Lecture 8

RW on the Exclusion process

$(\eta_t)_{t\geq 0}$ Exclusion with $\boxed{\text{density } \rho\in(0,1)}$ $\boxed{\text{rate } \gamma\geq 0}$

$\mathbf{P}^\rho$

$(X_n)_{n\geq 0}$: P^η : quenched law of (X_n)

$\mathbb{P}^\rho = \mathbf{E}^\rho \times P^\eta(\cdot)$.

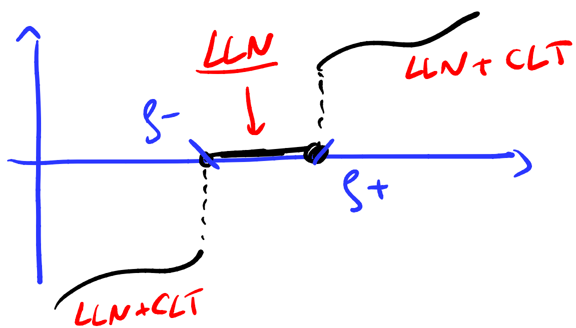
$\boxed{0 < p_\circ \leq p_\bullet < 1}$

Th: $\forall \rho \in (0,1) \setminus \{\rho_-, \rho_+\}$

$$\frac{X_n}{n} \to v(\rho) \quad \mathbb{P}^\rho\text{-a.s.}$$

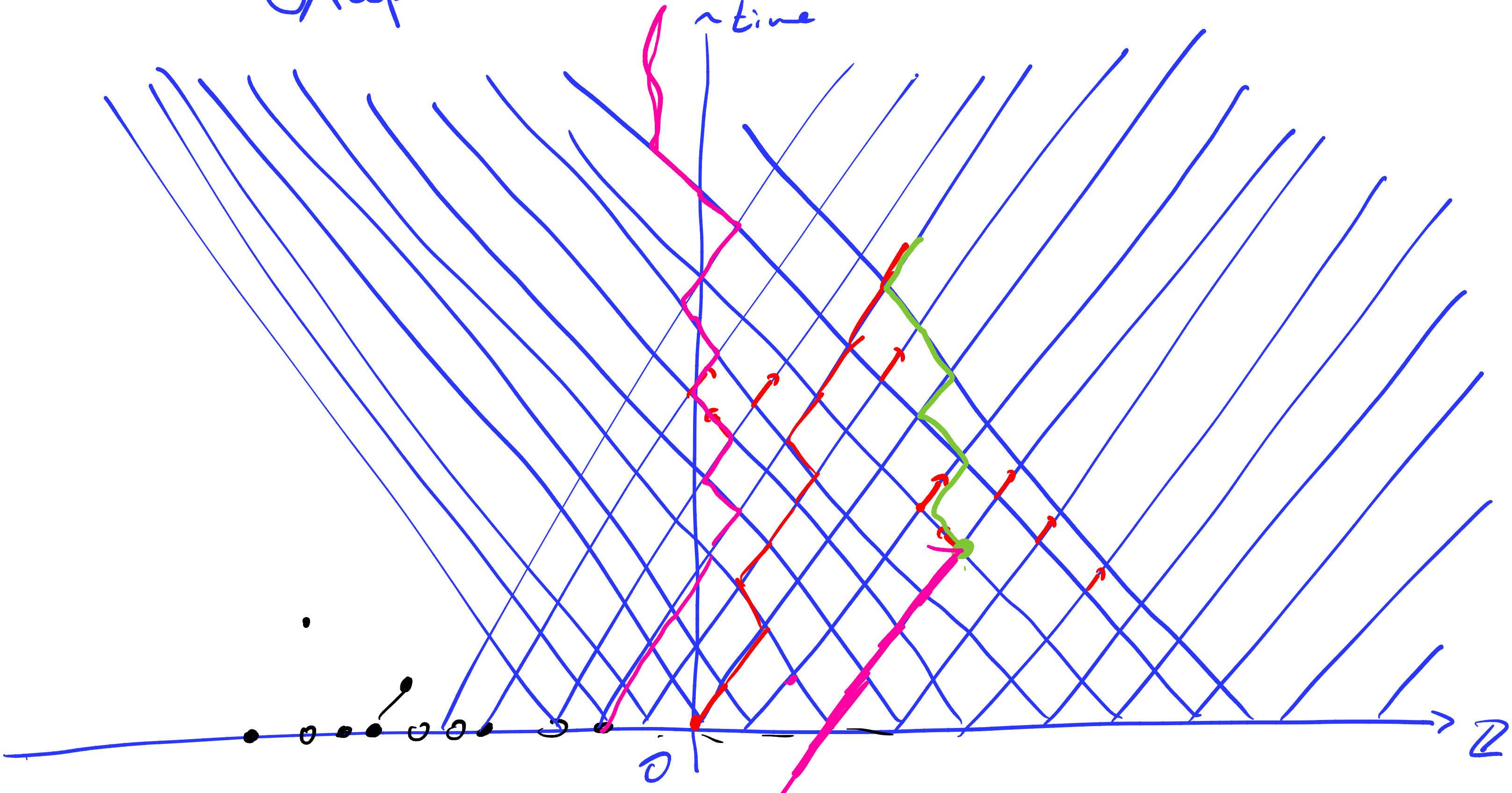
$+$ CLT for $\rho \in [0,1) \setminus [\rho_-, \rho_+]$

$$\rho_- = \sup\{\rho : v(\rho) < 0\}$$
$$\rho_+ = \inf\{\rho : v(\rho) > 0\}$$



Let's "prove" the LLN.

Graphical construction / Harris const.

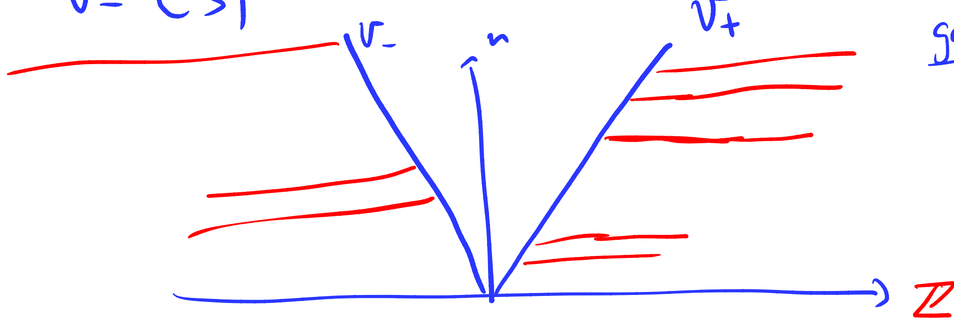


This provides a family of coupled random walks with a monotonicity property if we $(X_n^y)_{n\geq 0}$ at y, then $X_n^y \leq X_n^{y'}$ $\forall n\geq 0$ iff $y \leq y'$

I) Good candidates for the speed

$$v_+(s) = \inf\{v\in\mathbb{R} : \liminf_{n\to\infty} \mathbb{P}^s(X_n \geq v\cdot n) = 0\}$$

$$v_-(s) = \sup\{v\in\mathbb{R} : \liminf_{n\to+\infty} \mathbb{P}^s(X_n \leq v\cdot n) = 0\}$$



good: $v_\pm$ exist.

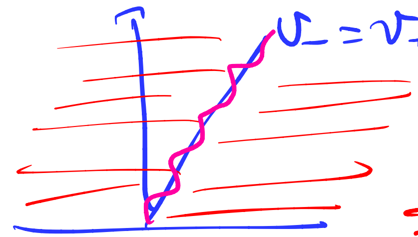
bad: * non-quantitive

* we do not even know that $v_- \leq v_+$

II) Idea to obtain a LLN

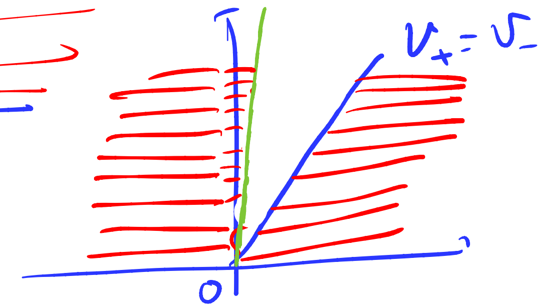
1) Prob. to go faster than $v_+(s) \vee 0$ (resp. slower than $v_-(s) \wedge 0$) decays fast with n.

2) $v_-(s) = v_+(s)$



3) Assume $v_+ = v_- > 0$

Prove $\mathbb{P}(X_n \le \varepsilon \cdot n) \longrightarrow 0$ fast

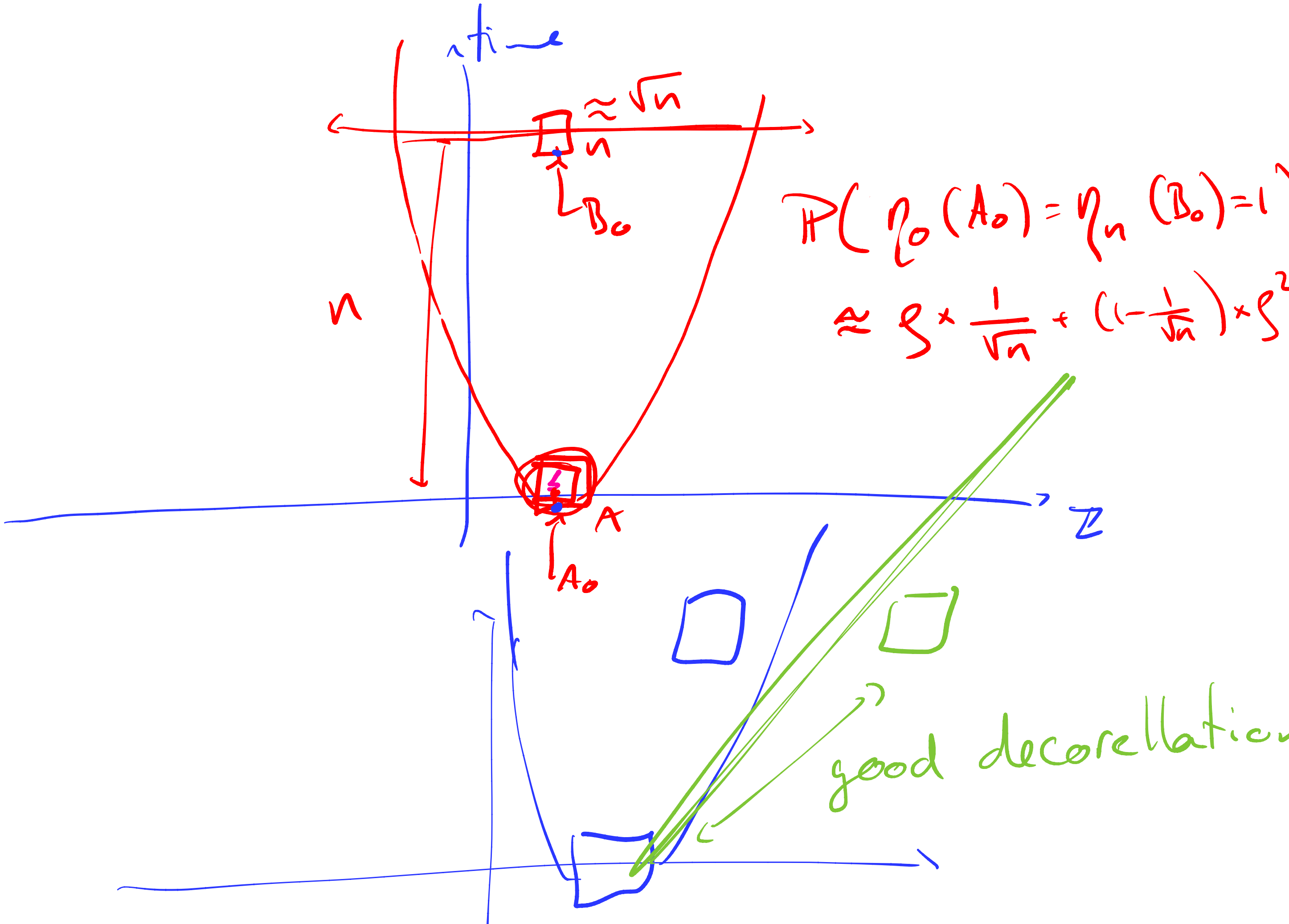


Using that, we will be able to use a regeneration structure

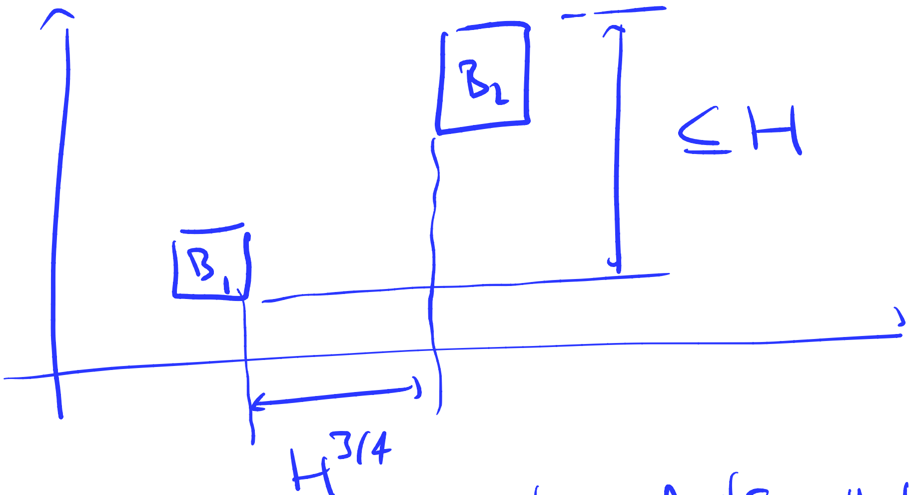
$\Longrightarrow$ LLN

$$\mathbb{P}(\eta_0(A_0)=\eta_n(B_0)=1)$$

$$\approx \rho \times \frac{1}{\sqrt{n}} + \left(1-\frac{1}{\sqrt{n}}\right)\times \rho^2$$



Lateral decoupling



$\forall f_1, f_2$ non-negative fct°, $\|\cdot\|_\infty \leq 1$, supported on B_1 & B_2 respective

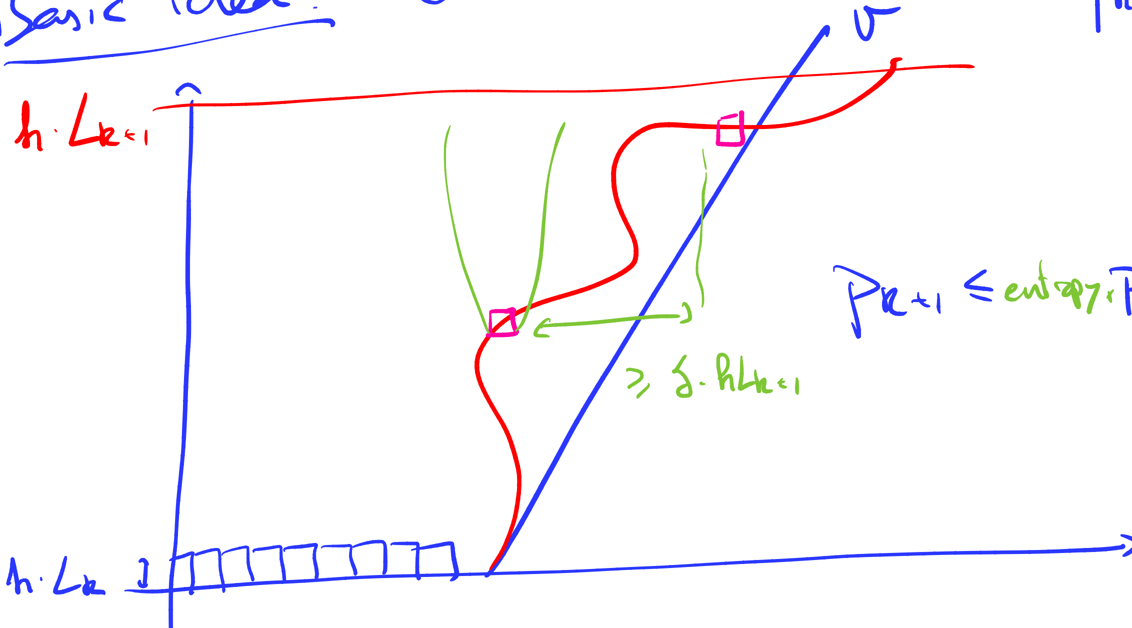
$$\mathrm{Cov}_\beta(f_1, f_2) \leq C_1(\beta)\, e^{-H^{1/4}}.$$

Very useful to control the probab. of events of the form $\{X_n \geq v\cdot n\}$ with $v>0$.

We will use renormalisation. $\boxed{v>0}$

<u>Basic idea:</u> control $\mathbb{P}(X_n \geq v\cdot n)$

$$p_k = \mathbb{P}(X_{hL_k} \geq h\cdot L_k\cdot v)$$



$$p_{k+1} \leq \text{entropy}\cdot p_k^2 + \underbrace{\text{error}_k}_{\text{control via lateral decoupling.}}$$

* The following choice works:

$L_0 = 10^{10}$. $\quad L_{k+1} \approx L_k^{5/4} = L_k^{1/4} \cdot L_k$ grows super-exponentially.

Assume that $p_k \le e^{-4(\log L_k)^{3/2}}$ ← very small in L_k (smaller than polynomial)

$h \geq 1$.

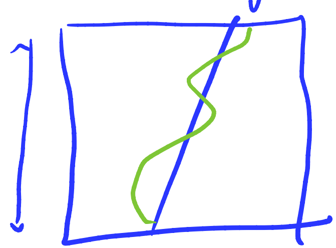
Assume

$$p_{k+1} \le L_k \times p_k^2 + c \cdot e^{-L_{k+1}^{1/4}}$$

$$\le \dots \le e^{-4(\log L_{k+1})^{3/2}}$$

We need to prove $p_0 \le e^{-4(\log L_0)^{3/2}}$

v



$h \cdot L_0$

choos $v > v_*(s) + \varepsilon$

$\Rightarrow \liminf \mathbb{P}(X_n > v \cdot n) = 0$

Choose h so that: $P_0 = \mathbb{P}^{\varrho}(X_{h \cdot L_n} = (h \cdot L_n) \cdot v) \leq \underbrace{e^{-4(\log L_0)^{3/2}}}_{\text{does not depend on } h.}$

"seed estimating / triggering"

+ recursive step.

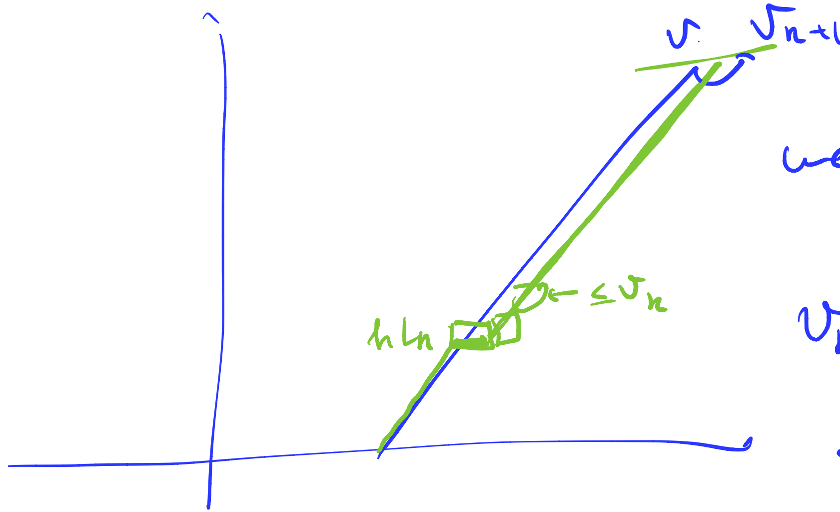
$$\Rightarrow \quad \mathbb{P}(X_{hL_n} \geq hL_n \cdot v) \leq e^{-4(\log L_n)^{3/2}}, \ \forall k \geq 0.$$

For $v > v_+(\varrho)$.

- To obtain the results for all $n \geq 0$, it's a straightforward interpolation argument.

$$\Rightarrow \quad \mathbb{P}^{\varrho}(X_n > (v_+(\varrho) + \varepsilon) \times n) \leq c \, e^{-(\log(n))^{3/2}}$$

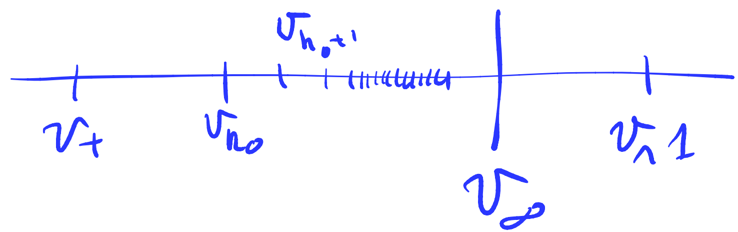
$\underline{\forall n \geq 0.}$



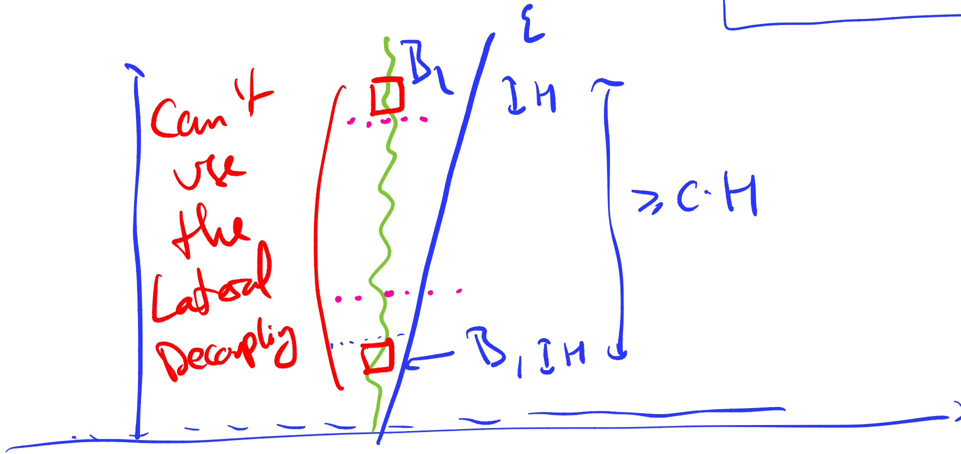
instead of v, we consider a sequence $(v_n)_{n\geq 0}$

$$v_{k+1} = v_k + \frac{v-(v_+ \vee 0)}{2^{n+1}}$$

$$v_{k_0} = \frac{v + v_+ \vee 0}{2}$$



For the third step, $\mathbb{P}^{\rho}(X_n \le \varepsilon \cdot n) \to 0$ when $v_+(\rho) = v_-(\rho) > 0$

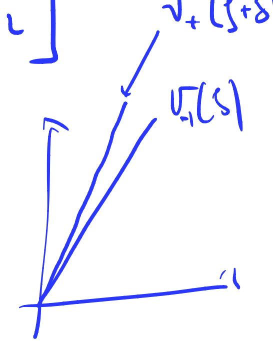


$$\varepsilon_H \geq \frac{1}{H^{1/16}}$$

For decorrelation, we use sprinkling:

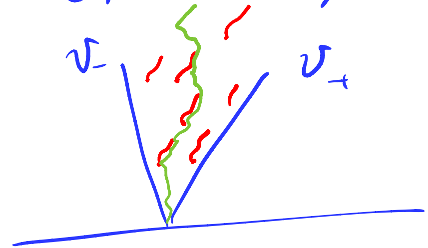
$$E^{\rho+\varepsilon_H}[f_1 f_2] \leq E^{\rho+\varepsilon_H}[f_1] \times E^{\rho}[f_2]$$

$f_1, f_2 \in [0,1]$ non-increasing.



Second step: $v_-(\xi) = v_+(\xi)$

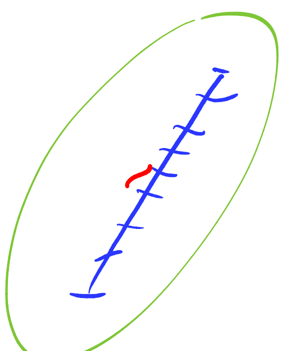
Assume $v_- < v_+$



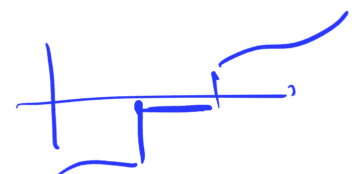
$\rightsquigarrow$ good chance to go at speed close to v_+

$\rightsquigarrow$ this will create delays that prevent the walk from approaching, ever, v_-.

That's a contradiction



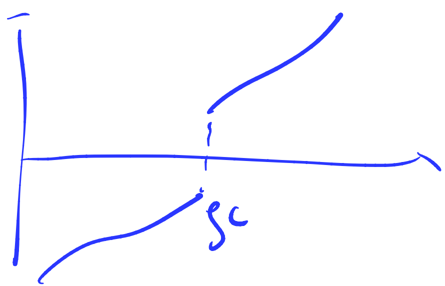
ξ_- ξ_+ :



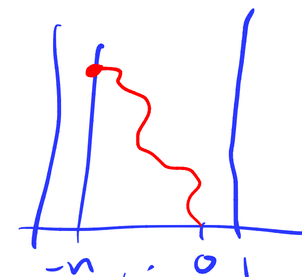
Th (Conchon-Kerjan, K., Rodriguez, '24)

$\rho_- = \rho_+ = \rho_c$.

More precisely $\sigma(\cdot)$ is strictly monotonic



Sharpness result



Cor: $B_n = \{X \text{ hits } -n \text{ before } 1\}$

1. $\mathbb{P}^\rho(B_n) \leq c_1 \cdot e^{-(\ln(n))^{3/2}}$, $\forall n \geq 0$, $\forall \rho > \rho_c$,
2. $\mathbb{P}^\rho(B_n) \geq \underset{\substack{\shortparallel \\ c_2(\rho)}}{c_2}$, $\forall n \geq 0$, $\forall \rho < \rho_c$

critical case p_c is hard. $\mathbb{P}^{p_c}(B_n)$??

* Our proof strategy is inspired by:

* Duminil-Copin, Goswami, Rodriguez & Severo ('23)

level set percolation for the GFF.

Gaussian Free Field

Same authors + Teixeira ('24)

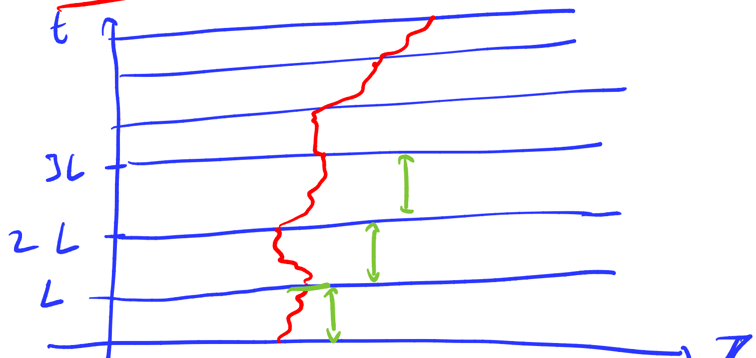
Random interlacement

Strategy:

1. Define a finite-range model with good sharpness

2. Compare the finite-range model to the full-range model, via interpolation scheme

Finite-range model



refresh the environment every L steps

The increment $\left(X^{\rho,L}_{(k+1)L} - X^{\rho,L}_{kL}\right)_{k\geq 0}$ are i.i.d.

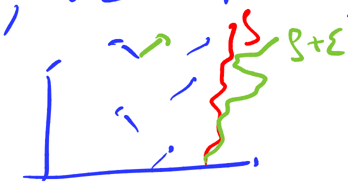
$$\frac{X_n^{\rho,L}}{n} \longrightarrow v_L(\rho) = \frac{E^\rho\left[X_L^\rho\right]}{L}$$

Prop. Approx: $\forall \rho \in (0,1)$, $\forall L$ large (ρ), $\alpha_L = \dfrac{\log^{100} L}{L}$

$$v_L\left(\rho - \frac{1}{\log L}\right) - \alpha_L \leq v(\rho) \leq v_L\left(\rho + \frac{1}{\log L}\right) + \alpha_L .$$

Prop. Quantitative Mon.: $\forall \rho \in (0,1)$, $\forall \varepsilon \in (0,1)$, $\forall L \geq L_1(\rho, \varepsilon)$

$$v_L(\rho + \varepsilon) - v_L(\rho) \geq 3 \cdot \alpha_L$$



Proof of Th:

$$v(\rho + 3\varepsilon) \overset{\text{Approx.}}{\geq} v_L\left(\rho + 3\varepsilon - \frac{1}{\log L}\right) - \alpha_L$$

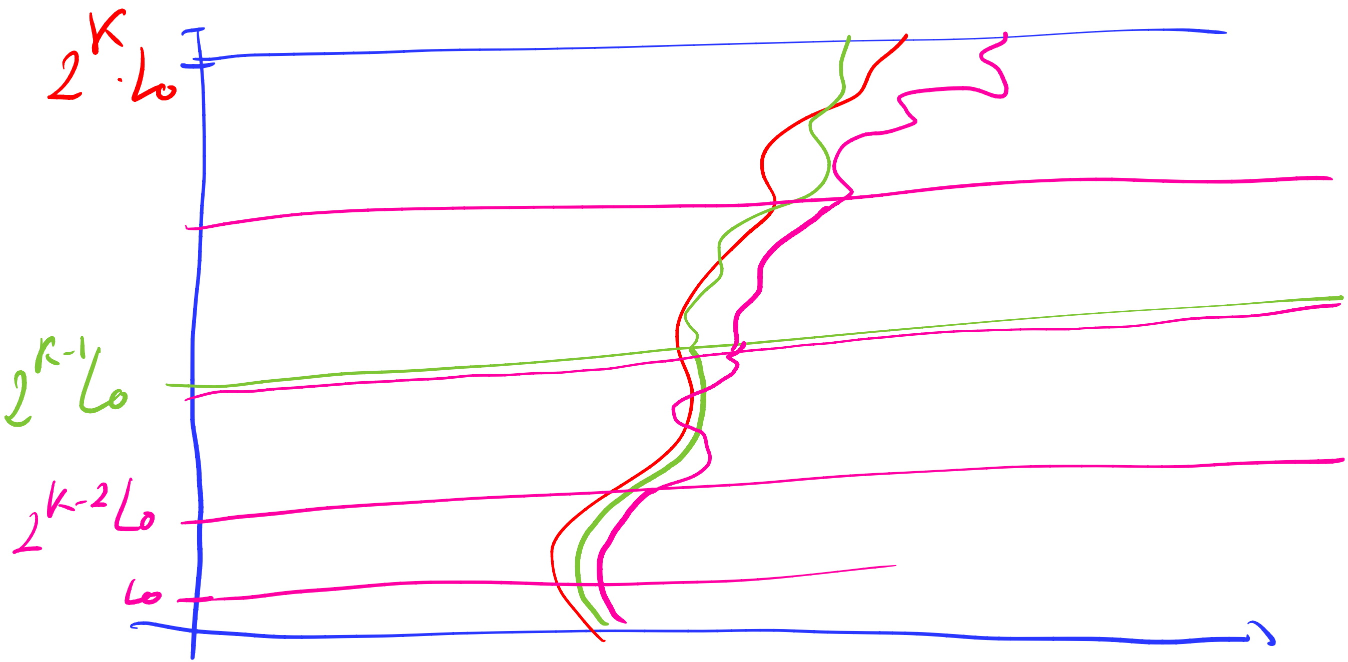
$$\geq v_L(\rho + 2\varepsilon) - \alpha_L$$

$$\overset{\text{Q.M.}}{\geq} v_L(\rho + \varepsilon) + 2\alpha_L \geq v_L\left(\rho + \frac{1}{\log L}\right) + 2\alpha_L$$

$$\overset{\text{Approx}}{\geq} v(\rho) + \alpha_L > v(\rho) \;\#$$

$X_{2^K L_0} = X^{s,2^K L_0}_{2^K L_0}$

$X^{2^{K-1} L_0}$



To control the discrepancy between X & $X^{2^K L_0}$ we need to control the discrepancy between $X^{2 \cdot 2^k L_0}$ & $X^{2^k L_0}$ for $k = 0, \dots, K$.

Compare $X^{2L,s}$ to $X^{L,s+\varepsilon_L}$