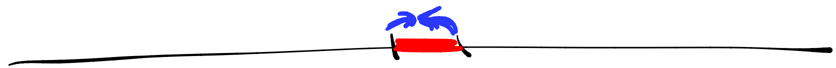
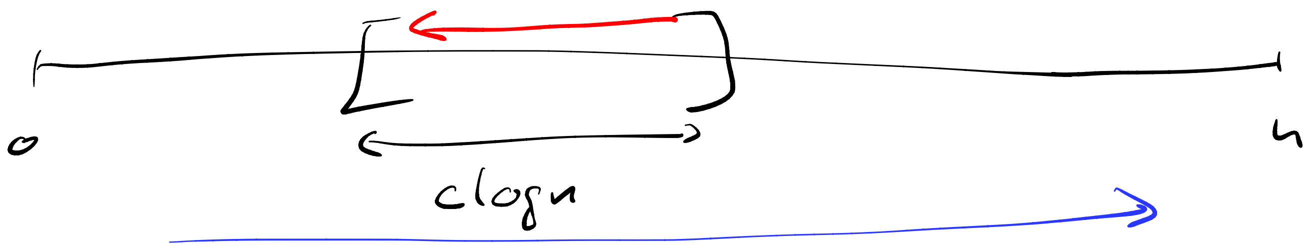
Lecture 5

Past weeks: RWRE in $d=1$.

* Trapping phenomena — **local traps**



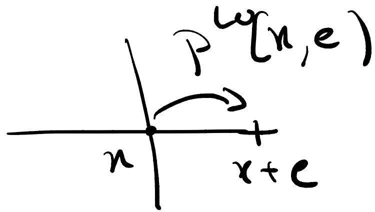
(non-local) **spread traps**

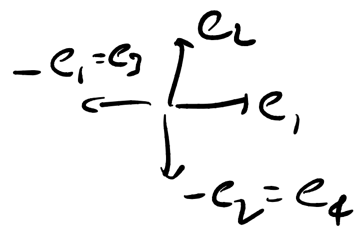


Even if the environment is uniformly elliptic. $(P_n^{\omega} \in [\kappa, 1-\kappa])$

RWRE in $\mathbb{Z}^d$, $d \geq 2$

Basis $\mathcal{B} = (e_i)_{1 \leq i \leq d}$ of $\mathbb{Z}^d$





$e_{i+d} = -e_i$ for $i \in \{1, \dots, d\}$.

ω: Collection $(p^\omega(x,\cdot))_{x \in \mathbb{Z}}$ s.t., $\forall x \in \mathbb{Z}$,

$$\sum_{i=1}^{2d} p^\omega(x, e_i) = 1.$$

We assume $\boxed{(UE)}$: $\forall i \in \{1, \dots, 2d\}$, $p^\omega(x, e_i) \geq \kappa$ a.s., for some $\kappa > 0$ fixed.

We choose $(p^\omega(x,\cdot))_{x \in \mathbb{Z}^d}$ i.i.d. $\sim \mathbf{P}$

Keep the same notation:

P_0^ω: quenched distribution of $(X_n)_{n\geq 0}$ in ω.

$\mathbb{P}_0$: annealed law.

Fix some direction $\vec{\ell} \in \mathbb{S}^{d-1}$ (unit vector of $\mathbb{R}^d$).

and assume:

Directional transience:

$\forall \ell' \in \mathcal{N}(\ell)$, $\mathbb{P}_0\left(\lim_{n\to+\infty} X_n \cdot \vec{\ell'} = +\infty\right) = 1$ $\quad \boxed{(DT|\vec{\ell})}$

Conjecture (Sznitman, 2001) On $\mathbb{Z}^d$, $d \geq 2$,

$\boxed{(UE)} + \boxed{(T|\vec{\ell})} \Rightarrow$ ballisticity, i.e.

$$\liminf \frac{X_n \cdot \vec{\ell}}{n} > 0$$

$$\Rightarrow \frac{X_n}{n} \to v \neq \vec{0}$$

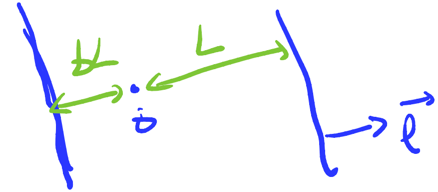
(deterministic.

Still open!!

$\tilde{c}(\log n)^d$, if $d \geq 2$.

Cst: $e^{\tilde{c}(\log n)^d} \gg n^a$, $\forall a > 0$.

*What is proved? Sznitman (2001) proved LLN under a stronger transience assumption, called Condition (T).



Def (Condition (T))

Fix $\vec{\ell} \in \mathbb{S}^{d-1}$, $b>0$, $L>0$.

Define the slab $U_{\vec{\ell},b,L} := \{x \in \mathbb{Z}^d : -bL < x\cdot\vec{\ell} < L\}$

Exit time $T_{U_{\vec{\ell},b,L}} = \inf\{n \ge 0 : X_n \notin U_{\vec{\ell},b,L}\}$.

* Condition (T)$|\vec{\ell}$ is satisfied if, $\forall \vec{\ell'} \in \mathcal{N}(\ell)$, $\forall b>0$,

$$\limsup_{L\to+\infty} \frac{1}{L} \log \mathbb{P}_0\left(X_{T_{U_{\vec{\ell'},b,L}}} \cdot \vec{\ell'} < 0\right) < 0$$

L large: $\mathbb{P}_0(\cdots < 0) \le e^{-cL}$

Sznitman proved:
(UE) + Condition (T) $\Rightarrow$ LLN
$\frac{X_n}{n} \to v \neq \vec{0}$. $\mathbb{P}_0$-a.s.,

Over the years, Condition (T) got weakened.

Sznitman: Cond. $(T)_\gamma$, (T')

$\hookrightarrow e^{-cL^\gamma}$ $\hookrightarrow \underbrace{\dots \leq e^{-cL^\gamma}}$, $\forall \gamma \in (0,1)$

Conjecture $(T) \Leftrightarrow (T') \Leftrightarrow (T)_\gamma$ for some $\gamma \in (0,1)$.

N. Berger, Drewitz & Ramirez ('14?)

Cond (P) $\left[\dots \leq \frac{1}{L^M}\right] \Rightarrow (T')$

In 2020, Guerra & Ramirez finally proved

$$(T) \Longleftrightarrow (T').$$

Conjecture still open. (HARD).

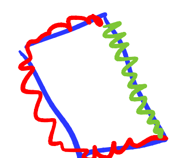
$$(UE)_K + (DT|\ell') \Longrightarrow LLN.$$

~~not~~ optimal

needed "optimal".

local requirement to avoid local traps

Global condition on transience

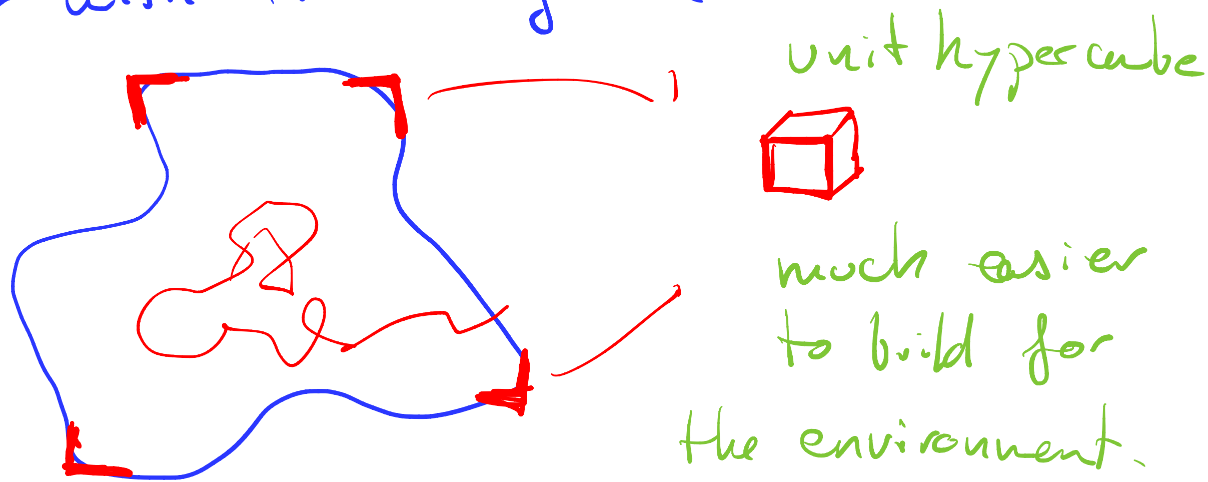


effective criterion.

* **Q°**: What is the minimal local requirement to ensure ballisticity / LLN ? (Assuming cond. (T)).

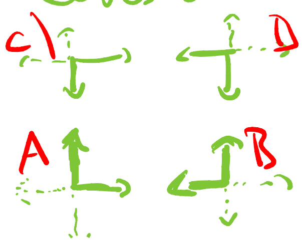
* **Q°**: Dropping (VE), what would be the geometry of a trap?

In a paper with A. Fribergh (2016), we argued.



Conj: A typical trap is included in a unit hypercube.

Remark: Considering 1 edge is not enough:



$\bullet\!\rightarrow\;\leftarrow\!\bullet$ not allowed.

$p^\omega(0,e_3) = p^\omega(0,e_4)$

A) $\mathbb{P}\left(p^\omega(0,e_1) = p^\omega(0,e_2) \geq y\right) = \frac{c}{4}\, y^{-\gamma}, \quad \gamma \in (0,1)$

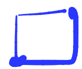
B) $\mathbb{P}\left(p^\omega(0,e_3) = p^\omega(0,e_2) \geq y\right) = \dots$

C) e_1 $\quad$ e_4

D) e_3 $\quad$ e_4

$$E[T_c] = +\infty \implies \frac{X_n}{n} = \vec{0}.$$

[unit square as (0,0) as bottom-left corner



(0,0)

No traps on single edges.

But $E[T_{\{(0,0),(0,1)\}}] < +\infty$.

<u>Conj</u>: Let H a unit hypercube of $\mathbb{Z}^d$ containing 0,

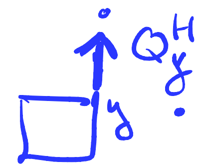
[then if $\max_{x \in H} E_x[T_H^{exit}] < \infty$ then the LLN holds under condition (T).

Prop (K., Fribergh, '14)

[If $\exists x \in H$ s.t. $E_x[T_H^{exit}] = +\infty$ then $\frac{X_n}{n} \to \vec{0}$.

To support the conjecture we proved another sufficient condition, more restrictive, more technical but close in spirit.

Let's introduce this condition:



H: unit hypercube

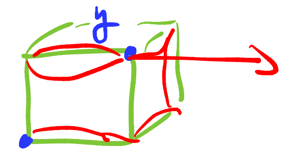
$$\forall y \in H, \quad Q_y^H = \max_{z \in \partial_y^+ H} p^\omega(y, z-y)$$

$$\partial_y^+ H = \{z \notin H : |z-y| = 1\}.$$

$$\forall x,y \in H, \quad \tilde{Q}_{x,y}^H = P_x^\omega[T_H^{exit} < T_x^+, X_{T_H^{exit}} \in \partial_y H]$$

[return time

$\tilde{Q}^H_{n,y}$



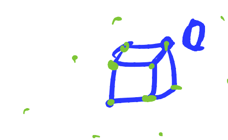
Define $\mathcal{H}_x = \left\{ x + \sum_{i=1}^{d} \varepsilon_i e_i,\ \varepsilon_j \in \{0,1\},\ \forall i \in \{1,\dots,d\} \right\}$.

$$\overline{\mathcal{H}}_0 = \{ \mathcal{H}_x,\ x \in \mathbb{Z}^d,\ 0 \in \mathcal{H}_x \}$$

We need to define a "Markovian hypercube"

<u>Idea</u>: Construct a hypercube (random) in a Markovian way, depending the environment we explore:

$h(\omega) = \{ f_0(\omega), \dots, f_{2^d-1}(\omega) \}$ constructed recursively as follows:



(1) $f_0(\omega) = 0_{\mathbb{Z}^d}$ $\mathbb{P}$-a.s.

(2) $\forall i \geq 0$, $f_{i+1}(\omega)$ is measurable w.r.t.

$\{p^\omega(x,\cdot),\ x \in \{f_0(\omega), \dots, f_i(\omega)\}\}$.

(3) $\forall i \geq 0$, $f_{i+1}(\omega) \in \partial \{f_0(\omega), \dots, f_i(\omega)\}$.

(4) $\forall i \geq 0$, $\{f_0(\omega), \dots, f_{i+1}(\omega)\}$ is contained in a unit hypercube.

Call $x_0(\omega)$ the vertex such that $h(\omega) = H_{x_0(\omega)}$.

["root", "bottom-left"

A marked Markovian hypercube is a pair $(h(\omega), (\alpha_x(\omega))_{x \in H_0})$ s.t.

* $h(\omega)$ is a Markovian hypercube

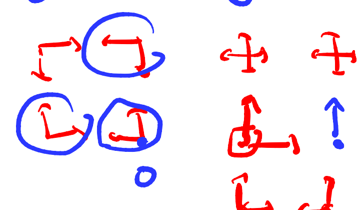
* $\forall x \in \mathcal{H}_0$, $\alpha_x(\omega) \geq 0$ a.s.
* $\forall x \in \mathcal{H}_0$, $\alpha_x(\omega)$ is measurable w.r.t. $\{p^\omega(y,\cdot),\ y \in h(\omega)\}$.

<u>Rk</u>: $\left(h(\omega), (\alpha_x(\omega))_{x\in\mathcal{H}_0}\right) \perp\!\!\!\perp \{p^\omega(y,\cdot),\ y \notin h(\omega)\}$.

<u>Definition</u>: Condition (K) is satisfied if

(1) $\forall x \in \mathcal{H}_0$, $\exists\, \gamma_x \geq 0$ s.t.

$$\mathbb{E}\left[\left(Q_x^{\mathcal{H}_0}\right)^{-\gamma_x}\right] < +\infty$$



(2) $\exists$ a marked Markovian hypercube s.t.

$$\mathbb{E}\left[\prod_{x\in\mathcal{H}_0}\left(\tilde{Q}^{h(\omega)}_{0,\,x_0(\omega)+x}\right)^{-\alpha_x(\omega)}\right] < +\infty,$$

(3) $\exists\, \varepsilon > 0$ s.t.

$$\sum_{x \in \mathcal{H}_0} \left(\gamma_x \wedge \alpha_x(\omega) \right) \geq \underline{1 + \varepsilon} \quad \mathbb{P}\text{-a.s.}$$

<u>Th</u>: Cond (T) + Cond (K) $\Rightarrow$ <u>LLN</u>.

<u>Idea</u>: Close in spirit to $\mathbb{E}_x\left[\left(T^{exit}_{\mathcal{H}_0}\right)^{1+\varepsilon} \right] < \infty$, $\forall x \in \mathcal{H}_0$.

$$\shortparallel$$

$$\mathbf{E}\left[E^{\omega}_x\left[\left(T^{exit}_{\mathcal{H}_0}\right)^{1+\varepsilon} \right] \right].$$

$\boxed{(1)}$ If $\mathbb{E}_x\left[T^{exit}_{\mathcal{H}_0} \right] < \infty$, $\forall x \in \mathcal{H}_0$, then

$$\mathbf{E}\left[\min_{y \in \mathcal{H}_0} \frac{1}{Q_y^{\mathcal{H}_0}} \right] < +\infty$$

under the quenched measure,
bound $T^{exit}_{\mathcal{H}_0}$ by a $\mathcal{G}eo\left(\max_{y\in\mathcal{H}_0} Q_y^{\mathcal{H}_0}\right)$.

Hence (1) requires that

$$E\left[\prod_{x\in\mathcal{H}_0}\left(\frac{1}{Q_x^{\mathcal{H}_0}}\right)^{\gamma_x}\right] < \infty \ \& \ \sum \gamma_x \geq 1+\varepsilon.$$

$$\overset{(*)}{\Longrightarrow} E\left[\left(\min_{x\in\mathcal{H}_0}\frac{1}{Q_x^{\mathcal{H}_0}}\right)^{1+\varepsilon}\right] < +\infty$$

$\boxed{(2)}$ If $E_x\left[T^{exit}_{\mathcal{H}_0}\right] < \infty$, $\forall x \in \mathcal{H}_0$ then

$$E\left[\min_{y\in H}\frac{1}{\tilde{Q}^{H}_{0,y}}\right] < +\infty, \text{ for } H \text{ s.t. } 0\in H$$

Turn the minimum to a product:

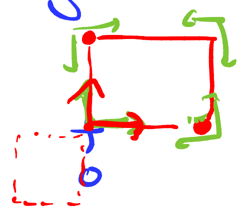
$$E\Big[\prod_{y\in H} \big(\tilde{Q}^{H}_{0,y}\big)^{-\alpha_y}\Big] < +\infty$$

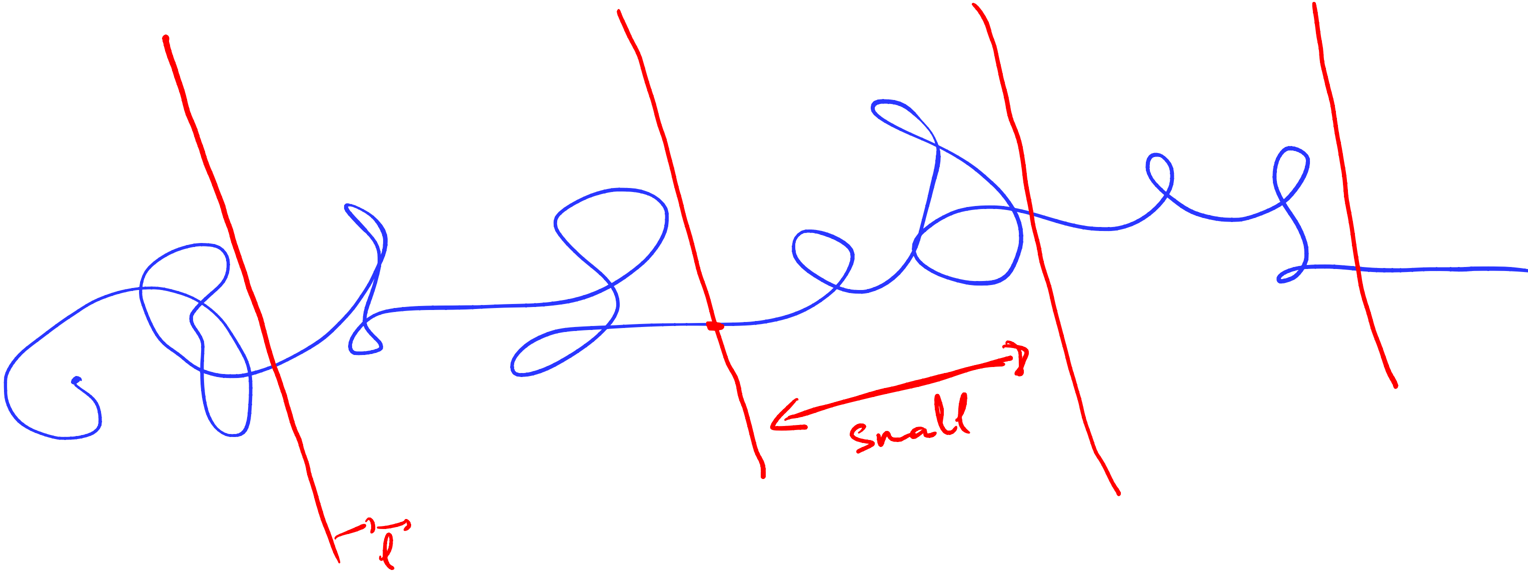
$$\overset{(**)}{\Longrightarrow} \quad E\Big[\Big(\min_{y\in H} \frac{1}{\tilde{Q}^{H}_{0,y}}\Big)^{1+\varepsilon}\Big] < \infty, \quad \sum \alpha_y \geq 1+\varepsilon.$$

For the conditions in (*) & (**), above to equivalent we have to assume "smooth" tails, e.g. polynomial.

$$(3) \quad \Big[\underbrace{\sum \big(\alpha_n(\omega) \wedge \gamma_n\big) \geq 1+\varepsilon}\Big]$$

It is needed: It prevents from choosing a useless Markovian hypercube.





regeneration times / slabs.

we need to prove that

* size of the slab is small ((Condition (T)))

* time spent inside is small


clog(n)
n