Lecture 7

Recap: $\mathcal{T}$: Galton-Watson tree with mean $m > 1$. (super-critical)

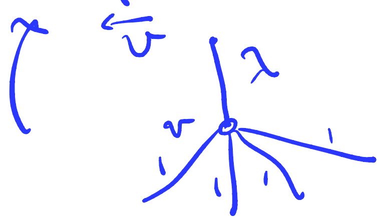
$q = \mathbb{P}[|\mathcal{T}| < \infty]$

f : generating function of offspring.

$[f(q) = q]$

(X_n) : biased RW on $\mathcal{T}$, with bias $\lambda > 0$

$\frac{\lambda}{\lambda + d_v}$



(X_n) is transient for $\forall \lambda \in (0,m)$

Th (Lyons, Pemantle & Peres, '96)

$$\frac{|X_n|}{n} \to v \text{ a.s. on a.e. } T \text{ upon non-extinction,}$$

where:

(i) $v > 0$ if $\overbrace{f'(q)}^{<1} < \lambda < m$

(ii) $v = 0$ if $0 < \lambda < f'(q)$ if $\mathbb{P}(Z_1 = 0) > 0$

We were in the case $1 < \lambda < m$.

We proved

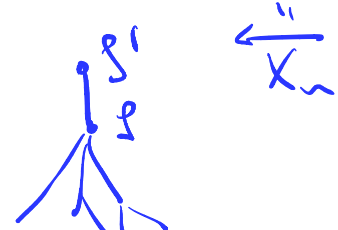
$$\frac{\mathbb{E}[R_n]}{n} \geq \frac{1}{n} + \frac{\lambda-1}{2\lambda G(\varrho,\varrho)+\lambda-1}$$

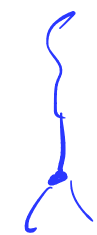
$R_n = \#\{x \in T : T_x \leq n\}$, $G(x,y) = \sum_{i \geq 0} P_x^T(X_i = y)$.

* **Fresh epochs:** $n > 0$ s.t. $X_n \neq X_k$, $\forall k < n$
* **Regeneration epochs:** fresh epoch n s.t. $X_k \neq X_{n-1}$ $\forall k > n$.

$\gamma(T) = P_\varrho^{T'}[T_{\varrho'} = +\infty]$

T': augmented tree =





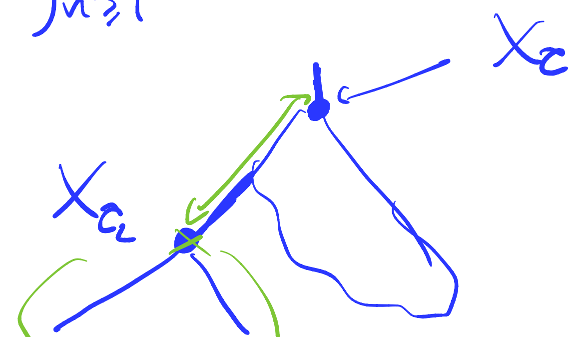
$\mathbb{P}_{non} : \mathbb{P}(\cdot \mid |T| = +\infty)$

Lemma: $\exists$ infinitely many regen. epochs

Prop: $\forall 1 < \lambda < m$, on the event of non-extinction

$(\tau_{n+1} - \tau_n)_{n \geq 1}$ are i.i.d.

$(|X_{\tau_{n+1}}| - |X_{\tau_n}|)_{n \geq 1}$ are i.i.d.



Prop: $\mathbb{E}_{non}[\tau_2 - \tau_1] < +\infty$

Proof:

$$\mathbb{E}_{non}[\#\{i \geq 1 : 1 \leq \tau_i \leq n\}] = \mathbb{E}_{non}\Big[\sum_{k=1}^{n} \mathbb{1}_{\{k \text{ is a regen epoch}\}}\Big]$$

$$= \sum_{k=1}^{n} \mathbb{P}_{non}[k \text{ is a fresh epoch}]\, \mathbb{E}_{non}[\gamma(T)]$$

$$= \mathbb{E}_{non}[\gamma(T)]\big(\mathbb{E}_{non}[R_n] - 1\big)$$

$$\Rightarrow \frac{\mathbb{E}_{non}[\#\{i \geq 1 : 1 \leq \tau_i \leq n\}]}{n} \geq \mathbb{E}_{non}[\gamma(T)] \times \mathbb{E}_{non}\Big[\frac{\lambda - 1}{3\lambda G(s,s)}\Big]$$

$$= \mathbb{E}_{non}[\gamma(T)]^2 \times \frac{\lambda - 1}{3\lambda} \boxed{> 0}$$

By the Renewal theorem and LLN we have $E_{non}[\tau_2-\tau_1] < \infty$.

#

Proof of $\frac{|X_n|}{n} \to v > 0$.

$$\tau_n = \sum_{n=1}^{n-1}(\tau_{n+1}-\tau_n) + \tau_1$$

$$\frac{\tau_n}{n} \to E_{non}[\tau_2-\tau_1] < \infty \text{ a.s.}$$

$$\frac{|X_{\tau_n}|}{n} \to E_{non}[|X_{\tau_2}| - |X_{\tau_1}|] \text{ a.s.}$$

$$\frac{|X_{\tau_n}|}{\tau_n} \to \underbrace{\frac{E_{non}[|X_{\tau_2}| - |X_{\tau_1}|]}{E_{non}[\tau_2-\tau_1]}}_{=v} > 0 \text{ a.s.}$$

To have $\frac{|X_n|}{n}$, use that $\frac{\tau_{n+1}}{\tau_n} \to 1$ a.s. and an inversion argument.

#

If $\mathbb{P}(Z_1 = 0) = 0$ ($\mathcal{T}$ cannot die), then the previous argument can be adapted to $0 < \lambda < 1$.

$\left[\text{for } \mathbb{E}[R_n] \geq n \cdot \frac{1-\lambda}{1+\lambda} \right]$

local drift.

Assume now $\mathbb{P}(Z=0) > 0$

Prop: $\frac{|X_n|}{n} \to v$ a.s. : $\begin{cases} v > 0 & \text{if } f'(q) < \lambda < 1 \\ v = 0 & \text{if } 0 < \lambda < f'(q). \end{cases}$

Define $$g(s) = \frac{f((1-q)s+q)-q}{1-q}$$

$$h(s) = \frac{f(qs)}{q}$$

An f-GW tree $\tilde{T}_f \equiv \begin{cases} \text{1) Generate a } g\text{-GW tree } T_g \text{ (with } g\text{)} \\ \text{2) Add } N_x \ h\text{-shrubs to vertex } x. \end{cases}$

Law of N_x = explicit but not relevant depending only on $d_{T_g}(x)$. See Lyons ('92).

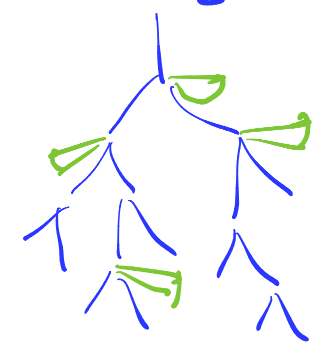
Observe: $\mathbb{P}^g\left(Z_1^{T_g}=0\right) = g(0) = \dfrac{f(q)-q}{1-q} = 0$

$$\mathbb{E}^h\left[Z_1^{T_h}\right] = h'(1) = \frac{q\cdot f(q)}{q} = q < 1$$

$\Rightarrow$ T_h dies out a.s.

* Construct the sequence of stopping times

$\tilde{T}_g$



$\sigma_0 = 0$,

$\sigma_{n+1} = \inf\{k > \sigma_n : X_k \in \tilde{T}_g\}$

$Y_n = X_{\sigma_n}$ RW on $\tilde{T}_g$

Between σ_n & σ_{n+1}, $X_\cdot$ does an excursion inside a shrub, of random duration $\sigma_{n+1}-\sigma_n$.

Consider a single shrub and let us compute the expected time to return to its root

On a tree T: $E_\varrho^T[T_\varrho^+] = \dfrac{2}{T_1} \sum_{n\geq 1} T_n \lambda^{-n+1}$

reciprocal of the stationary probability.

T_n = # vertices at generation n.

For a h-GW tree, we obtain:

$$E\left[E^{T_n}_{\xi_n}\left[T^+_{\xi_n}\right]\right] = 2\sum_{n\geq 0}\left(h'(1)\right)^{n-1}\cdot\lambda^{-n+1}$$

$$= \begin{cases} \dfrac{2}{1-f'(a)\cdot\lambda^{-1}} & \text{if } \dfrac{h'(1)}{\lambda}<1 \iff \lambda > f'(a) \\ \infty & \text{, otherwise.} \end{cases}$$

The expected time between two regen epochs is infinite when $0<\lambda\leq f'(a)$

$$LLN \Rightarrow v=0$$

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* Assume $f'(a)<\lambda<1$,

Between τ_n & τ_{n+1} : the walk does a number of excursions into the shrub $\sim$ Geom -1.

of average $$\frac{d_{T_f}(Y_n) - d_{T_g}(Y_n)}{\lambda + d_{T_f}(Y_n)}$$

We obtain:

$$E_{new}\left[\tau_{n+1} - \tau_n \mid Y_n\right] \le \text{Cste.}\ d_{T_f}(Y_n)$$

* Let $Z_1, \dots, Z_{K_n}$ the distinct vertices visited by $Y_1, \dots, Y_n$.

Define $$U_k = \sum_{j=1}^{n} \mathbb{1}\{Y_j = Z_k\}$$

$$\sum_{i=1}^{n} d_{\tilde{T}_f}(y_i) \leq \sum_{i=1}^{K_n} U_k \, d_{\tilde{T}_f}(z_k)$$

1-dim bound. ($\lambda<1$)

$$E_{\mathrm{non}}\left[U_k d_{\tilde{T}_f}(z_k)\right] \leq E_{\mathrm{non}}\left[d_{\tilde{T}_f}(z_k) \times \frac{2\lambda}{1-\lambda}\right]$$

$$\leq \frac{1}{1-q} \times m \times \frac{2\lambda}{1-\lambda}$$

$$E_{\mathrm{non}}\left[\sum_{i=1}^{n} d_{\tilde{T}_f}(y_i)\right] \leq n \times \frac{m}{1-q} \times \frac{2\lambda}{1-\lambda}.$$

$$\Rightarrow E_{\mathrm{non}}\left[\frac{\sigma_n}{n}\right] \leq \text{cste} \xRightarrow{\text{Fatou}} \liminf \frac{\sigma_n}{n} < +\infty \text{ a.s.}$$

To conclude use that the regenerations

occurs with positive frequence for Y,

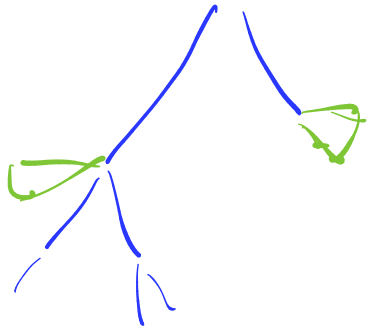
hence $\lim \frac{\tau_k^Y}{k} < +\infty$ a.s.

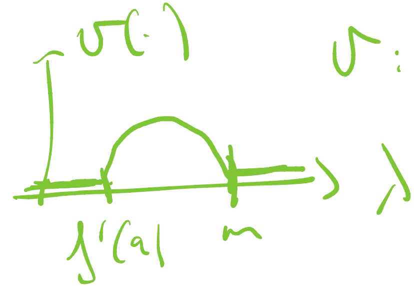
If σ_k^Y is a regen. for $Y \Rightarrow$ it is a regeneration for X with $\tau_{k'}^X$, $k' \geq k$.

$$\frac{\tau_{k'}^X}{k'} = \frac{\sigma_{\tau_k^Y}}{k'} \leq \frac{\frac{\sigma_{\tau_k^Y}}{k} \times k}{k}$$

$$\Rightarrow \liminf \frac{\tau_k^X}{k} < +\infty$$

$\hookrightarrow$ turns into a limit since $E[\tau_{k+1} - \tau_k] < +\infty$. #





v : non-monotone

Q° / Conjecture : Prove that

Lyons, Pemantle?

if $\mathbb{P}(Z_1=0)=0$ then v is $\downarrow$ in λ,

$\forall \lambda \in (0,m)$.

double check

Still open

Partial results of Aïdékon $\left(\lambda \in \left(0,\frac{1}{2}\right)\right)$

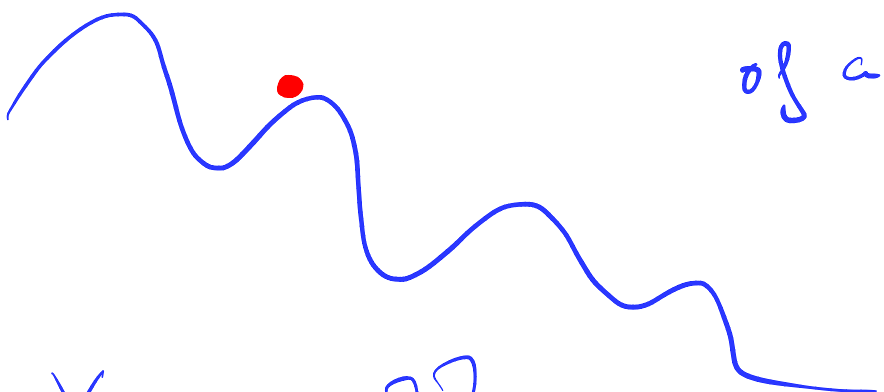
* Ben Arous, Fribergh & Sidoravicius

$\lambda < \frac{1}{712}$ ← very nice & easy.

Random walks in dynamic random environment

We will focus on $d=1$, $\mathbb{Z}$.

Moving particle on top of a fluctuating potential



Q°: 1) LLN for (X_n)

2) Fluctuations?

$\hookrightarrow$ CLT, ... or NOT $\to$ atypical behaviour.

$\frac{X_n}{n^a}$ $a > \frac{1}{2}$??

Definition of the model

I) The environment: Simple symmetric Exclusion process

Markov process $(\eta_t)_{t\geq 0}$, $\eta_t \in \{0,1\}^{\mathbb{Z}}$, $\forall t \geq 0$.

* initial law: $\eta_0 \sim \text{Prod. Ber}(\rho)$, $\boxed{\rho \in (0,1) \text{ density}}$

* Dynamics:

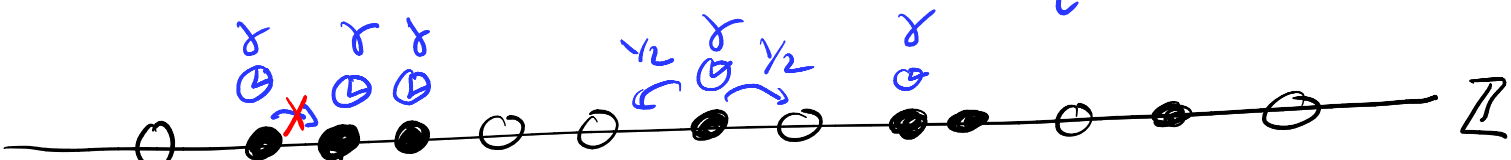
$$\Omega f(\eta) = \sum_{\substack{x,y\in\mathbb{Z} \\ x\sim y}} \mathbb{1}_{\{\eta(x)=1,\, \eta(y)=0\}} \cdot \frac{\gamma}{2}\left[f(\eta^{xy}) - f(\eta)\right]$$

where $\eta_{xy}(\zeta) = \begin{cases} \eta(\zeta) & \text{if } \zeta \neq x,y \\ \eta(y) & \text{if } \zeta = x \\ \eta(x) & \text{if } \zeta = y \end{cases}$

$\boxed{\gamma > 0 \text{ rate}}$

$\mathbf{P}^\rho$: law of the env.

$\eta^\rho \sim \mathbf{P}^\rho$



Prod. Ber(ρ) is invariant for this dynamic.

II The random walk

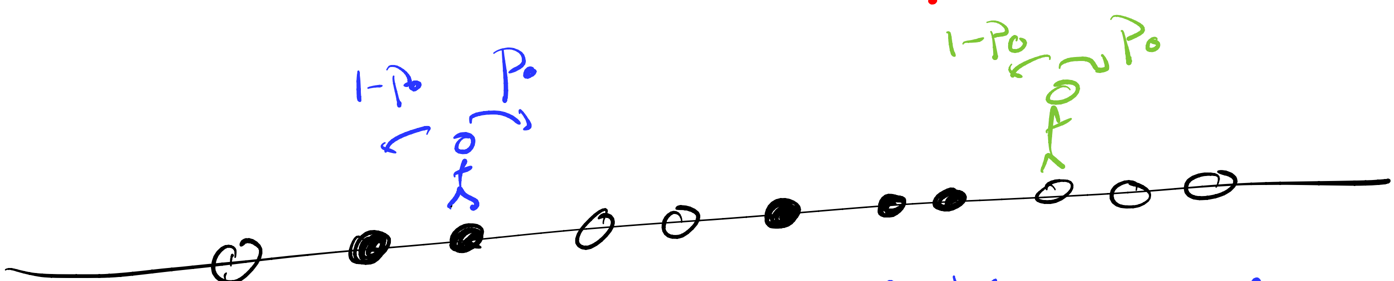
Fix a realisation η of η^ρ (call of $\eta_t,\ t \geq 0$)

Fix parameters $0 < p_\circ \underset{\text{w.l.o.g.}}{\leq} p_\bullet < 1$.

Given η, we define a Markov chain $(X_n)_{n\geq 0}$,

$X_0 = 0$, and $\forall n \geq 0$,

$$X_{n+1} = X_n \begin{matrix} +1 \\ -1 \end{matrix} \text{ w.p. } \begin{cases} \begin{matrix} p_\bullet \\ 1-p_\bullet \end{matrix} & \text{if } \eta_n(X_n) = 1 \\[2ex] \begin{matrix} p_\circ \\ 1-p_\circ \end{matrix} & \text{if } \eta_n(X_n) = 0 \end{cases}$$



P^η : quenched law of X_n given η.

$\mathbb{P}^\xi$: annealed law $\mathbb{P}^\xi \times P^{\eta^\xi}(\cdot)$

Th (Hilário, K., Teixeira, '19/'20)

$\exists$ a non-decreasing function, deterministic, $v : (0,1) \to [-1,1]$

s.t.

1) $\frac{X_n}{n} \to v(\rho)$ $\mathbb{P}^\rho$-a.s., $\forall \rho \in (0,1) \setminus \{\rho_-, \rho_+\}$

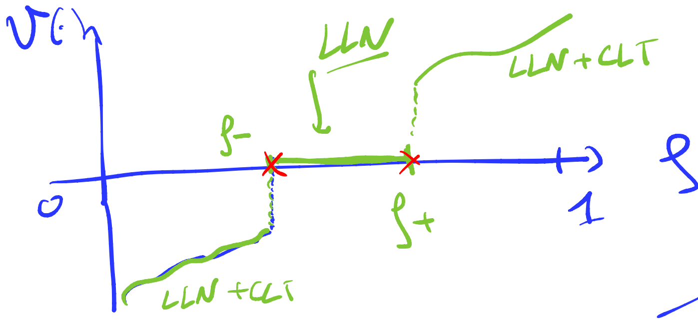
where $\rho_- = \sup\{\rho \in (0,1) : v(\rho) < 0\}$

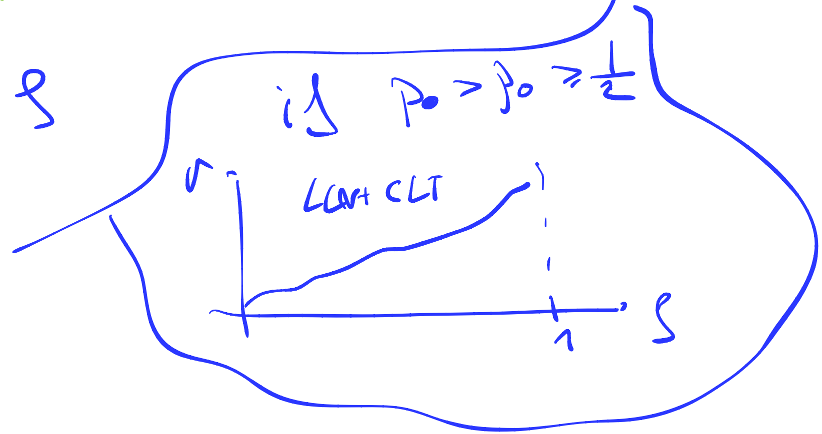
$\rho_+ = \inf\{\rho \in (0,1) : v(\rho) > 0\}$.

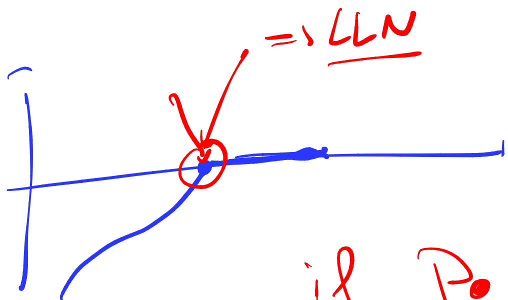
2) $\left(\frac{X_{\lfloor nt \rfloor} - nt \cdot v(\rho)}{\sqrt{n}} \right)_{t \geq 0} \Rightarrow (B_t)$ under $\mathbb{P}^\rho$

$\forall \rho \in (0,1) \setminus [\rho_-, \rho_+]$

think: $p_0 < \frac{1}{2} < p_\bullet$







if $p_\bullet = 1 - p_0 > \frac{1}{2}$ then at $\rho = \frac{1}{2}$

$\Rightarrow v\left(\frac{1}{2}\right) = 0$ & LLN holds.

but no fluctuations.

Two questions come from the Th. above

1) Fluctuations when $v(\rho) = 0$.

Physics papers: super-diffusivity.

$\dfrac{X_n}{n^{2/3}}$ has a limit. ???

This question is still widely open.

This may depend on the value of all parameters.

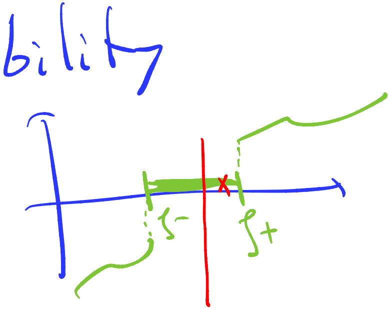
$p_0 = 1 - p_c > \frac{1}{2}$

$\rho = \frac{1}{2}$

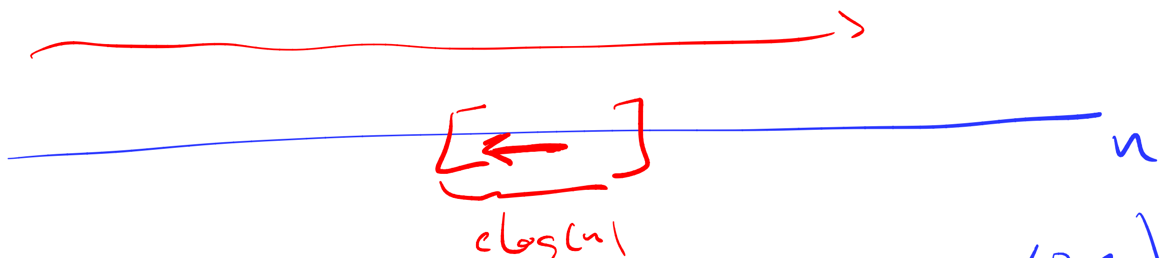
$\Rightarrow v = 0$

γ? p_0?

2) We left open the possibility for $\underbrace{\rho_- < \rho_+}_{\hookrightarrow \text{ transient zero-speed regime.}}$



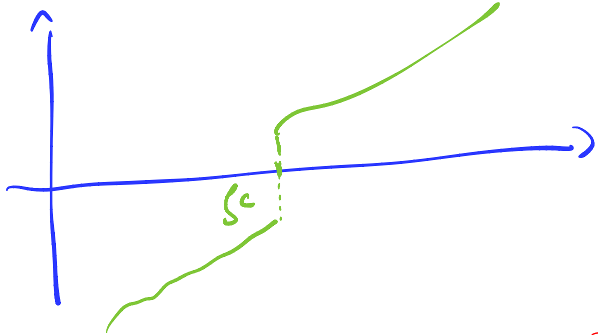
ρ_c : recurrent

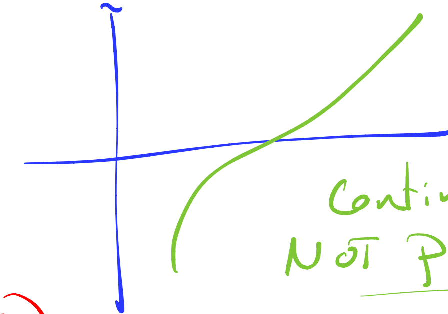


Th (Conchon-Kerjan, K., Rodrigues, '24)

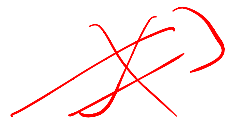
$$\rho_- = \rho_+ = \rho_c$$

More precisely $v(\cdot)$ is strictly increasing.

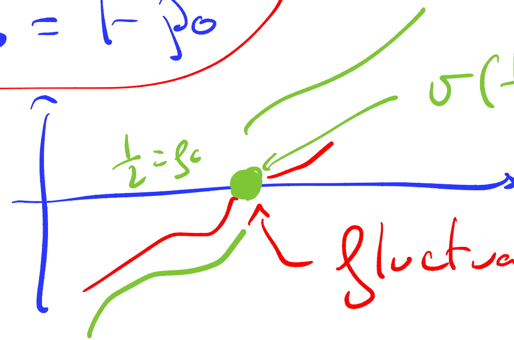




Continuity.
Not proved.



$\boxed{\text{if } \rho_\bullet = 1 - \rho_\circ}$



$\sigma(\frac{1}{2}) = 0$ & LLN

fluctuations??