1. Let $G$ be a finite group acting on a finite set $X$, and for any $a \in G$ let $X^a = \{ x \in X : a \cdot x = x \}$ be the set of fixed points of the action of $\langle a \rangle \trianglelefteq G$ on $X$. Show that
\[
\sum_{a \in G} |X^a| = \sum_{x \in X} |\text{Stab}_G(x)|
\]

**Hint:** Consider $F = \{ (a, x) \in G \times X : a \cdot x = x \}$ and count its elements in two different ways.

2. Show that the set $\mathbb{P}(\mathbb{F}_5^2)$ of 1-dimensional subspaces of the standard 2-dimensional vector space $\mathbb{F}_5^2$ (over the field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ of integers modulo 5) has 6 elements.

**Hint:** Any 1-dimensional subspace is spanned by a vector $v = \left( \begin{smallmatrix} 1 \\ x \end{smallmatrix} \right)$ where either $x_1 = 0$ or $x_1 = 1$.

3. True or False?
   (i) Any subgroup of the centre $Z(G)$ of a group $G$ is normal in $G$.
   (ii) The kernel of any homomorphism is a normal subgroup.
   (iii) Any subgroup is the kernel of a homomorphism.
   (iv) For any finite group $G$, any $N \trianglelefteq G$, and any $a \in G$, the order $o(aN)$ of $aN$ in $G/N$ divides the order $o(a)$ of $a$ in $G$.

**Hint:** Note (ii) and (iii) can’t both be true unless all subgroups are normal subgroups. In (iv) observe that $o(x)$ divides $n$ if and only if $x^n = 1$.

4. Let $G$ be a finite group with a normal subgroup $N$ and let $a \in G$. Show that if $n = o(a)$ is coprime to the index $[G : N]$, then $a \in N$.

**Hint:** Consider $o(aN)$ and apply Lagrange’s Theorem.

5. Let $G$ be a finite group acting on a finite set $X$, and for any $a \in G$ let $X^a = \{ x \in X : a \cdot x = x \}$ be the set of fixed points of action of $\langle a \rangle \trianglelefteq G$ on $X$. Show that the number $N$ of orbits of the $G$ action is given by
\[
N = \frac{1}{|G|} \sum_{a \in G} |X^a|
\]

**Hint:** Observe that for any orbit $X_i \subseteq X$ of the $G$-action $\sum_{x \in X_i} \frac{1}{|\text{orb}_G(x)|} = 1$. This orbit counting lemma is often attributed to Burnside.

6. Let $X = \mathbb{P}(\mathbb{F}_5^2)$ be the set of all 1-dimensional subspaces of $\mathbb{F}_5^2$ and let $G = \text{GL}_2(\mathbb{F}_5)/K$ where $K = \{ \lambda I : \lambda \in \mathbb{F}_5^\times \} = \{ I, 2I, 3I, 4I \}$. [Thus elements of $G$ are left cosets $AK = \{ \lambda A : \lambda \in \mathbb{F}_5^\times \}$ of $K$ in $\text{GL}_2(\mathbb{F}_5)$ and by (i), $G$ is a group, called the projective general linear group $\text{PGL}_2(\mathbb{F}_5)$.]
   (i) Show that $K \trianglelefteq \text{GL}_2(\mathbb{F}_5)$.
   (ii) Show that $AK \cdot \text{span}(v) = \text{span}(Av)$ defines a faithful transitive action of $G$ on $X$. 

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**MA30237 Group Theory : Exercises 6**

Upload answers by end of Wednesday 11 November for the Seminar of Friday 13 November

Homepage: http://moodle.bath.ac.uk/course/view.php?id=55692

(W) = Warmup, (H) = Homework, (A) = Additional, (D) = Discussion
(iii) Show that the stabilizer of span($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) has order 20 and compute the order of $G$.

(iv) Let $Y$ be the set of bijections $f: \{1, 2, 3, 4, 5, 6\} \to X$ and let $G$ act on $Y$ by $AK \cdot f = \phi(AK) \circ f$, where $\phi: G \to \text{Sym}(X)$ is the action homomorphism of the action in (ii). Show that the action of $G$ on $Y$ has 6 orbits.

[Hint: (ii) is essentially a rephrasing of previous exercises, while in (iii) you can compute explicitly for which $A \in \text{GL}_2(\mathbb{F}_5)$, $AK$ is in the stabilizer. In (iv), one approach is to observe that the action of $G$ on $Y$ is free, another is to apply Burnside’s orbit counting lemma.]

7 (A/D). The fact that $G = \text{PGL}_2(\mathbb{F}_5)$ acts with 6 orbits on the set $Y$ of bijections $\{1, 2, 3, 4, 5, 6\} \to X$, where $X = \mathbb{P}(\mathbb{F}_5^2)$, has several interesting consequences. First show that $S_6$ also acts on $Y$ by $(\sigma \cdot f)(j) = f(\sigma^{-1}(j))$ for $j \in \{1, 2, 3, 4, 5, 6\}$, $f \in Y$ and $\sigma \in S_6$. Then show that this action is free and transitive and commutes with the action of $G$. Hence $S_6$ acts on the 6 element set of orbits of $G$ on $Y$. This action is different from the action of $S_6$ on $\{1, 2, 3, 4, 5, 6\}$ as the stabilizer of a bijection $f \in Y$ does not fix any $j \in \{1, 2, 3, 4, 5, 6\}$. It follows also that $G$ is isomorphic to $S_5$, providing another explanation of the fact that $A_5$ has a transitive action on a 6 element set.

[Hint: You might compute how a transposition in $S_6$ acts on the 6 element set of orbits of $G$ on $Y$, or look up the “exceptional outer automorphism of $S_6$” on Wikipedia.]