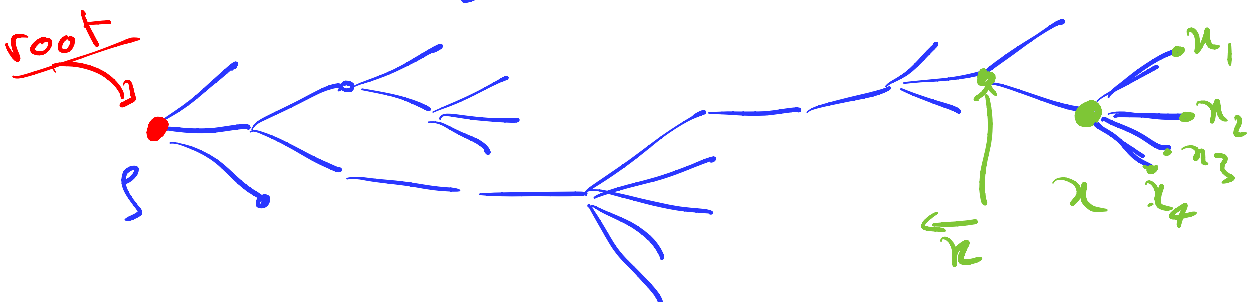
Lesson 6

* Random walks on Galton–Watson trees.

biased.

↪ RW on random graphs.

Fix a bias $\lambda > 0$, and define a RW (X_n) s.t. $X_0 = \rho$, where ρ is the root of a tree T:



For all $n \geq 0$, $\forall x \in T$, if $X_n = x$, then letting $\overleftarrow{x}$ be the parent of x, & $x_1, \ldots, x_d$ are the offspring of x, then

(i) if $x = \rho$ then $X_{n+1} = x_i$ w.p. $\frac{1}{d}$, $\forall i \in \{1, \ldots, d\}$

(ii) if $x \neq \rho$ then

$$X_{n+1} = \begin{cases} \overleftarrow{x} & \text{w.p. } \dfrac{\lambda}{d_x + \lambda} \\ x_i & \text{w.p. } \dfrac{1}{d_x + \lambda} \end{cases}.$$

We denote E^T the expectation given T.

We can T to be picked as a Galton-

Watson tree, with law $\mathbf{P}$.

Define the annealed measure $\mathbb{P} = \mathbf{E}[P^T(\cdot)]$.

* Galton-Watson trees : random trees, branching processes.

L_i | Bienaymé - G-W
| Bienaymé-Watson
| Bienaymé

Start with a single particle (individual, the root)

Consider L an integer-valued r.v., $L \geq 0$, whose distribution is called the offspring distribution

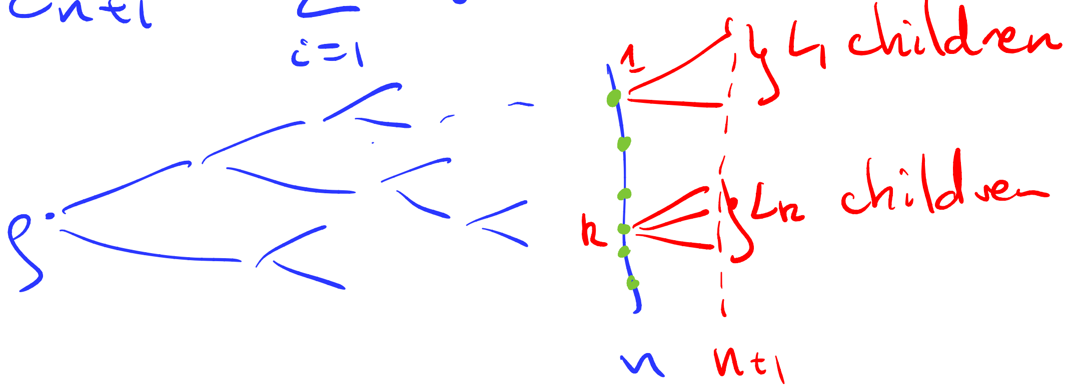
Goal: Define (Z_n)

Let $Z_0 = 1$.

For $n \geq 0$, given Z_n, we consider $L_1, \dots, L_{Z_n}$ i.i.d. copies of L, indep. of everything else.

We say that the i-th individual at generation n, for $1 \leq i \leq Z_n$, has L_i children and set

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_i .$$



That inductively define a branching process $\mathcal{T}$.
This tree $\mathcal{T}$ can be finite or infinite.
What is $\mathbb{P}(|\mathcal{T}| = +\infty) \begin{cases} > 0 \\ = 0 \end{cases}$?
This determined by $m = \mathbb{E}[L]$

$$\boxed{\text{If } m > 1 \text{ then } \mathbb{P}(|\mathcal{T}| = +\infty) > 0}$$

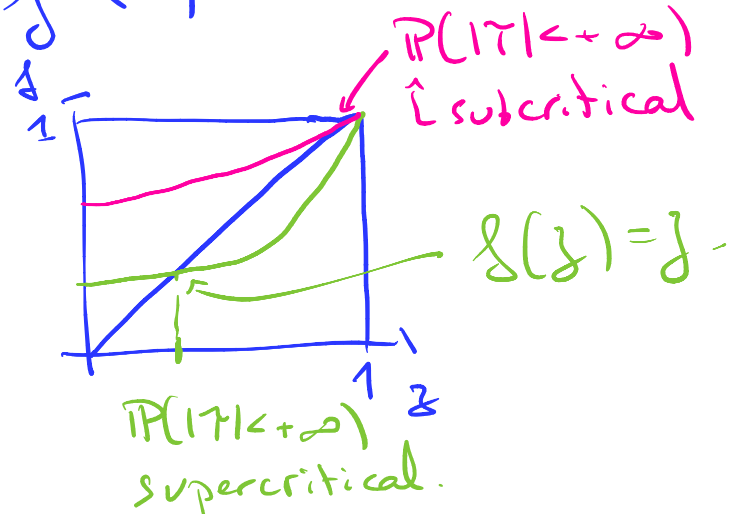
if $m \leq 1$ then $\mathbb{P}(|\mathcal{T}| = +\infty) = 0$.
↳ $\mathcal{T}$ finite a.s.

<u>Rk</u>: $\mathbb{E}[Z_n] = m^n$.

When studying random trees, the generating function of L is an important quantity:

$$f(z) = \sum_{k \geq 0} \mathbb{P}(L=k) z^k = E[z^L], \quad z \geq 0.$$

$$f'(1) = m = E[L], \quad f(0) = P(L=0).$$



Let $\mathcal{T}$ be a BGW-tree in the super-critical regime $m>1$. Let (X_n) be a λ-biased RW on $\mathcal{T}$.

<u>Th</u> (R. Lyons, 1990). For $m>1$, a.s. on the event of non-extinction, the λ-biased RW on $\mathcal{T}$ is

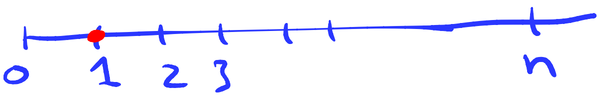
1) Transient if $0<\lambda<m$, $\leftarrow$ it visits the root finitely many times

2) Recurrent if $\lambda \geq m$ $\leftarrow$ returns to ϱ a.s.

We will prove the (non-critical) cases using the proof of A. Collevecchio (2006).

Reminder: The Gambler's ruin

Consider a RW on $\{0,1,2,\dots\}$ s.t. if $Y_n = j \geq 1$, $Y_{n+1} = j+1$ w.p. $p \in (0,1)$, or to $j-1$ w.p. $1-p$.



Let $Y_0 = 1$, then:

$$\mathbb{P}_1(T_n < T_0) = \begin{cases} \dfrac{\left(\frac{1-p}{p}\right) - 1}{\left(\frac{1-p}{p}\right)^n - 1} & \text{if } p \neq 1-p \quad (\neq 1/2) \\[2ex] \dfrac{1}{n} & \text{if } p = \dfrac{1}{2}. \end{cases}$$

Proof (of Lyons's result)

(2) Recurrence: First moment method.

Fix $\underline{\lambda > m > 1}$.

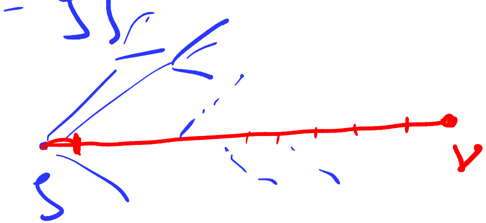
$\forall\, v \in T$ s.t. $|v| = n$ ← generation of v / distance to ρ.

Define: $T_v = \inf\{k \geq 0 : X_k = v\}$

$$H_v = \inf\{k > T_v : X_k = \overleftarrow{v}\}$$

$$\tau = \inf\{k \geq 1 : X_k = \rho\}$$

$$P^T[T_v < \tau] \leq \frac{\lambda - 1}{\lambda^n - 1}$$



$$P^{\tau}(X \text{ hits level } n \text{ before } \tau)$$

$$= P^{\tau}\Big(\bigcup_{v:|v|=n} \{T_v < \tau\}\Big)$$

$$\leq \sum_{v:|v|=n} P^{\tau}(T_v < \tau)$$

$1 < m < \lambda$

$$\leq Z_n \cdot \frac{\lambda - 1}{\lambda^n - 1}$$

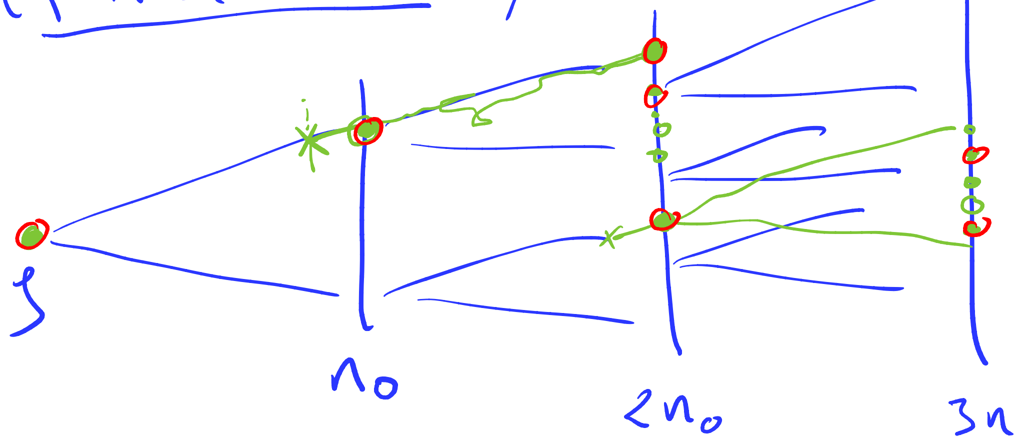
$\mathbf{E}[\,\cdot\,]$

$$\mathbb{P}(X \text{ hits level } n \text{ before } \tau) \leq \frac{m^n}{\lambda^n - 1} \cdot (\lambda - 1)$$

$$\leq C \cdot \left(\frac{m}{\lambda}\right)^n \text{ summable.}$$

By Borel-Cantelli Lemma X returns to ς almost surely.

(1) Transience, $0 < \lambda < m$, $m > 1$.



Assume that $\tau < +\infty$ a.s. (i.e. X recurrent).

This would imply $T_\gamma < +\infty$ a.s. & $H_v < +\infty$ a.s. $\forall v \in T$.

Besides, starting from T_v and up to H_v, the random behaves like a new indep. λ-biased RW $(X_k)_{1 \le k \le \tau}$

Choose n_0 s.t.

$$m^n \left(\frac{\lambda - 1}{\lambda^{n+1} - 1} \right) > 1, \quad \forall n \geq n_0 .$$

For a vertex v, with $|v| = (k-1)n_0$, define x_v the number of sites at level $k \cdot n_0$ visited between times T_v & H_v: $(x_v)_{v:|v|=(k-1)n_0}$ is an i.i.d. family.

We will add a parent $\overleftarrow{\rho}$ to the root ρ

$\Rightarrow T_{\overleftarrow{\rho}} < +\infty$ a.s.

Follow the coloring scheme where:

- ρ is colored green
- For any vertex γ, $|\gamma| = k \cdot n_0$, we colour γ green if: 1) the ancestor of γ at generation $(k-1) \cdot n_0$ is green, and 2) γ is hit between time T_v & H_v.

This gives us a branching process with offspring dist X_v.

$\boxed{\text{Assume } \lambda \neq 1}$

$$E^{T}[X_v] = \#\{\text{children of } v \text{ at generation } k \cdot n_0\} \times \frac{\lambda - 1}{\lambda^{n_0+1} - 1}$$

$$\mathbb{E}[x_v] \geq m^{n_0} \times \frac{\lambda - 1}{\lambda^{n_0+1} - 1} > 1.$$

$\uparrow$ def of n_0.

$\Rightarrow$ The colouring scheme is a super-critical branching process.

$\Rightarrow (X_n)$ is transient.

#

Transient if $m > 1$, $0 < \lambda < m$

Th (R. Lyons, Pemantle, & Y. Peres) Let $0<\lambda<m$.

The speed $\lim_{n\to+\infty} \frac{|X_n|}{n} = v$ exists a.s. upon non-extinction. Moreover

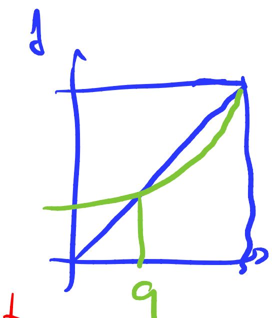
(i) $v>0$ if $f'(q)<\lambda<m$

(ii) $v=0$ if $0<\lambda<f'(q)$

f: generating fct° of L

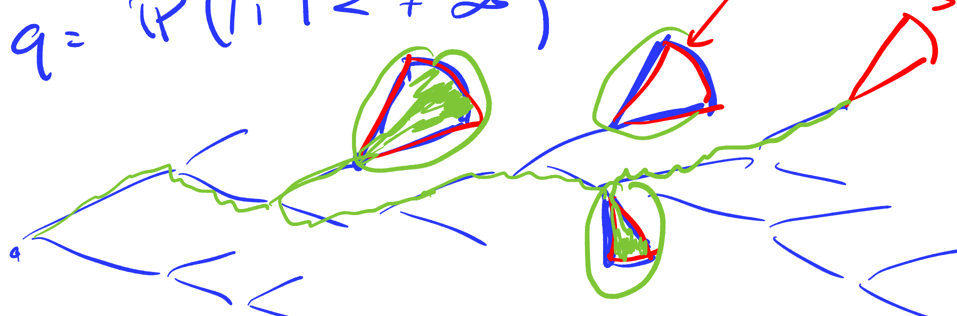
$q = \mathbb{P}(|T|<+\infty)$

Think: $f'(q)<1$



finite sub-trees

$=$ Traps.



We will distinguish $\lambda > 1$ (bias to the root) and $\lambda < 1$ (bias towards the bottom of the tree)

We will not consider $\lambda = 1$ (obtained by adapting the case $f'(q) < \lambda < 1$.

I) The case $1 < \lambda < m$

1) The range grows linearly in time

↳ vertices visited by the walk.

Define the Green function of X on $\mathcal{T}$

$$G(x,y) = \sum_{i=0}^{+\infty} P_x^{\mathcal{T}}(X_i = y)$$

← average number of visits to y starting from x.

$d(x) = \#$ children of x.

Prop: Fix $\lambda > 1$ and $\mathcal{T}$ s.t. X is transient

$\forall x \in \mathcal{T}$,

$$G(x,x) \leq \frac{d(x)+\lambda}{\lambda - 1} \cdot G(\varrho,\varrho).$$

Proof: $\hat{G}(x,x) = \sum_{i=0}^{+\infty} P_x^{\mathcal{T}}\left(X_i = x,\ i < T_\varrho \right)$

RW killed at ϱ.

$$f(x,y) = P_x^{\mathcal{T}}\left(\exists n > 0 : X_n = y \right)$$

$$\hat{f}(x,y) = P_x^{\mathcal{T}}\left(\exists n > 0 : X_n = y,\ n < T_\varrho \right).$$

$$G(x,x) \leq \hat{G}(x,x) + f(x,\varrho) \times f(\varrho,x)\, G(x,x)$$

$$\Rightarrow \quad G(x,x) \leq \frac{\hat{G}(x,x)}{1 - f(\varrho,\varrho)} = \hat{G}(x,x) \times G(\varrho,\varrho).$$

$f(\varrho,\varrho) < 1$ $\quad \sim$ Geom

$\overleftarrow{x}$ parent of x.

$$1-\hat{g}(x,x) \geq \frac{\lambda}{\lambda+d(x)}\left(1-g(\overleftarrow{x},x)\right)$$

Gambler's ruin: $\hat{g}(\overleftarrow{x},x) \leq \frac{1}{\lambda}$

$$\Rightarrow \quad 1-\hat{g}(x,x) \geq \frac{\lambda}{\lambda+d(x)} \times \frac{\lambda-1}{\lambda} = \frac{\lambda-1}{\lambda+d(x)}.$$

As $\hat{G}(x,x) = \dfrac{1}{1-\hat{g}(x,x)}$ (E of Geom)

$$G(x,x) \leq \frac{\lambda+d(x)}{\lambda-1}\, G(\rho,\rho).$$

#

Prop: $\lambda > 1$. Γ s.t. X is transient. $\forall n \geq 1$,

$$\frac{E^{\Gamma}[R_n]}{n} \geq \frac{1}{n} + \frac{\lambda - 1}{2\lambda G(\rho,\rho) + \lambda - 1}$$

where $R_n = |\{x \in \Gamma : T_x \leq n\}|$.

Proof: $\forall k \leq n$,

$$P^{\Gamma}(\forall j \in (k,n],\ X_j \neq X_k \mid X_k) \geq \frac{1}{G(X_k, X_k)}$$

Note that:

$$E^{\hat{\top}}[R_n] = 1 + E^{\hat{\top}}\left[\sum_{k=0}^{n-1} \mathbb{1}_{\{\forall j \in [k,n],\, X_j \neq X_k\}}\right]$$

Towerprop.

$$\geq 1 + E^{\hat{\top}}\left[\sum_{k=0}^{n-1} \frac{1}{G(X_k, X_k)}\right].$$

$$E^{\top}[R_n] \geq 1 + \frac{\lambda - 1}{G(\S,\S)} E^{\hat{\top}}\left[\sum_{k=0}^{n-1} \frac{1}{d(X_k)+\lambda}\right].$$

This bound is good for small, typical degrees.

Otherwise, for large degree, we can use

$$E^{\hat{\top}}\left[\,|X_{n+1}| - |X_n| \;\middle|\; X_n = x\right] = \frac{d(x)-\lambda}{d(x)+\lambda}.$$

$$E^{\hat{\top}}[R_n] \geq 1 + E^{\hat{\top}}[\,|X_n|\,] \geq 1 + E^{\hat{\top}}\left[\sum_{k=0}^{n-1} \frac{d(X_k)-\lambda}{d(X_k)+\lambda}\right]$$

$$\left(\frac{2\lambda G(\rho,\rho)}{\lambda-1}+1\right)E^{\tau}(R_n)$$

$$\geq \frac{2\lambda G(\rho,\rho)}{\lambda-1}+1+\underbrace{2\lambda E^{\tau}\left[\sum_{k=0}^{n-1}\frac{1}{d(X_k)+\lambda}\right]+E^{\tau}\left[\frac{d(X_n)-\lambda}{d(X_n)+\lambda}\right]}_{=n}$$

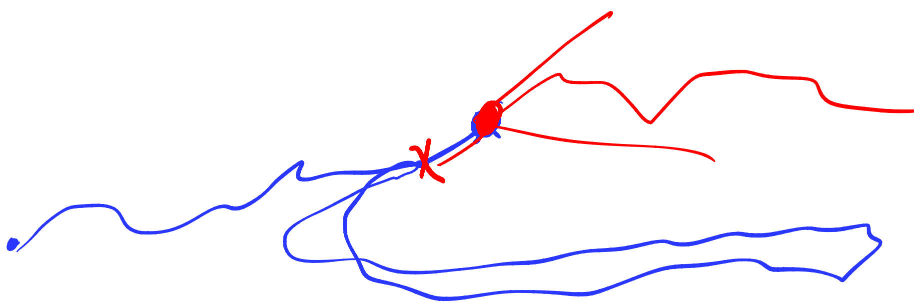
$$\Rightarrow E^{\tau}[R_n] \geq 1+\frac{\lambda-1}{2\lambda G(\rho,\rho)+\lambda-1}\times n.$$

#

To obtain the limit, we will consider:

Fresh epochs: $n \geq 0$ s.t. $X_k \neq X_n$, $\forall k < n$

Regeneration epoch: if, moreover,
$X_k \neq \underbrace{X_{n-1}}_{= \overleftarrow{X}_n}$, $\forall k > n$.



Consider T, add a parent ξ to ρ to obtain T'.

augmented tree

$\gamma(\tau) = P_{\varsigma}^{\tau'}\left[T_{\varsigma}^{\leftarrow} = +\infty\right].$

$\mathbb{P}_{non} = \mathbb{P}\left[\,\cdot \;\middle|\; |\tau| = +\infty\right].$

Lemma: Let A be a measurable set of infinite trees.
Let $\mathcal{F}_n$ be the σ-field generated by $\{X_i \neq X_j\}$, $0 \leq i < j \leq n$. Let α be an $(\mathcal{F}_n)$-stopping time such that α is a fresh epoch. τ^{α}: subtree at X_{α}.
Then

$$\mathbb{P}_{non}\left[\tau^{\alpha} \in A \,\middle|\, \mathcal{F}_{\alpha}\right] = \mathbb{P}_{non}\left[\tau \in A\right].$$

Proof: left as exercise.

Lemma: $\exists$ infinitely many regeneration epochs.

Proof:

$$\lim_{k \to +\infty} \mathbb{P}_{non}\left[\exists \text{ regeneration} \geq N \,\middle|\, \tilde{\mathcal{F}}_{N+k}\right]$$

$$\geq \liminf \mathbb{P}_{non}\left[\exists \text{ reg.} \geq N+k \,\middle|\, \tilde{\mathcal{F}}_{N+k}\right]$$

$$\geq \mathbb{E}_{non}\left[\gamma(T)\right]$$

prob that ξ is a regen. epoch

#