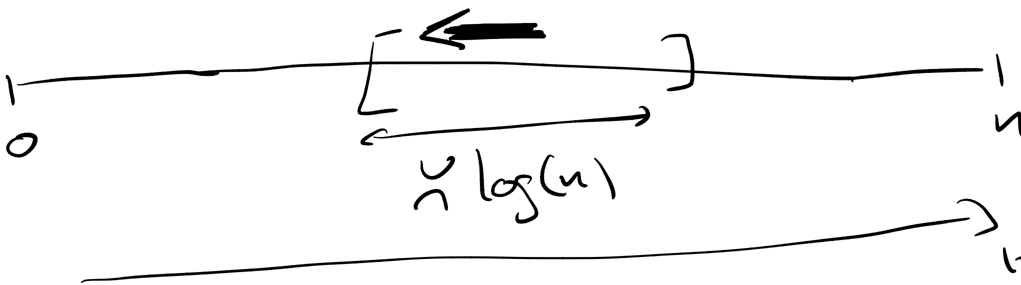
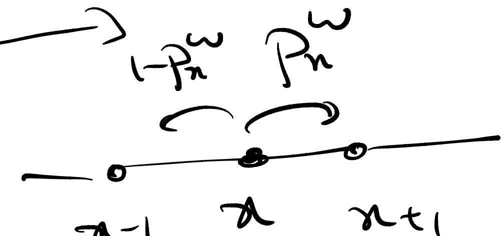
Lecture 4

RWRE, $d=1$. $(p_x^\omega)_{x\in\mathbb{Z}}$ i.i.d., $p_0^\omega \in [\kappa, 1-\kappa]$ $\mathbb{P}$-a.s.

uniformly elliptic.



$$\rho_x^\omega = \frac{1-p_x^\omega}{p_x^\omega},$$



Let's prove part B of Solomon's Theorem

↪ Use the environment seen from the particle.

$\Omega = [\kappa, 1-\kappa]^{\mathbb{Z}}$, with $\kappa \in (0, \frac{1}{2})$, endowed with canonical product σ-field $\mathcal{B}$.

* For $\omega \in \Omega$, we define, $\forall x \in \mathbb{Z}$

$$\begin{cases} \omega(x,1) = p_x^\omega \\ \omega(x,-1) = 1 - p_x^\omega \end{cases}$$

t_x : canonical translation on Ω

$t_y p_x^\omega = p_{x+y}^\omega$

Preserves $\mathbb{P}$

Note $(\Omega, \mathcal{B}, (t_x)_{x\in\mathbb{Z}}, \mathbb{P})$ is ergodic

($\forall A$ s.t. $t_x^{-1} A = A$ then $\mathbb{P}(A) \in \{0,1\}$)

* Define $\overline{\omega}_n = t_{X_n}\omega$, $\forall n \geq 0$, Ω-valued process

Ex: Prove $(\overline{\omega}_n)_{n\geq 0}$ under P_0^ω, $\omega \in \Omega$, is a Markov chain with state space Ω and translation kernel

$R f(\omega) = p_0^\omega \, f \circ t_1(\omega) + (1 - p_0^\omega) \, f \circ t_{-1}(\omega)$,

$\forall f$ bounded, measurable on Ω,
and initial law δ_ω.

$\rho_x = \frac{\omega(x,-1)}{\omega(x,+1)} = \frac{1-p_x^\omega}{p_x^\omega}$

1) Assume $E[\rho] < 1$ $\left(\Rightarrow E[\ln(\rho)] < 0\right)$.

* Find an invariant probability $\mathbb{Q} = f\,\mathbf{P}$ for R

that is $\int Rh \cdot d\mathbb{Q} = \int h \cdot d\mathbb{Q}$, $\forall h$ bounded measurable

$\prod_{z=1}^{y} \rho_z$

Define $\mathbb{Q} = f(\omega)P$, with:

$$f(\omega) = \frac{1 - E[\rho]}{1 + E[\rho]} \left(1 + \rho_0^\omega\right)\left(\sum_{y \geq 0} \Pi_{0,y}\right)$$

* $\mathbb{Q}$ is an invariant probab. measure for R.

Note that $f > 0$ for P-a.e. ω, and, left as exercise, $\int f(\omega)\, dP = 1$.

$$\int Rh \cdot dQ = \int f(\omega)\left(p_0^\omega\, h\circ t_1(\omega) + (1-p_0^\omega)\, h\circ t_{-1}(\omega)\right) dP$$

$$\overset{(1)}{=} \int h(\omega) \underbrace{\left(p_{-1}^\omega\, f\circ t_{-1}(\omega) + (1-p_1^\omega)\, f\circ t_1(\omega)\right)}_{\shortparallel} dP$$

$$\frac{1-E(\rho)}{1+E[\rho]}\left[\underbrace{p_{-1}^\omega(1+\rho_{-1}^\omega)}_{\underset{1}{\shortparallel}}\Big(\sum_{y\geq -1}\underbrace{\Pi_{-1,y}}_{\underset{}{\shortparallel}\ \prod_{j=0}^{y}\rho_j}\Big) + \underbrace{(1-p_1^\omega)(1+\rho_1^\omega)}_{\rho_1^\omega}\Big(\sum_{y\geq 1}\underbrace{\Pi_{1,y}}_{\prod_{j=2}^{y}\rho_j}\Big)\right]$$

$$= \frac{1-E[\rho]}{1+E[\rho]}\left(\left[1+\rho_0^\omega+\rho_0\rho_1^\omega+\dots\right) + \left(\rho_1^\omega+\rho_1\rho_2^\omega+\dots\right)\right]$$

$$= \frac{1-E[\rho]}{1+E[\rho]} \Big((1+\rho_0^\omega)(1+\rho_1^\omega+\rho_1^\omega\rho_2^\omega+\dots) \Big] = f(\omega)$$

$$(1) = \int h(\omega) f(\omega)\, d\mathbb{P} = \int h \cdot d\mathbb{Q}.$$

$\Longrightarrow$ $\mathbb{Q}$ invariant for R.

As $f>0$ $\mathbb{P}$-a.s., we have $\mathbb{Q} \sim \mathbb{P}$.

* $\mathbb{Q}$ is ergodic

Define $\tilde{\Omega} = \Omega^{\mathbb{N}}$: space of Ω-valued trajectories.

$\tilde{\theta}$: canonical shift on $\tilde{\Omega}$

$\tilde{\mathcal{S}}$: σ-field

$\tilde{P}_\omega$: law of the process associated to R started in ω. $\left(\begin{smallmatrix}(\tilde{\omega}_n)\\ \text{process}\end{smallmatrix}\right)$

We have to prove:

$\forall A \in \tilde{\mathcal{B}}$ invariant set $(\tilde{\Theta}^{-1}(A) = A)$

$$\tilde{\mathbb{P}}_{\mathbb{Q}}(A) = \int_{\Omega} \tilde{P}_{\omega}(A)\, d\mathbb{Q} \in \{0,1\}.$$

Define $\varphi(\omega) = \hat{P}_{\omega}(A)$, for A invariant set

Exercise: Prove that $(\varphi(\tilde{\omega}_n))_{n\geq 0}$ is a $\tilde{P}_{\mathbb{Q}}$-martingale

$$\tilde{E}_{\mathbb{Q}}\left[\mathbb{1}_A \,\middle|\, \tilde{\omega}_0, \dots, \tilde{\omega}_n\right] = \varphi(\tilde{\omega}_n)$$

$$\xrightarrow[n \to +\infty]{\tilde{P}_{\mathbb{Q}}\text{-a.s.}} \mathbb{1}_A \qquad \text{(Lévy's 0-1 law).}$$

Let us prove that $\begin{cases} \varphi(\omega) \in \{0,1\} & \mathbb{Q}\text{-a.s.} \\ R\varphi = \varphi & \mathbb{Q}\text{-a.s.} \end{cases}$

First, assume that $\exists\, 0<a<b<1$, s.t. $\mathbb{Q}[\varphi(\omega)\in[a,b]] > 0$, and not that

Birkhoff's ergodic Th. implies

$$\frac{1}{n}\sum_{n=0}^{n-1} \mathbb{1}_{\{\varphi(\tilde{\omega}_n)\in[a,b]\}} \xrightarrow{\tilde{P}_{\mathbb{Q}}\text{-a.s.}} \psi$$

$$\psi = \tilde{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{\varphi(\tilde{\omega}_0)\in[a,b]\}} \,\middle|\, \mathcal{I}\right]$$

invariant sets

Note that $\tilde{E}_{\mathbb{Q}}[\psi] = \tilde{P}_{\mathbb{Q}}[\varphi(\tilde{\omega}_0)\in[a,b]]$
$= \mathbb{Q}[\varphi(\omega)\in[a,b]] > 0.$

This contradicts $\varphi(\bar{\omega}_n) \longrightarrow \mathbb{1}_A$ $\tilde{P}_{\mathbb{Q}}$ a.s..

$\Rightarrow \varphi(\omega) \in \{0,1\}$ $\mathbb{Q}$-a.s.

The second point is obtained by the martingale property $\varphi = R\varphi$ $\mathbb{Q}$-a.s.

Now, let's prove $\underbrace{\int \varphi(\omega) d\mathbb{Q}}_{\mathbb{Q}[\varphi(\omega)=1]} \in \{0,1\}$.

This will imply:

$$\tilde{P}_{\mathbb{Q}}(A) = \int_{\Omega} \tilde{P}_{\omega}(A) d\mathbb{Q} = \int_{\Omega} \varphi(\omega) d\mathbb{Q} \in \{0,1\}.$$

To this end, note that $\mathbb{Q}$ - a.s., hence $\mathbb{P}$-a.s.,

$$\varphi(\omega) \geq R\varphi(\omega) \geq K\varphi\circ t_{-1}(\omega) + K\varphi\circ t_1(\omega)$$

As $\varphi(\omega) \in \{0,1\}$, one can prove that

$\varphi(\omega) = \varphi\circ t_n(\omega), \forall n \in \mathbb{Z}$, $\mathbb{P}$-a.s. $\Big|$ $\varphi = \mathbb{1}_B$, B invariant for (t_n)

As $\mathbb{P}$ is ergodic for (t_x), we have

$$\mathbb{P}(\varphi(\omega)=1) \in \{0,1\} \iff \mathbb{Q}[\varphi(\omega)=1] \in \{0,1\}.$$

Finally, one can prove that $\mathbb{Q}$ is unique

(use Birkhoff's ergodic Th with fct° g

$$\mathbb{E}_0\Big[\frac{1}{n}\sum_{k=0}^{n-1} g(\tilde{\omega}_k)\Big] \longrightarrow \int g\,d\mathbb{Q}$$

<u>To sum up</u>: We know that $\mathbb{Q}$ is an ergodic invariant probab. measure (unique) for R.

Define the local drift: $d(x,\omega) = p_x^\omega - (1-p_x^\omega)$
$$= 2p_x^\omega - 1$$
$$= E_x^\omega[X_1 - X_0]$$

By Birkhoff's ergodic Th, for $\mathbb{Q}$-a.e. ω

$$\frac{1}{n}\sum_{k=0}^{n-1} d(X_k,\omega) = \frac{1}{n}\sum_{k=0}^{n-1} d(0, t_{X_k}\omega) \xrightarrow{P_0^\omega\text{-a.s.}} \int d(0,\omega)\, d\mathbb{Q}$$

This also holds for P-a.e. ω.

<u>Ex</u>: Prove that $\int d(0,\omega)\, d\mathbb{Q} = \dfrac{1-E[\rho]}{1+E[\rho]}$.

To conclude, define the martingale (with bounded increments)

$$M_n = X_n - X_0 - \sum_{k=0}^{n-1} d(X_k, \omega). \quad (\text{under } P_0^\omega).$$

By Azuma's inequality:

$$P_0^\omega\left[\pm M_n > \lambda \cdot 2\sqrt{n}\right] \leq e^{-\frac{\lambda^2}{2}}.$$

Choose $\lambda = n^{1/4}$ and apply Borel-Cantelli Lemma

$$\frac{M_n}{n} \longrightarrow 0 \quad P_0^\omega\text{-a.s., for } P\text{-a.e. } \omega.$$

This finally implies:

$$\frac{X_n}{n} \longrightarrow \int d(0,\omega)\, dQ = \frac{1-E[S]}{1+E[S]} \quad P_0^\omega\text{-a.s.}$$

2) $E[\rho^{-1}] < 1$: Same arguments.

3) Assume $\dfrac{1}{E[\rho^{-1}]} \le 1 \le E[\rho]$.

We'll use a coupling.

We say that $\omega \ll \omega'$ if, $\forall x \in \mathbb{Z}$, $P_x^{\omega} \le P_x^{\omega'}$.

For two such environments, define a Markov chain on $\mathbb{Z}^2$ with law $\overline{P}^{\omega,\omega'}_{(0,0)}$ s.t. $(X_0, X_0') = (0,0)$

$$\begin{cases} \text{under } \overline{P}^{\omega,\omega'}_{(0,0)},\ (X_n) \text{ has the same law as under } P_0^{\omega}; \\ \text{———},\ (X_n') \text{ ——— } P_0^{\omega'} \\ \overline{P}^{\omega,\omega'}_{(0,0)}\left(X_n \le X_n',\ \forall n \in \mathbb{N}\right) = 1. \end{cases}$$

$P_x^{\omega'} \geq P_x^{\omega}$

x

$= X_n = X_n'$

if $X_n' = X_n$: $X_{n+1} = 1 + X_n \Rightarrow X'_{n+1} = 1 + X'_n$ [w.p. P_x^{ω}]

$X'_{n+1} = X'_n - 1 \Rightarrow X_{n+1} = X_n - 1$ [w.p. $1 - P_x^{\omega'}$]

w.p. $1 - (P_x^{\omega} + 1 - P_x^{\omega'})$: $X_{n+1} = X_n - 1$, $X'_{n+1} = X'_n + 1$

When at different sites

X walks according to ω

X' ——— ω', independently.

Sample ω from P, and define ω' s.t., $\forall x \in \mathbb{Z}$,

$$P_x^{\omega'} = P_x^{\omega} \cdot (1-\eta) + \eta \cdot (1-k), \text{ for } \eta \in (0,1) \ (\text{recall } k \in (0, \tfrac{1}{2}))$$

We can choose η close to 1 so that $E[\rho'] < 1$.

Therefore for P-a.e. ω, $\overline{P}^{\omega,\omega'}_{(0,0)}$-a.s.

$$\limsup \frac{X_n}{n} \leq \limsup \frac{X_n'}{n} = \frac{1 - E[\rho']}{1 + E[\rho]},$$

As we can make the RHS vanish by taking η closer to 0, we have that, $\mathbb{P}_0$-a.s.

$$\limsup \frac{X_n}{n} \leq 0.$$

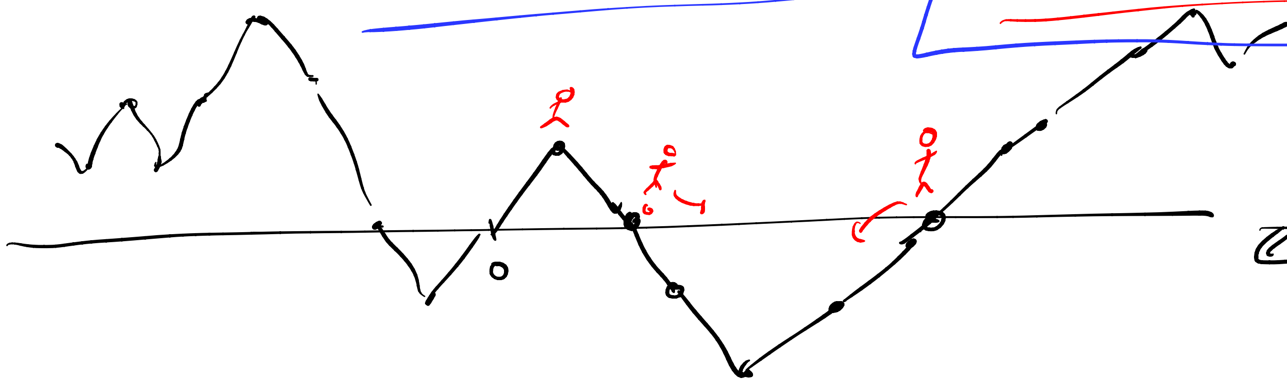
By the symmetric argument:

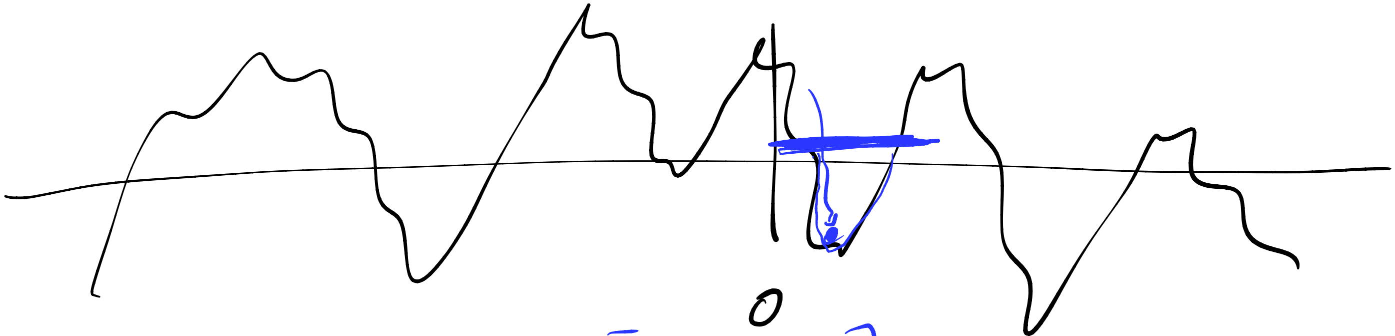
$$\liminf \frac{X_n}{n} \geq 0 \quad \mathbb{P}_0\text{-a.s.}$$

$$\Rightarrow \quad \frac{X_n}{n} \longrightarrow 0 \quad \mathbb{P}_0\text{-a.s.}$$

\# We can say more in the recurrent case: $\boxed{E[\log(\rho_0)] = 0.}$

Sinai's walk





Let $\sigma_p^2 = E\left[\log(\rho_0)^2\right]$

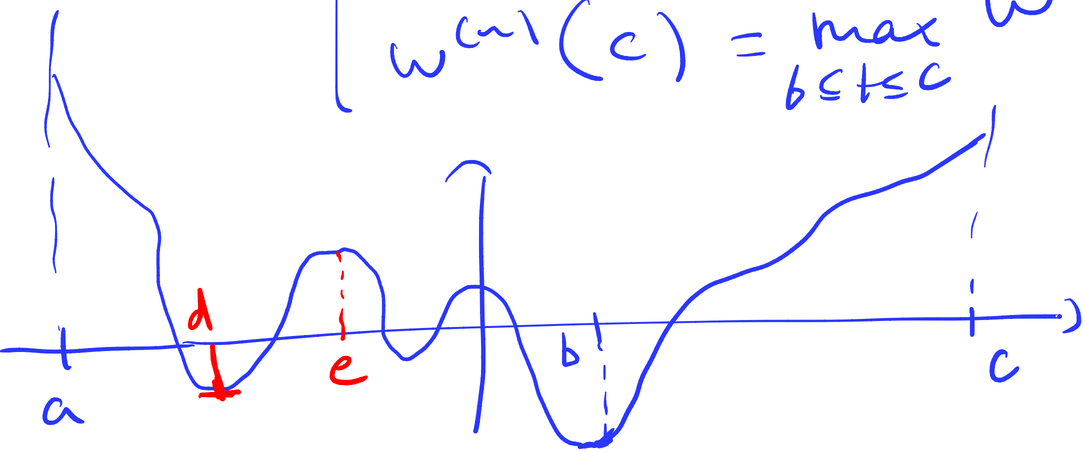
$$W^{(n)}(t) = \frac{1}{\log(n)} \sum_{i=0}^{\lfloor \log(n)^2 \cdot t \rfloor} \log(\rho_i) \times \mathrm{sign}(t), \quad t \in \mathbb{R}.$$

$\left(W^{(n)}(t)\right)$ converges weakly to $\left(\sigma_p B_t\right)_{t \in [0,1]}$

$B_\cdot$ = standard Brownian motion

Next, we call (a,b,c), with $a<b<c$,
a valley of $W^{(n)}$ if:

$$\begin{cases} W^{(n)}(b) = \min_{a \le t \le c} W^{(n)}(t) \\ W^{(n)}(a) = \max_{a \le t \le b} W^{(n)}(t) \\ W^{(n)}(c) = \max_{b \le t \le c} W^{(n)}(t) \end{cases}$$



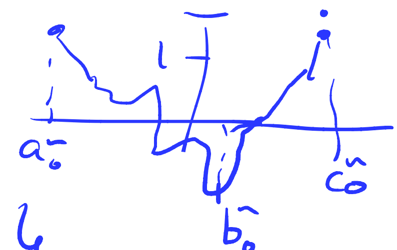
Depth of (a,b,c)

$$d(a,b,c) = \min\{W^{(n)}(a) - W^{(n)}(b),\ W^{(n)}(c) - W^{(n)}(b)\}$$

If $a<d<e<b$ are such that

$$W^{(n)}(e)-W^{(n)}(d)=\max_{a\le x<y\le b}\left(W^{(n)}(y)-W^{(n)}(x)\right)$$

Then (a,d,e) & (e,b,c) are valleys, obtained by a left refinement. Similarly, one can define right refinements.



Define:

$$c_0^n=\min\{t\ge 0: W^{(n)}(t)\ge 1\}$$

$$a_0^n=\max\{t\le 0: W^{(n)}(t)\ge 1\}$$

$$b_0^n: W^{(n)}(b_0^n)=\min_{a_0^n\le t\le c_0^{(n)}} W^{(n)}(t)$$

Proceed to a finite sequence of refinements to find the smallest valley $(\bar a_n, \bar b_n, \bar c_n)$ s.t.

$\bar a_n < 0 < \bar c_n$ and $d(\bar a_n, \bar b_n, \bar c_n) \geq 1$.

(Sinai)

Th: For any $\eta > 0$,

$$\mathbb{P}\left(\left| \frac{X_n}{(\log(n))^2} - \bar b_n \right| > \eta \right) \xrightarrow[n \to +\infty]{} 0$$

Kesten identified the limit in law of $\bar b_n$