then $d\boldsymbol{R} = \tilde{\boldsymbol{R}}^* d\Psi \circ d\tilde{\boldsymbol{R}}$, and so the system transforms to $\langle \tilde{\boldsymbol{R}}^* \tilde{A}, d\boldsymbol{t}\rangle \wedge d\tilde{\boldsymbol{R}} = 0$ where $\tilde{A} = (d\Psi)^{-1} \circ \Psi^* A \circ d\Psi$.

The fundamental case for hydrodynamic integrability is when A can be simultaneously diagonalized by such a diffeomorphism Ψ, with the eigenvectors tangent to coordinate lines on $\mathscr{V}$. Thus there are coordinates $r_a : a \in \mathcal{A} = \{1, \dots N\}$ on $\mathscr{V}$ called *Riemann invariants*, and functions $\kappa_a\colon \mathscr{V} \to \mathfrak{t}^*$ called *characteristic momenta* such that $A = \sum_{a\in\mathcal{A}} \kappa_a dr_a \otimes \partial_{r_a}$. These coordinates may be assumed (for local questions) to realise $\mathscr{V}$ as an open subset of $\mathbb{R}^N$. Setting $\theta_a = \langle \kappa_a, d\boldsymbol{t}\rangle$ for $a \in \mathcal{A}$, the above EDS is generated by the 2-forms $\theta_a \wedge dr_a : a \in \mathcal{A}$ on $\mathscr{M} \times \mathscr{V}$, and the QLS $\mathscr{E}$ is spanned by $\kappa_a \otimes \partial_{r_a} : a \in \mathcal{A}$.

Note that there is some residual gauge freedom in the system: each κ_a may be scaled by a function f_a on $\mathscr{V}$ (*scaling equivalence*), and each r_a may be replaced by $r'_a = \rho_a(r_a)$

for functions ρ_a of one variable (*reparametrization equivalence*); these do not change $\mathscr{E}$, or the differential ideal generated by $\theta_a \wedge dr_a : a \in \mathcal{A}$.

I now turn to the examples that this paper seeks to unify. These examples have in common that they all come from a PDE system, for a function $\boldsymbol{w}\colon \mathscr{M} \to \mathscr{W}$ with values in an affine space $\mathscr{W}$, which is *not a priori* a QLS, but which may be reformulated as a QLS on the derivative of $\boldsymbol{w}$. There are three cases: general grassmannian equations [6, 7], in which the derivative of $\boldsymbol{w}$ satisfies an algebraic constraint; hessian or Hirota type equations [8, 19, 20], in which $\boldsymbol{w}$ is the derivative of a scalar function φ whose second derivative satisfies an algebraic constraint, and invariant wave equations [3], in which the equation on $\boldsymbol{w}$ is linear in the second derivative with coefficients depending only on the first derivative (thus the equation is invariant under translation of $\boldsymbol{w}$). These three classes of examples will be referred to as types G, H and I respectively.

Examples 2.3. Let $\mathscr{W}$ be an affine space modelled on a vector space V with $\dim V = q$. Then the derivative of a smooth map $\boldsymbol{w}\colon \mathscr{M} \to \mathscr{W}$ may be viewed as a V-valued 1-form $d\boldsymbol{w} \in \Omega^1_{\mathscr{M}}(V)$ (strictly speaking this is $\boldsymbol{w}^* d\boldsymbol{s} \circ d\boldsymbol{w}$ where $d\boldsymbol{s}\colon T\mathscr{W} \to V$ is the tautological 1-form of $\mathscr{W}$)—hence it has an affine derivative $\boldsymbol{u} = d\boldsymbol{w}/d\boldsymbol{t}\colon \mathscr{M} \to \mathfrak{t}^* \otimes V$

Alternatively, for each $z \in \mathscr{M}$, the graph of the linear map $\boldsymbol{u}(z) \in \mathfrak{t}^* \otimes V \cong \mathit{Hom}(\mathfrak{t}.V)$ is in $Gr_n(\mathfrak{t} \oplus V)$, the grassmannian of n-dimensional subspaces $\mathfrak{t} \oplus V$, so $\boldsymbol{u}$ may be viewed as a map $\Gamma_{\boldsymbol{u}}\colon \mathscr{M} \to Gr_n(\mathfrak{t} \oplus V); z \mapsto (d\boldsymbol{t}_z, d\boldsymbol{w}_z)(T_z\mathscr{M})$. This viewpoint reveals $PGL(\mathfrak{t} \oplus V)$ as a natural symmetry group [6, 7], with $\mathfrak{t}^* \otimes V$ being identified as the open subset of n-dimensional subspaces of $\mathfrak{t} \oplus V$ on which the projection to $\mathfrak{t}$ is an isomorphism.

Now an arbitrary $\boldsymbol{u}\colon \mathscr{M} \to \mathfrak{t}^* \otimes V$ arises locally in this way from $\boldsymbol{w}\colon \mathscr{M} \to \mathscr{W}$ if and only if $d\boldsymbol{u}/d\boldsymbol{t}\colon \mathscr{M} \to \mathfrak{t}^* \otimes \mathfrak{t}^* \otimes V$ takes values in $S^2\mathfrak{t}^* \otimes V$. With respect to affine coordinates $\boldsymbol{t} = (t_1, \dots t_n)$ on $\mathscr{M}$, $\boldsymbol{u}$ has components $\boldsymbol{u}_i\colon \mathscr{M} \to V$, and this is simply the statement that for all $i, j \in \{1, \dots n\}$, $\partial_{t_i}\boldsymbol{u}_j = \partial_{t_j}\boldsymbol{u}_i$, i.e., the integrability condition to write $\boldsymbol{u}_i = \partial_{t_i}\boldsymbol{w}\colon \mathscr{M} \to V$ for each $i \in \{1, \dots n\}$.

There are now two ways to impose equations on $\boldsymbol{u}$. First $\Gamma_{\boldsymbol{u}}$ can be required to take values in an m-dimensional submanifold of $Gr_n(\mathfrak{t} \oplus V)$. Let $\mathscr{U}$ be the corresponding submanifold of the open subset $\mathfrak{t}^* \otimes V$. The integrability condition on $\boldsymbol{u}$ may be conveniently described using the map $(id_{\mathscr{M}}, \boldsymbol{u})\colon \mathscr{M} \to \mathscr{M} \times \mathscr{U}$ as follows: let $\boldsymbol{p} \in \Omega^0_{\mathscr{M}\times\mathscr{U}}(\mathfrak{t}^* \otimes V)$ be the tautological coordinate given by projection onto $\mathscr{U} \subseteq \mathfrak{t}^* \otimes V$ and let $\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle \in \Omega^2_{\mathscr{M}\times\mathscr{U}}(V)$ be the wedge product of $d\boldsymbol{p}$ with $\pi^*_{\mathscr{M}} d\boldsymbol{t} \in \Omega^1_{\mathscr{M}\times\mathscr{U}}(\mathfrak{t})$ where their values are contracted using the natural map $\mathfrak{t}^* \otimes V \times \mathfrak{t} \to V$. Then the symmetry of $d\boldsymbol{u}/d\boldsymbol{t}$ means equivalently that $(id_{\mathscr{M}}, \boldsymbol{u})^*\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle = 0$. This motivates a second class of quasilinear constraints on $\boldsymbol{u}$ that $(id_{\mathscr{M}}, \boldsymbol{u})^*\langle d\boldsymbol{p} \wedge \Phi\rangle = 0$ for further differential forms $\Phi \in \Omega^k_{\mathscr{M}\times\mathscr{U}}(\mathfrak{t})$ of the form $\Phi = \sum_{I,j} F_{I,j}(\boldsymbol{p}) dt_I \otimes \partial_{t_j}$, where $I = \{i_1, \dots i_k\}$ is a multi-index with $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and $dt_I = dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_k}$. Thus the QLS is the EDS generated by components of $\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle$ and the additional differential forms $\langle d\boldsymbol{p} \wedge \Phi\rangle$.

The three classes G,H,I of QLS are special cases of these EDS.

(G) This is the case of a general submanifold $\mathscr{U}$, with no additional differential forms Φ. Since $\mathscr{U}$ has dimension m, it has codimension $m' = qn - m$ and so may be described locally by equations $F_k(\boldsymbol{p}) = 0$ for $k \in \{1, \dots m'\}$. If $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$ with $(id_{\mathscr{M}}, \boldsymbol{u})^*\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle = 0$ then $\boldsymbol{u}$ comes from a smooth map $\boldsymbol{w}\colon \mathscr{M} \to \mathscr{W}$ which solves the system

$$F_k(d\boldsymbol{w}/d\boldsymbol{t}) = 0 \qquad \text{for all} \qquad k \in \{1, \dots m'\}.$$

For any $p \in \mathscr{U}$, the fibre $\mathscr{E}_p$ of the QLS is the intersection of $\mathfrak{t}^* \otimes T_p\mathscr{U}$ with $S^2\mathfrak{t}^* \otimes V$.

(H) This is an important special case of (G) where $\mathscr{W} = V = \mathfrak{t}^*$ and the submanifold is a hypersurface in the lagrangian grassmannian of maximal isotropic subspaces in $\mathfrak{t} \oplus V = \mathfrak{t} \oplus \mathfrak{t}^*$ with respect to the natural symplectic form given by $\omega(X_1 + \xi_1, X_2 + \xi_2) = \langle \xi_1, X_2\rangle - \langle \xi_2, X_1\rangle$. The lagrangian grassmannian has $S^2\mathfrak{t}^*$ as an open subset which, given affine coordinates $\boldsymbol{t} = (t_1, \dots t_n)$, has coordinates $p_{ij} = p_{ji}$ $(i, j \in \{1, \dots n\})$. Hence the hypersurface $\mathscr{U}$ in $S^2\mathfrak{t}^*$ is given locally by $F(\ldots, p_{ij}, \ldots) = 0$ for some function F of $\frac{1}{2}n(n+1)$ variables. Any $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$ with $(id_{\mathscr{M}}, \boldsymbol{u})^*\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle = 0$ now comes from a scalar function φ on $\mathscr{M}$ (via $\boldsymbol{w} = d\varphi/d\boldsymbol{t}$) which solves the equation

$$F(\ldots, \partial^2_{t_i,t_j}\varphi, \ldots) = 0.$$

In this case, for any $p \in \mathscr{U}$, $T_p\mathscr{U} \leqslant S^2\mathfrak{t}^*$ is the kernel of dF_p, so $\mathscr{E}_p$ is the intersection of $\mathfrak{t}^* \otimes T_p\mathscr{U}$ with $S^3\mathfrak{t}^*$.

(I) Let $q = 1$ (so that V is one-dimensional), with $\mathscr{U}$ open in $\mathfrak{t}^* \otimes V \subseteq Gr_n(\mathfrak{t} \oplus V)$, and suppose $Q(\boldsymbol{p}) \in \Omega^0_{\mathscr{M}\times\mathscr{U}}(S^2\mathfrak{t})$ is a quadratic form on $\mathfrak{t}^*$ with coefficients depending only on $\boldsymbol{p}$. Thus in affine coordinates $\boldsymbol{t} = (t_1, \ldots t_n)$ with $\boldsymbol{p} = (p_1, \ldots p_n)$, $Q = \sum_{i,j} F_{ij}(p_1, \ldots p_n)\partial_{t_i} \otimes \partial_{t_j}$ with $F_{ij} = F_{ji}$, and defines $\Phi \in \Omega^{n-1}_{\mathscr{M}\times\mathscr{U}}(\mathfrak{t})$ by the partial contraction $\Phi = \sum_{i,j} F_{ij}(p_1, \ldots p_n)\partial_{t_i} \lrcorner (dt_1 \wedge \cdots \wedge dt_n) \otimes \partial_{t_j}$. Now if $(id_{\mathscr{M}}, \boldsymbol{u})$ pulls back both $\langle d\boldsymbol{p} \wedge d\boldsymbol{t}\rangle$ and $\langle d\boldsymbol{p} \wedge \Phi\rangle$ to zero, then $\boldsymbol{u}$ comes from a scalar function w on $\mathscr{M}$ which solves the equation

$$\sum_{i,j} F_{ij}(\partial_{t_1}w, \ldots \partial_{t_n}w)\partial^2_{t_i,t_j}w = 0.$$

At each $p \in \mathscr{U}$, $\mathscr{E}_p$ is the kernel of $Q(p) \otimes id_V$ in $\mathfrak{t}^* \otimes T_p\mathscr{U} \cap S^2\mathfrak{t}^* \otimes V$.

Remark 2.4. Up to differential coverings, the dKP equation can be reformulated as a QLS in each of these ways [3, 6, 8]. For example, the equation

$$w_{xt} + w_t w_{tt} = w_{yy} \tag{2.2}$$

has type I and so defines a QLS on the first derivatives $f = w_t$, $g = w_x$, and $h = w_y$:

$$f_x = g_t, \quad f_y = h_t, \quad g_y = h_x \quad \text{and} \quad f_x - ff_t = h_y.$$

On the other hand, by differentiating (2.2) with respect to t, it follows that $u = w_t$ satisfies the dKP equation $(u_x + uu_t)_t = u_{yy}$. Finally, writing (2.2) as $(w_x + \frac{1}{2}w_t^2)_t = w_{yy}$ yields a first order form

$$w_x + \tfrac{1}{2}w_t^2 = v_y, \quad v_t = w_y$$

of type G, and a further potential φ with $\varphi_y = v$ and $\varphi_t = w$ reduces this system to the equation $\varphi_{xt} + \frac{1}{2}\varphi_{tt}^2 = \varphi_{yy}$ of type H.

2.2. The characteristic correspondence. The projective bundle $P(\mathfrak{t}^* \otimes T\mathscr{U}) \to \mathscr{U}$ has a subbundle $\mathcal{R}$ whose fibre at $p \in \mathscr{U}$ is

$$\mathcal{R}_p := \{[\xi \otimes Z] \in P(\mathfrak{t}^* \otimes T_p\mathscr{U}) \mid \xi \in \mathfrak{t}^*,\, Z \in T_p\mathscr{U}\}.$$

This is the image of the *Segre embedding* of $P(\mathfrak{t}^*) \times P(T_p\mathscr{U})$, consisting of the projectivizations of rank one tensors.

Definition 2.5. Let $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ be a QLS.

- The *rank one variety* of $\mathscr{E}$ is $\mathcal{R}^{\mathscr{E}} := \mathcal{R} \cap P(\mathscr{E})$.
- The *characteristic* and *cocharacteristic varieties* of $\mathscr{E}$ are the projections $\mathcal{X}^{\mathscr{E}}$ and $\mathcal{C}^{\mathscr{E}}$ of $\mathcal{R}^{\mathscr{E}}$ to $\mathscr{U} \times P(\mathfrak{t}^*)$ and $P(T\mathscr{U})$ respectively.
- The *characteristic correspondence* of $\mathscr{E}$ is the diagram

$$\begin{array}{ccc} & \mathcal{R}^{\mathscr{E}} & \\ {}^{\pi_{\mathcal{X}}}\swarrow & & \searrow^{\pi_{\mathcal{C}}} \\ \mathscr{U} \times P(\mathfrak{t}^*) \supseteq \mathcal{X}^{\mathscr{E}} & & \mathcal{C}^{\mathscr{E}} \subseteq P(T\mathscr{U}) \end{array}$$

where $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are the natural projection maps.

For convenience, $\mathcal{R}^{\mathscr{E}}$, $\mathcal{X}^{\mathscr{E}}$ and $\mathcal{C}^{\mathscr{E}}$ will be assumed to be fibre bundles over $\mathscr{U}$ whose fibres are projective varieties, so that the characteristic correspondence is a double fibration.

Remark 2.6. For (complex, or real hyperbolic) determined systems, the characteristic variety $\mathcal{X}^{\mathscr{E}}$ is a hypersurface, hence of dimension $n-2$. Thus $P(\mathscr{E}_p)$ does not meet generic fibres of $\mathcal{R}_p$ over $P(\mathfrak{t}^*)$, and the fibres it does meet, it generically meets in dimension zero, hence in a single point because these fibres are projective subspaces in the Segre embedding. For $m \geqslant n$, the same is true for the fibres over $P(T_p\mathscr{U})$, and hence the characteristic correspondence maps are isomorphisms, and $\mathcal{C}^{\mathscr{E}}_p$ has codimension $(m-1)-(n-2) = m-n+1$ in $P(T_p\mathscr{U})$.

Example 2.1 *bis*. For the generalized dKP equation (1.3), the fibre of the rank one variety $\mathcal{R}^{\mathscr{E}}$ at (u,v) consists of the nonzero elements with (u_t,u_x,u_y) and $(u_y,v_x,u_x+\tau(u)u_t)$ linearly dependent, which forces

$$u_tv_x = u_yu_x, \quad u_yv_x = (u_x+\tau(u)u_t)u_x, \quad \text{and} \quad u_y^2 = u_t(u_x+\tau(u)u_t).$$

The last equation is solved by

$$u_t = c\lambda_0^2, \quad u_x = c(\lambda_1^2-\tau(u)\lambda_0^2), \quad \text{and} \quad u_y = c\lambda_0\lambda_1$$

for constants c,λ_0,λ_1. Hence $c\lambda_0^2v_x = c^2\lambda_0\lambda_1(\lambda_1^2-u\lambda_0^2)$ and $c\lambda_0\lambda_1v_x = c^2\lambda_1^2(\lambda_1^2-\tau(u)\lambda_0^2)$, so either $c\lambda_0 = c\lambda_1 = 0$ or $\lambda_0v_x = c\lambda_1(\lambda_1^2-u\lambda_0^2)$. Now if $\lambda_0 = 0$, $c\lambda_1 = 0$, otherwise c is divisible by λ_0; thus, without loss, $c = \lambda_0$, $v_x = \lambda_1^2(\lambda_1^2-\tau(u)\lambda_0^2)$ and

$$\mathcal{R}^{\mathscr{E}}_{(u,v)} = \{(\lambda_0^2, \lambda_1^2-\tau(u)\lambda_0^2, \lambda_0\lambda_1)\otimes(\lambda_0,\lambda_1) : \lambda_0,\lambda_1\in\mathbb{R}\},$$

the cocharacteristic variety is $\mathcal{C}^{\mathscr{E}} = P^1$, and the characteristic variety $\mathcal{X}^{\mathscr{E}}$ is a u-dependent conic in P^2.

Example 2.2 *bis*. For a (diagonalizable) hydrodynamic system, with Riemann invariants and characteristic momenta $r_a,\kappa_a : a\in\mathcal{A}$, the description of $\mathscr{E}$ immediately yields that $[\kappa_a\otimes\partial_{r_a}]$ are in the rank one variety. It follows generically (if the κ_a are pairwise independent) that

$$\mathcal{X}^{\mathscr{E}} = \{[\kappa_a] : a\in\mathcal{A}\}, \quad \mathcal{C}^{\mathscr{E}} = \{[\partial_{r_a}] : a\in\mathcal{A}\} \quad \text{and} \quad \mathcal{R}^{\mathscr{E}} = \{[\kappa_a\otimes\partial_{r_a}] : a\in\mathcal{A}\}.$$

Examples 2.3 *bis*. For a QLS $\mathscr{E}$ of type G, H or I, it is straightforward to describe the characteristic correspondence using the description of $\mathscr{E}$ given previously.

(G) At each $p\in\mathscr{U}$,

$$\begin{aligned}\mathcal{X}^{\mathscr{E}}_p &= \{[\xi]\in P(\mathfrak{t}^*) \mid \xi\otimes v\in T_p\mathscr{U} \text{ for some nonzero } v\in V\}\\ \mathcal{C}^{\mathscr{E}}_p &= \{[\xi\otimes v]\in P(T_p\mathscr{U})\}\\ \mathcal{R}^{\mathscr{E}}_p &= \{[\xi\otimes\xi\otimes v]\in P(\mathfrak{t}^*\otimes T_p\mathscr{U})\}.\end{aligned}$$

Now $P(\mathfrak{t}^*)\times P(V)$ has dimension $n+q-2$, so $P(T_p\mathscr{U})$ generically meets it in a variety of dimension $n+q-m'-2$, so the system is determined for $m' = q$, in which case for any $[\xi]\in\mathcal{X}^{\mathscr{E}}$ there is a unique $[v]\in P(V)$ such that $[\xi\otimes v]\in\mathcal{C}^{\mathscr{E}}$. The references [6, 7] concern the cases $q = m' = 2$, with $n\in\{3,4\}$.

(H) In this case dF_p induces a quadratic form Q_p on $\mathfrak{t}^*$ for each $p\in\mathscr{U}$, and

$$\begin{aligned}\mathcal{X}^{\mathscr{E}}_p &= \{[\xi]\in P(\mathfrak{t}^*) \mid Q_p(\xi) = 0\}\\ \mathcal{C}^{\mathscr{E}}_p &= \{[\xi\otimes\xi]\in P(S^2\mathfrak{t}^*) \mid Q_p(\xi) = 0\}\\ \mathcal{R}^{\mathscr{E}}_p &= \{[\xi\otimes\xi\otimes\xi]\in P(S^3\mathfrak{t}^*) \mid Q_p(\xi) = 0\}.\end{aligned}$$

(I) Again there is a quadratic form Q_p (up to scale) on $\mathfrak{t}^*$ for each $p\in\mathscr{U}$, and

$$\begin{aligned}\mathcal{X}^{\mathscr{E}}_p &= \{[\xi]\in P(\mathfrak{t}^*) \mid Q_p(\xi) = 0\}\\ \mathcal{C}^{\mathscr{E}}_p &= \{[\xi\otimes v]\in P(\mathfrak{t}^*\otimes V) \mid Q_p(\xi) = 0\}\\ \mathcal{R}^{\mathscr{E}}_p &= \{[\xi\otimes\xi\otimes v]\in P(S^2\mathfrak{t}^*\otimes V) \mid Q_p(\xi) = 0\}.\end{aligned}$$

Types H and I are thus determined equations wherever $Q_p \neq 0$.

3. Algebraic geometry: compliant quasilinear systems

3.1. Projective embeddings. Motivated by the previous section, I focus now on the case that the fibres over each $p \in \mathscr{U}$ of the characteristic, cocharacteristic and rank one varieties are projective embeddings of the same abstract variety. This is a standard situation in elementary algebraic geometry, where projective embeddings are described using line bundles (see e.g. [15]).

Let $\Xi \subseteq P(V)$ be a projective variety and let $L \to \Xi$ be the restriction to Ξ of the dual tautological line bundle $\mathcal{O}_V(1) \to P(V)$, whose fibre at $\ell \in P(V)$ is the dual vector space to $\ell \leqslant V$: $\mathcal{O}_V(1)_\ell = \ell^*$. Since the space of regular global sections $H^0(P(V), \mathcal{O}_V(1))$ is V^*, restriction to Ξ defines a canonical linear map $V^* \to H^0(\Xi, L)$. Furthermore, this linear map is injective unless Ξ is contained in a hyperplane in $P(V)$.

Definition 3.1. A *linear system* on an abstract variety Ξ is a line bundle $L \to \Xi$ together with a linear subspace W of $H^0(\Xi, L)$, and its *base locus* is $B = \{x \in \Xi \mid \forall\, w \in W,\ w(x) = 0\}$. The linear system is said to be *complete* if $W = H^0(\Xi, L)$ and *basepoint-free* if $B = \varnothing$.

The reason for these definitions is that if (L, W) is a basepoint-free linear system on Ξ, then there is a map $\Xi \to P(W^*)$ sending $x \in \Xi$ to the element of $P(W^*)$ corresponding to the hyperplane $\{w \in W : w(x) = 0\}$ (the annihilator of this hyperplane). If this map is an embedding, the linear system is said to be *very ample*. In particular, if the complete linear system yields an embedding, the line bundle L is said to be very ample.

Lemma 3.2. *Let L_1 and L_2 be line bundles over Ξ. Then there is a canonical linear map*

$$H^0(L_1) \otimes H^0(L_2) \to H^0(L_1 \otimes L_2); \quad \ell_1 \otimes \ell_2 \mapsto \ell$$

defined by pointwise multiplication: $\ell(x) = \ell_1(x) \otimes \ell_2(x)$.

Proof. The map sending $(\ell_1, \ell_2) \in H^0(L_1) \times H^0(L_2)$ to $\ell_1(x) \otimes \ell_2(x)$ is regular in x, so induces a section of $L_1 \otimes L_2$; this is bilinear in (ℓ_1, ℓ_2), and so induces a map on the tensor product. □

Similarly, using the canonical isomorphism $(L_1^* \otimes L_2) \otimes L_1 \cong L_2$, there is a canonical pointwise multiplication map

$$H^0(\Xi, L_1^* \otimes L_2) \otimes H^0(\Xi, L_1) \to H^0(\Xi, L_2)$$

with transpose

$$\Phi\colon H^0(\Xi, L_2)^* \to H^0(\Xi, L_1^* \otimes L_2)^* \otimes H^0(\Xi, L_1)^*. \tag{3.1}$$

Proposition 3.3. *Suppose L_1 and L_2 are very ample line bundles on Ξ. For $j \in \{1, 2\}$, let $W_j = H^0(\Xi, L_j)$, $\phi_j\colon \Xi \to P(W_j^*)$ the corresponding projective embeddings, $V = H^0(\Xi, L_1^* \otimes L_2)$, and $\Phi\colon W_2^* \to W_1^* \otimes V^*$ be given by* (3.1)*. Then for any $x \in \Xi$, Φ maps $\phi_2(x)$ into $\phi_1(x) \otimes L(x)$ for some one dimensional subspace $L(x)$ of V^*.*

Proof. Let $\ell_1 \in W_1$ and $\ell \in V$; then for any $x \in \Xi$ and $\alpha \in \phi_2(x)$, $\langle \Phi(\alpha), \ell_1 \otimes \ell\rangle = \langle \alpha, \ell_1(x)\ell(x)\rangle$, which vanishes whenever $\ell_1(x) = 0$, i.e., $\ell_1 \in \ker \phi_1(x)$. Let $e_1, \dots e_{k-1}$ be a basis of $\ker \phi_1(x)$ and extend by e_k to a basis of W_2. Write $\Phi(\alpha) = \sum_{j=1}^k \varepsilon_j \otimes \alpha_j$ where $\varepsilon_1, \dots \varepsilon_k$ is the dual basis and $\alpha_j \in V^*$; it then follows by evaluating on $e_j \otimes v$ for $j \in \{1, \dots k-1\}$ and any $v \in V$ that $\Phi(\alpha) = \varepsilon_k \otimes \alpha_k$. But ε_k vanishes on $\ker \phi_1(x)$ and so $\Phi(\alpha) \in \phi_1(x) \otimes \langle \alpha_k\rangle$. □

In general, there is no reason to suppose that Φ here injects; indeed V could even be zero. The injectivity can be viewed as a relative ampleness condition.

Definition 3.4. Let L_1 and L_2 be line bundles on Ξ. If the canonical map Φ in (3.1) is injective (equivalently Φ^* surjects), L_2 is said to be *more ample* than L_1.

Example 3.1. The degree d line bundles $\mathcal{O}_V(d)$ over a projective space $P(V)$ are very ample for $d \geqslant 1$ and then $\mathcal{O}_V(d_1)$ is more ample than $\mathcal{O}_V(d_2)$ iff $d_1 \geqslant d_2$.

These ideas apply fibrewise to the bundles of projective varieties $\mathcal{X}^{\mathscr{E}}$ and $\mathcal{C}^{\mathscr{E}}$: the corresponding fibrewise dual tautological line bundles pull back to line bundles $\mathcal{L}_{\mathcal{X}} \to \mathcal{X}^{\mathscr{E}}$ and $\mathcal{L}_{\mathcal{C}} \to \mathcal{C}^{\mathscr{E}}$.

For a line bundle $\mathcal{L}$ over a bundle of projective varieties $\mathcal{Y} \to \mathscr{U}$, let $H^0(\mathcal{L}) \to \mathscr{U}$ denote the bundle of fibrewise regular sections. Then there are canonical maps

$$\mathscr{U} \times \mathfrak{t} \to H^0(\mathcal{L}_{\mathcal{X}}) \quad \text{and} \quad T^*\mathscr{U} \to H^0(\mathcal{L}_{\mathcal{C}}) \tag{3.2}$$

given by restricting fibrewise sections of the dual tautological line bundles over $U \times P(\mathfrak{t}^*)$ and $P(T\mathscr{U})$ to $\mathcal{X}^{\mathscr{E}}$ and $\mathcal{C}^{\mathscr{E}}$ respectively.

If $\mathcal{X}^{\mathscr{E}}$ and $\mathcal{C}^{\mathscr{E}}$ are not contained (fibrewise) in any hyperplane, these maps are injective, hence fibrewise linear systems, and surjectivity means that these linear systems are complete. In this situation, the projective embeddings can be recovered from the line bundles, and the relative ampleness of Definition 3.4 can be used (fibrewise) to compare them.

3.2. Compliant quasilinear systems. The following definition covers a wide class of QLS, yet makes the algebraic geometry of the characteristic and cocharacteristic projective embeddings as straightforward as possible, allowing an easy comparison between them.

Definition 3.5. A QLS $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ is *compliant* if all of the following conditions hold:

(1) the characteristic correspondence maps $\pi_{\mathcal{X}}$ and $\pi_{\mathcal{C}}$ are isomorphisms, with $\zeta^{\mathscr{E}} = \pi_{\mathcal{X}} \circ \pi_{\mathcal{C}}^{-1} \colon \mathcal{C}^{\mathscr{E}} \to \mathcal{X}^{\mathscr{E}}$ denoting the induced isomorphism;

(2) the canonical maps $\mathscr{U} \times \mathfrak{t} \to H^0(\mathcal{L}_{\mathcal{X}})$ and $T^*\mathscr{U} \to H^0(\mathcal{L}_{\mathcal{C}})$ of (3.2) are isomorphisms;

(3) $\mathcal{V}^{\mathscr{E}} := H^0(\mathcal{L}_{\mathcal{C}} \otimes (\zeta^{\mathscr{E}})^*\mathcal{L}_{\mathcal{X}}^*)^* \to \mathscr{U}$ is a vector bundle over $\mathcal{C}^{\mathscr{E}}$, and $\mathcal{L}_{\mathcal{C}}$ is more ample than $(\zeta^{\mathscr{E}})^*\mathcal{L}_{\mathcal{X}}$, so that the canonical vector bundle map

$$\Phi^{\mathscr{E}} \colon T\mathscr{U} \to \mathfrak{t}^* \otimes \mathcal{V}^{\mathscr{E}},$$

defined fibrewise as in (3.1), using the isomorphisms in (2), is injective.

(4) if $\operatorname{rank}(\mathcal{V}^{\mathscr{E}}) \geqslant 2$, no 2-dimensional submanifold Σ of $\mathscr{U}$ has $\Phi^{\mathscr{E}}(T\Sigma) \subseteq \mathfrak{t}^* \otimes \mathcal{V}^{\mathscr{E}}$ everywhere decomposable (i.e., with all elements of rank one).

Compliancy may seem rather restrictive due to the number of conditions involved. However, by Remark 2.6, condition (1) is expected for determined QLS. A significant part of condition (2) is that the characteristic variety $\mathcal{X}^{\mathscr{E}}$ does not lie in a projective hyperplane bundle, but this could be taken as part of what it means for a QLS to be dispersionless (I have not found a rigorous definition of "dispersionless" for arbitrary QLS in the literature).

The key condition here is (3), giving a tensor product decomposition of $T\mathscr{U}$. This fails in Example 1.1 because the characteristic embedding is more ample than the cocharacteristic one, rather than vice-versa. However all of the QLS in Examples 2.3 are generically compliant. Condition (4) implies that the generic tangent space to any 2-dimensional submanifold Σ of $\mathscr{U}$ contains indecomposable elements (via the injection $\Phi^{\mathscr{E}}$). This is a technical condition needed to ensure that the decomposition in (3) is enough to recover the compatiblity of the hydrodynamic system from the cocharacteristic variety.

The following is an immediate consequence of Proposition 3.3 (fibrewise), and allows the recovery of the characteristic embedding from the cocharacteristic one.

Proposition 3.6. *Let $\mathscr{E}$ be a compliant QLS. Then for any $[X] \in \mathcal{C}^{\mathscr{E}}$, there exist nonzero $\kappa \in \zeta([X])$ and $v \in \mathcal{V}^{\mathscr{E}}$ such that $\Phi^{\mathscr{E}}(X) = \kappa \otimes v$.*

4. Differential geometry: cocharacteristic nets

4.1. Nets and conjugate nets.

Definition 4.1. Let $\mathscr{N}$ be an N-manifold and $\mathcal{A} = \{1, \dots N\}$.

- A *pre-net* on $\mathscr{N}$ is a direct sum decomposition $T\mathscr{N} = \bigoplus_{a\in\mathcal{A}} \mathcal{D}_a$ into rank one distributions $\mathcal{D}_a \leqslant T\mathscr{N}$ for $a \in \mathcal{A}$; thus each $\mathcal{D}_a$ is tangent to a foliation of $\mathscr{N}$ with one dimensional leaves.
- A pre-net $\mathcal{D}_a : a \in \mathcal{A}$ on $\mathscr{N}$ is *integrable* if for every subset $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D}_\mathcal{B} := \bigoplus_{b\in\mathcal{B}} \mathcal{D}_b$ is an integrable distribution (i.e., tangent to a foliation with $\#\mathcal{B}$ dimensional leaves); an integrable pre-net is called a *net*.

Recall that by the Frobenius Theorem (see e.g. [16]), a constant rank distribution is tangent to a foliation if and only if its sheaf of smooth sections is closed under Lie bracket.

Proposition 4.2. *For a pre-net $\mathcal{D}_a : a \in \mathcal{A}$ on $\mathscr{N}$, the following are equivalent.*

(1) *$\mathcal{D}_a : a \in \mathcal{A}$ is integrable, i.e., a net.*

(2) *For every $\mathcal{B} \subseteq \mathcal{A}$ with $\#\mathcal{B} = 3$, $\mathcal{D}_\mathcal{B}$ is an integrable distribution, and if $N = 3$, the same is true when $\#\mathcal{B} = 2$.*

(3) *For every $\mathcal{B} \subseteq \mathcal{A}$ with $\#\mathcal{B} = 2$, $\mathcal{D}_\mathcal{B}$ is an integrable distribution.*

(4) *For each $a \in \mathcal{A}$, $\mathcal{D}_{\mathcal{A}\setminus\{a\}}$ is an integrable distribution.*

(5) *Near any $p \in \mathscr{N}$, there are local coordinates $r_a : a \in \mathcal{A}$ such that $\mathcal{D}_a = \mathrm{span}\{\partial_{r_a}\}$.*

Proof. It is trivial that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). This is vacuous for $N \leqslant 2$ and built into the statement for $N = 3$. For $N \geqslant 4$ and $\#\mathcal{B} = 2$, there exist distinct $b, c \in \mathcal{A} \setminus \mathcal{B}$, so that $[\mathcal{D}_\mathcal{B}, \mathcal{D}_\mathcal{B}] \leqslant \mathcal{D}_{\mathcal{B}\cup\{b\}} \cap \mathcal{D}_{\mathcal{B}\cup\{c\}} = \mathcal{D}_\mathcal{B}$.

(3) $\Rightarrow$ (4). For any $b, c \in \mathcal{A} \setminus \{a\}$ and any local sections X_b and X_c of $\mathcal{D}_b$ and $\mathcal{D}_c$ respectively, $[X_b, X_c]$ is a section of $\mathcal{D}_b \oplus \mathcal{D}_c \leqslant \mathcal{D}_{\mathcal{A}\setminus\{a\}}$, and hence $\mathcal{D}_{\mathcal{A}\setminus\{a\}}$ is integrable.

(4) $\Rightarrow$ (5). The r_a are pullbacks of local coordinates on the one dimensional local leaf spaces of $\mathcal{D}_{\mathcal{A}\setminus\{a\}}$.

(5) $\Rightarrow$ (1). For $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{D}_\mathcal{B}$ is the joint kernel of $dr_a : a \in \mathcal{A} \setminus \mathcal{B}$, hence integrable. □

Example 4.1. Consider the hydrodynamic system $\boldsymbol{R}^*A \wedge (d\boldsymbol{R}/d\boldsymbol{t}) = 0$ where A is a section of $\mathfrak{t}^* \otimes \mathrm{End}(T\mathscr{V})$ over $\mathscr{V}$, as in Example 2.2. If A is everywhere simultaneously diagonalizable with distinct eigenfunctions, then the rank one eigendistributions define a pre-net on $\mathscr{V}$. The condition that these eigendistributions are tangent to coordinate lines (leading to the QLS $\mathscr{E}$ spanned by $\kappa_a \otimes \partial_{r_a} : a \in \mathcal{A}$) is precisely the condition that this pre-net is a net.

4.2. Conjugate and cocharacteristic nets. For application to hydrodynamic integrability, a special class of nets will be needed. Suppose that $\mathcal{D}_a : a \in \mathcal{A}$ is a pre-net on $\mathscr{N}$, and that $T\mathscr{N} \leqslant \mathfrak{t}^* \otimes \mathcal{V}$ for a vector space $\mathfrak{t}^*$ and a line bundle $\mathcal{V} \to \mathscr{N}$; then each $\mathcal{D}_a$ defines a line subbundle M_a of $\mathscr{N} \times \mathfrak{t}^*$ with $\mathcal{D}_a = M_a \otimes \mathcal{V}$.

One may then require, for all $a, b \in \mathcal{A}$ and for any sections X_b of $\mathcal{D}_b$ and σ_a of M_a, that $d_{X_b}\sigma_a$ is a section of $M_a \oplus M_b \leqslant \mathscr{N} \times \mathfrak{t}^*$. If this holds then for all $a, b \in \mathcal{A}$, $[X_a, X_b]$ is a section of $\mathcal{D}_a \oplus \mathcal{D}_b$ and $\mathcal{D}_a : a \in \mathcal{A}$ is therefore a net, which will be called a *conjugate net.*

Example 4.2. The terminology here comes from the classical situation (see e.g. [1]) that $\mathscr{N}$ is an affine space with translations $\mathfrak{t}^*$ (or indeed a flat affine manifold). Then the tangent bundle of $\mathscr{N}$ is isomorphic to $\mathscr{N} \times \mathfrak{t}^*$ and a net $\mathcal{D}_a : a \in \mathcal{A}$ is conjugate if for all $a, b \in \mathcal{A}$, the coordinate lines along the any surface tangent to $\mathcal{D}_a \oplus \mathcal{D}_b$ are conjugate, i.e., orthogonal with respect to the second fundamental form of the surface.

Definition 4.3. Let $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ be a compliant QLS with $T\mathscr{U} \leqslant \mathfrak{t}^* \otimes \mathcal{V}^\mathscr{E}$ and cocharacteristic variety $\mathcal{C}^\mathscr{E} \leqslant P(T\mathscr{U})$. An N-dimensional *cocharacteristic net* in $\mathscr{U}$ is a parametrized submanifold $U \colon \mathscr{V} \to \mathscr{N} \subseteq \mathscr{U}$, with $\mathscr{V}$ open in $\mathbb{R}^N$, such that:

(1) the net on $\mathscr{N}$ spanned by $\partial_a U : a \in \mathcal{A} = \{1, \dots N\}$ satisfies $[\partial_a U] \in \mathcal{C}^{\mathscr{E}}$; and
(2) if $\mathcal{V}^{\mathscr{E}}$ has rank one, the net is conjugate.

5. Hydrodynamic integrability

5.1. Integrable hydrodynamic systems. As a warm-up for the proof of the main theorem, and to motivate the definition of hydrodynamic integrability, consider again Example 2.2 in the diagonalizable case, where the QLS $\mathscr{E}$ is spanned by $\kappa_a \otimes \partial_{r_a} : a \in \mathcal{A}$. Since the system is equivalently the EDS generated by the 2-forms $\theta_a \wedge dr_a$, where $\theta_a = \langle \kappa_a, d\boldsymbol{t}\rangle$, it is compatible if and only if these 2-forms algebraically generate a differential ideal, i.e., for all $a \in \mathcal{A}$, $d\theta_a \wedge dr_a = 0 \bmod (\theta_b \wedge dr_b)_{b\in\mathcal{A}}$. This holds if and only if

$$\text{for all } a \in \mathcal{A},\ d\kappa_a \wedge dr_a = 0 \bmod (\kappa_b\, dr_b)_{b\in\mathcal{A}} \text{ (as } \mathfrak{t}^*\text{-valued 2-forms on } \mathscr{V}),$$

i.e., for all $a \in \mathcal{A}$, there are (scalar-valued) 1-forms $\beta_a = \sum_{b\in\mathcal{A}} \beta_{ab} dr_b$ and functions γ_{ab} on $\mathscr{V}$ such that

$$\begin{aligned} d\kappa_a \wedge dr_a &= \beta_a \wedge (\kappa_a\, dr_a) + \textstyle\sum_{b\in\mathcal{A}} (\kappa_b\, dr_b) \wedge (\gamma_{ab}\, dr_a) \\ \text{i.e.,}\quad \partial_b \kappa_a &= \beta_{ab}\kappa_a + \gamma_{ab}\kappa_b \quad \text{for all} \quad b \neq a. \end{aligned} \tag{5.1}$$

Fixing β_a and γ_{ab}, $\kappa_a : a \in \mathcal{A}$ may be viewed as a $\mathfrak{t}^*$-valued solution to a linear system on $\mathscr{V}$. This linear system in turn has a compatibility condition: differentiating once more,

$$\begin{aligned} 0 &= d\beta_a \wedge (\kappa_a\, dr_a) - \beta_a \wedge d\kappa_a \wedge dr_a + \textstyle\sum_{b\in\mathcal{A}} \big(\gamma_{ab}\, d\kappa_b + \kappa_b\, d\gamma_{ab}\big) \wedge dr_b \wedge dr_a \\ &= \kappa_a\, d\beta_a \wedge dr_a + \textstyle\sum_{b\in\mathcal{A}} \kappa_b \big(\gamma_{ab}(-\beta_a + \beta_b) + d\gamma_{ab}\big) \wedge dr_b \wedge dr_a \\ &\qquad + \textstyle\sum_{b,c\in\mathcal{A}} \kappa_c\, \gamma_{ab}\gamma_{bc}\, dr_c \wedge dr_b \wedge dr_a \\ &= \kappa_a\, d\beta_a \wedge dr_a + \textstyle\sum_{c\in\mathcal{A}} \kappa_c \big(\gamma_{ac}(\beta_c - \beta_a) + d\gamma_{ac} - \sum_{b\in\mathcal{A}} \gamma_{ab}\gamma_{bc} dr_b\big) \wedge dr_c \wedge dr_a \end{aligned}$$

for all $a \in \mathcal{A}$. This is satisfied for all $\kappa_a : a \in \mathcal{A}$ if and only if for all $a, c \in \mathcal{A}$ with $c \neq a$

$$\begin{aligned} d\beta_a \wedge dr_a = d(\beta_a \wedge dr_a) = d(r_a d\beta_a) &= 0 \\ \textstyle\sum_{b\in\mathcal{A}} \big(\partial_b \gamma_{ac} + \gamma_{ac}\beta_{cb} - \gamma_{ac}\beta_{ab} - \gamma_{ab}\gamma_{bc}\big) dr_b \wedge dr_c \wedge dr_a &= 0. \end{aligned} \tag{5.2}$$

If (5.2) holds, and $(\kappa_a)_{a\in\mathcal{A}}$ is a $\mathfrak{t}^*$-valued solution to (5.1) then $\theta_a \wedge dr_a : a \in \mathcal{A}$ generate a differential ideal, and $\mathscr{E}$ may be called an *integrable* hydrodynamic system. Such systems were introduced by Tsarev [23] as *semi-hamiltonian systems of hydrodynamic type.* Such systems are regarded as integrable because of the fundamental observation of Tsarev [23] that they admit *generalized hodograph* solutions. Interpreted in the affine geometry of Definition 2.1 and Remark 2.2 (using in particular the space of affine functions $\mathfrak{h} \to \mathfrak{t}^*$ and the tautological affine coordinate $\boldsymbol{t} \colon \mathscr{M} \to \mathfrak{h}^*$), his observation is as follows.

Proposition 5.1. *Let $\mathscr{E}$ be a hydrodynamic system with characteristic momenta $\kappa_a : a \in \mathcal{A}$. Then for any lift $\tilde{\kappa}_a : a \in \mathcal{A}$ of $\kappa_a : a \in \mathcal{A}$ to an $\mathfrak{h}$-valued solution of* (5.1)*, any function $\boldsymbol{R} \colon \mathscr{M} \to \mathscr{V}$ solving the implicit equations*

$$\langle \tilde{\kappa}_a(\boldsymbol{R}), \boldsymbol{t}\rangle = 0 \tag{5.3}$$

for all $a \in \mathcal{A}$, is a solution of $\mathscr{E}$.

Proof. The exterior derivative of (5.3) is $\langle \boldsymbol{R}^* d\tilde{\kappa}_a, \boldsymbol{t}\rangle + \boldsymbol{R}^*\theta_a = 0$, since $d\boldsymbol{t}$ takes values in $\mathfrak{t}$. Hence, writing $R^a = r_a \circ \boldsymbol{R}$ for the components of $\boldsymbol{R}$, it follows from (5.1) and (5.3) that

$$\begin{aligned} -R^*\theta_a \wedge dR^a &= \langle R^* d\tilde{\kappa}_a, \boldsymbol{t}\rangle \wedge dR^a \\ &= \big(\langle \tilde{\kappa}_a(\boldsymbol{R}), \boldsymbol{t}\rangle \beta_a + \textstyle\sum_{b\in\mathcal{A}} R^*\gamma_{ab} \langle \tilde{\kappa}_b(\boldsymbol{R}), \boldsymbol{t}\rangle dR^b\big) \wedge dR^a = 0. \end{aligned}$$

for all $a \in \mathcal{A}$. ∎

In the integrable case, the linear system (5.1) is compatible, so there are many such solutions. Tsarev originally presented these systems in particularly convenient gauge: under scaling equivalence $\kappa_a \mapsto f_a\kappa_a$, (5.1) is modified by adding $f_a^{-1}df_a$ to β_a and rescaling γ_{ab}. Tsarev used this freedom to ensure that constants are scalar solutions of (5.1). This forces $\beta_a = -\sum_b \gamma_{ab} dr_b \mod dr_a$ and (5.1) becomes

$$d\kappa_a = \textstyle\sum_{b\in\mathcal{A}} \gamma_{ab}(\kappa_b - \kappa_a)dr_b \mod dr_a, \quad \text{i.e.,} \quad \partial_b\kappa_a = \gamma_{ab}(\kappa_b - \kappa_a) \quad \text{for} \quad b \neq a.$$

Since $d\beta_a \wedge dr_a = \sum_{b,c}(\partial_c\gamma_{ab})dr_a \wedge dr_b \wedge dr_c$, the integrability condition (5.2) becomes $\partial_c\gamma_{ab} = \partial_b\gamma_{ac}$ and $\partial_b\gamma_{ac} + \gamma_{ac}\gamma_{cb} - \gamma_{ac}\gamma_{ab} - \gamma_{ab}\gamma_{bc}$ for a, b, c distinct.

5.2. Hydrodynamic reductions. One remaining task before proving the main theorem is to formalize the notion of a hydrodynamic reduction of a QLS $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ on maps $\boldsymbol{u}\colon \mathscr{M} \to \mathscr{U}$ (for a flat affine manifold $\mathscr{M}$). Recall from the introduction that the data defining an N-component hydrodynamic reduction are a map $U\colon \mathscr{V} \to \mathscr{U}$ and maps $\kappa_a\colon \mathscr{V} \to \mathfrak{t}^*$ for $a \in \mathcal{A} = \{1, \dots N\}$ and $\mathscr{V}$ open in $\mathbb{R}^N$ with coordinates $r_1, \dots r_N\colon \mathscr{V} \to \mathbb{R}$. These data are required to satisfy two properties:

(1) the hydrodynamic system on maps $\boldsymbol{R}\colon \mathscr{M} \to \mathscr{V}$ defined by $dR^a \wedge \langle\kappa_a(\boldsymbol{R}), d\boldsymbol{t}\rangle = 0$ (for all $a \in \mathcal{A}$) is compatible (where $R^a = r_a \circ \boldsymbol{R}$ are the components of $\boldsymbol{R}$);

(2) if $\boldsymbol{R}$ solves this system, then $\boldsymbol{u} = U \circ \boldsymbol{R}$ solves $\mathscr{E}$.

Note that the characteristic momenta $\kappa_a : a \in \mathcal{A}$ are only naturally defined up to scale, and the hydrodynamic system may be rephrased that for all $a \in \mathcal{A}$ there are scalar valued functions $f_a(r_1, \dots r_n)$ (depending on the solution) such that $dR^a/d\boldsymbol{t} = f_a(\boldsymbol{R})\kappa_a(\boldsymbol{R})$. As in the introduction, the chain rule for $\boldsymbol{u} = U \circ \boldsymbol{R}$ implies

$$\frac{d\boldsymbol{u}}{d\boldsymbol{t}} = \frac{\boldsymbol{R}^* dU \circ d\boldsymbol{R}}{d\boldsymbol{t}} = \sum_{a\in\mathcal{A}} \frac{dR^a}{d\boldsymbol{t}} \otimes \partial_a U(\boldsymbol{R}) = \sum_{a\in\mathcal{A}} f_a(\boldsymbol{R})\kappa_a(\boldsymbol{R}) \otimes \partial_a U(\boldsymbol{R}).$$

Under the compatibility of the hydrodynamic system (so that it has many independent solutions $\boldsymbol{R}$), criterion (2) above is therefore equivalent to the property that $\beta_a := \kappa_a \otimes \partial_a U$ is in $\mathscr{E}$ for all $a \in \mathcal{A}$. But then, where β_a is nonzero, $[\beta_a]$ is in the rank one variety $\mathcal{R}^{\mathscr{E}}$, i.e., $[\kappa_a] \in \mathcal{X}^{\mathscr{E}}$ is characteristic and $[\partial_a U] \in \mathcal{C}^{\mathscr{E}}$ is cocharacteristic. Geometrically, the hydrodynamic reduction provides many *N-secant solutions* $\boldsymbol{u}$ to $\mathscr{E}$, i.e., the projective image of $d\boldsymbol{u}$ meets the rank one variety $\mathcal{R}^{\mathscr{E}}$ in N points.

Definition 5.2. An *N-component hydrodynamic reduction* of a QLS $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ with characteristic variety $\mathcal{X}^{\mathscr{E}}$ is a map

$$(U, [\kappa_1], \dots [\kappa_N])\colon \mathscr{V} \to \mathcal{X}^{\mathscr{E}} \times_{\mathscr{U}} \cdots \times_{\mathscr{U}} \mathcal{X}^{\mathscr{E}},$$

where $\mathscr{V}$ is open in $\mathbb{R}^N$ and the codomain is the N-fold fibre product, such that $\kappa_a \otimes \partial_a U$ is in $\mathscr{E}$ for all $a \in \{1, \dots N\}$ and the hydrodynamic system with characteristic momenta $\kappa_1, \dots \kappa_N$ is compatible as described in (5.1).

Then [9, 11] $\mathscr{E}$ is *integrable by hydrodynamic reductions* if for all $N \geqslant 2$ it admits N-component hydrodynamic reductions parameterized by $N(n-2)$ functions of 1-variable.

For $N \leqslant m = \dim \mathscr{U}$, a hydrodynamic reduction generically and locally determines an N-dimensional submanifold $\mathscr{N}$ of $\mathscr{U}$ (the image of U) together with a net $\partial_a U : a \in \mathcal{A}$ on $\mathscr{N}$ with $[\partial_a U] \in \mathcal{C}^{\mathscr{E}}$. Thus a hydrodynamic reduction defines a net satisfying Definition 4.3 (1). Conversely, if $\mathscr{E}$ is a compliant QLS, then for any such net, Proposition 3.7 shows that the embedding of $\mathcal{C}^{\mathscr{E}}$ into $P(\mathfrak{t}^* \otimes \mathcal{V}^{\mathscr{E}})$ gives $\partial_a U = \kappa_a \otimes v_a$ for some local sections v_a of $U^*\mathcal{V}^{\mathscr{E}}$, where $[\kappa_a]$ are the characteristic momenta corresponding to $[\partial_a U]$ under the isomorphism $\zeta^{\mathscr{E}}\colon \mathcal{C}^{\mathscr{E}} \to \mathcal{X}^{\mathscr{E}}$.

Proof of Theorem 1.1. The discussion so far has estabilished a correpondence between hydrodynamic reductions, modulo the compatibility condition, and nets satisfying 4.3 (1). It therefore remains to show that under this correspondence, the compatibility of the hydrodynamic system is equivalent to 4.3 (2), i.e., is automatic if rank $\mathcal{V}^{\mathscr{E}} \geqslant 2$ and is equivalent to the net being conjugate otherwise.

The proof of this last step follows the line of argument in [6], cf. also [3] (QLS of type I) and [8] (QLS of type H). First choose a basis $\varepsilon_1, \dots \varepsilon_n$ for $\mathfrak{t}^*$ and rescale the characteristic momenta such that $\kappa_{a1} = 1$. Under the embedding of $\mathcal{C}^{\mathscr{E}}$ into $P(\mathfrak{t}^* \otimes \mathcal{V}^{\mathscr{E}})$, $\partial_b U = \kappa_b \otimes v_b$ for some local sections v_b of $\mathcal{V}^{\mathscr{E}}$ over $\mathscr{N}$, and hence, in $\mathfrak{t}^*$ components, $\partial_b U_k = \kappa_{bk} v_b = \kappa_{bk}\partial_b U_1$ for $k \in \{1, \dots n\}$. Taking the ∂_a derivative of this equation and commuting partial derivatives yields

$$(\partial_a \kappa_{bk})\partial_b U_1 - (\partial_b \kappa_{ak})\partial_a U_1 = (\kappa_{ak} - \kappa_{bk})\partial_a\partial_b U_1.$$

On dividing by $\kappa_{ak} - \kappa_{bk}$, the right hand side is independent of k and hence

$$\Big(\frac{\partial_a \kappa_{bk}}{\kappa_{ak} - \kappa_{bk}} - \frac{\partial_a \kappa_{b\ell}}{\kappa_{a\ell} - \kappa_{b\ell}}\Big) v_b = \Big(\frac{\partial_b \kappa_{ak}}{\kappa_{ak} - \kappa_{bk}} - \frac{\partial_b \kappa_{a\ell}}{\kappa_{a\ell} - \kappa_{b\ell}}\Big) v_a.$$

Thus both sides are zero unless v_a and v_b are linearly dependent, i.e., multiples of some $v \in \mathcal{V}^{\mathscr{E}}$, say. But then the span of $\partial_a U = \kappa_a \otimes v_a$ and $\partial_b U = \kappa_b \otimes v_b$ is $\mathrm{span}\{\kappa_a, \kappa_b\} \otimes \mathrm{span}\{v\}$, hence decomposable. For $\mathrm{rank}(\mathcal{V}^{\mathscr{E}}) \geqslant 2$, the set where this holds has empty interior by condition (4) of compliancy, and so the hydrodynamic compatibility criterion is satisfied on the dense complement, hence everywhere by continuity.

It remains to establish the equivalence in the case $\mathrm{rank}(\mathcal{V}^{\mathscr{E}}) = 1$.

If the compatibility condition $\partial_a \kappa_{bk} = \gamma_{ba}(\kappa_{ak} - \kappa_{bk})$ for $a \neq b$ holds, then

$$\begin{aligned}\partial_a\partial_b U_k &= (\partial_a \kappa_{bk})\partial_b U_1 + \kappa_{bk}\,\partial_a\partial_b U_1\\ &= \gamma_{ba}(\kappa_{ak} - \kappa_{bk})\partial_b U_1 + \kappa_{bk}(\gamma_{ab}\partial_a U_1 + \gamma_{ba}\partial_b U_1)\\ &= \gamma_{ab}(v_a/v_b)\partial_b U_k + \gamma_{ba}(v_b/v_a)\partial_a U_k.\end{aligned}$$

Thus $\partial_a\partial_b U$ is in the span of and $\partial_a U$ and $\partial_b U$, so the net is conjugate.

Conversely, if the net is conjugate with $\partial_a\partial_b U_k = \alpha_{ab}\partial_b U_k + \beta_{ab}\partial_a U_k$ for $a \neq b$, then taking $k = 1$,

$$\partial_a\partial_b U_1 = \alpha_{ab}\partial_b U_1 + \beta_{ab}\partial_a U_1 = \alpha_{ab} v_b + \beta_{ab} v_a.$$

On the other hand, the ∂_a derivative of $\partial_b U_k = \kappa_{bk}\partial_b U_1$ yields

$$\kappa_{bk}\,\partial_a\partial_b U_1 = \partial_a\partial_b U_k - (\partial_a \kappa_{bk})\partial_b U_1 = \alpha_{ab}\kappa_{bk} v_b + \beta_{ab}\kappa_{ak} v_a - (\partial_a \kappa_{bk}) v_b.$$

Eliminating $\partial_a\partial_b U_1$ between these equations, it follows that

$$\alpha_{ab}\kappa_{bk} v_b + \beta_{ab}\kappa_{bk} v_a = \alpha_{ab}\kappa_{bk} v_b + \beta_{ab}\kappa_{ak} v_a - (\partial_a \kappa_{bk}) v_b$$

and hence $\partial_a \kappa_{bk} = \beta_{ab}(v_a/v_b)(\kappa_{ak} - \kappa_{bk})$, which is the compatibility condition. □

Corollary 5.3. *A compliant QLS $\mathscr{E} \leqslant \mathfrak{t}^* \otimes T\mathscr{U}$ is integrable by hydrodynamic reductions if and only if $\mathscr{U}$ admits a family of 3-dimensional cocharacteristic nets parametrized by $3(n-2)$ functions of one variable.*

In particular, this Corollary applies to generic QLS of types G, H and I, unifying and extending results in [3, 6, 8].

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