Lecture 3

Last week: Considered RW in random conductances.

Heavy tails at 0 & $+\infty$

$\mathbb{P}(C(0,1) > t) = L_\infty(t)\, t^{-\alpha_\infty}$, $\mathbb{P}\left(\frac{1}{C(0,1)} > t\right) = L_0(t)\, t^{-\alpha_0}$,

L_∞, L_0 slowly varying, $\alpha_0, \alpha_\infty \in (0,1)$.

$\rightsquigarrow$ Convergence of the environment.

We coupled the limit environment with the discrete one:

$$\hat{c}(x, x+1) = G_\infty^{-1}\left(n^{\frac{1}{\alpha_\infty}}\left(S^{\alpha_\infty}\left(\frac{x+1}{n}\right) - S^{\alpha_\infty}\left(\frac{x}{n}\right)\right)\right)$$

$$G_\infty : \overline{\mathbb{P}}\left(S^{\alpha_\infty}(1) > G_\infty(y)\right) = \mathbb{P}\left(\bar{c}(0,1) > y\right)$$

$$\bar{c} \sim c(0,1) \mid c(0,1) \geq 1 .$$

We proved $\hat{c} \sim \bar{c}$.

$c(0,1) < 1$.

We can do the same for couple $\hat{r} \sim \bar{r}$ $\hat{\Downarrow}$

$\sim r(0,1) \mid r(0,1) > t$

Define: $\forall x \geq 0$,

$$\hat{c}(x, x+1) = \hat{c}(x^*, x^*+1)\,\mathbb{1}_{\{b_x = 1\}} + \left(\hat{r}(x-x^*, x-x^*+1)\right)^{-1}\mathbb{1}_{\{b_x=0\}}$$

$(b_x)_{x\in\mathbb{Z}}$ i.i.d. $\mathrm{Ber}\left(\mathbb{P}(c(a)\geq 1)\right)$

$$x^* = \sum_{j=0}^{x-1} \mathbb{1}_{\{b_j = 1\}}.$$

Similar for $x<0$.

Easy to prove $\hat{c} \sim c(0,1)$.

All of this implies:

$$S^{\alpha_0,(n)}(t) = \frac{1}{\underbrace{d_{n,0}}_{n^{\frac{1}{\alpha_0}}\,\simeq}}\ \underbrace{R([0, nt])}_{=\sum_{k=0}^{\lfloor nt\rfloor - 1} r(k,k+1)},\ t\geq 0$$

this converges in J_1 to $\left(S^{\alpha_0}(t)\right)_t$.

Similarly,

$$\frac{1}{\underbrace{2 d_{n,\infty}}_{\substack{\text{SS} \\ n^{\frac{1}{\alpha_\infty}}}}} \sum_{i \in [-K_n, K_n]} \delta_{\frac{i}{n}} \underbrace{\hat{c}(i)} \longrightarrow d\,\mathcal{S}^{\alpha_\infty} \cdot du \quad \text{as a finite measure.}$$

$$\hat{c}(i) = \hat{c}(i-1, i) + \hat{c}(i, i+1)$$

under the coupling

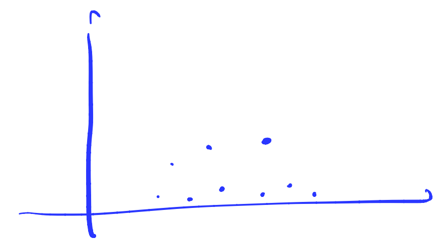
This will imply the GHP-convergence
Gromov-Hausdorff-Prokhorov

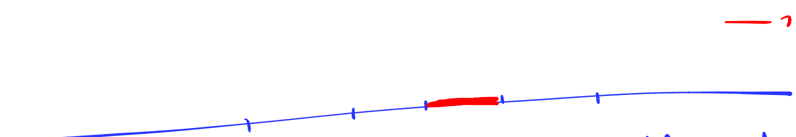
of $(\mathcal{X}_n, m_n, \mu, \beta_n, \Phi_n)$ to $(X, d, \mu, \mathcal{S}^{\alpha_0}(\beta), \phi)$

where
$$\begin{cases} \mathcal{X}_n = [-Kn, Kn] \cap \mathbb{Z} \\ m_n(a,b) = \dfrac{R(a,b)}{d_{n,0}} \\ \mu_n = \dfrac{1}{2d_{n,\infty}} \sum_{i \in \mathcal{X}_n} \delta_{i/n} \, \hat{c}(i) \\ \Phi_n(\cdot) = \dfrac{1}{n}(\cdot). \end{cases}$$

$\beta_n \to \beta$, $\hat{c}$ continuity point of S^{α_0}

$$\begin{cases} \mathcal{X} = S^{\alpha_0}([-K,K]) \\ d = \text{Euclidean} \\ \phi = (S^{\alpha_0})^{-1}(\cdot) \end{cases}$$



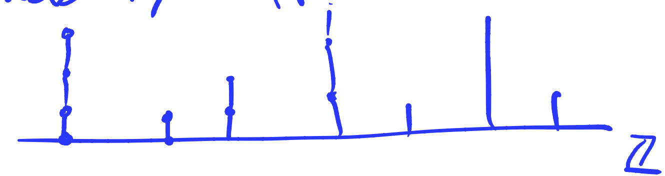


The RW remembers the traps. $\Rightarrow$ FIN

There are other examples of trap models where the random walk "forgets" the traps. That is the case of Bouchaud trap model or a RW in RC with heavy tails at $+\infty$, on $\mathbb{Z}^d$, $d \geq 3$, the limit will be a Fractional Kinetics → characteristic of trap models that forget the traps.

Toy example (see "randomly trapped RW")

RW on Comb graph:



Depending on the law of the length of a tooth, we can see either a BM or FK (or FIN).

(X_n) on comb graph. Observe $(X_n^{(1)})$
[coordinate along $\mathbb{Z}$.

Take length of a tooth to be

$$N \quad \text{s.t.} \quad \mathbb{P}(N=n) = \frac{1}{Z} n^{-1-\alpha},$$

If $\alpha > 1$ then we will BM in the limit.

If $\alpha \in (0,1)$ then the teeth can be long, and when exploring a long tooth, the RW will take a long time but it is not likely to hit the top of the tooth

$\rightsquigarrow$ the RW will scale to a Fractional Kinetics, defined by:

$(B_t)_t$ a Brownian motion.

$(V_t)_t$ α-stable subordinator $\perp\!\!\!\perp (B_t)$

$$\psi_t^\alpha = \inf\{s \geq 0 : V_s > t\}$$

$$FK_\alpha = \quad Z_t = B_{\psi_t^\alpha} \,, \quad t \geq 0.$$

$\perp\!\!\!\perp$ => forget the environment.

Rk: if we add a small drift towards the top of the teeth, we can observe FIN diffusion in the limit

This ends our discussion on local traps
small

There exist other trapping mechanisms, e.g. for RW in random environment. (RWRE)

$d = 1$.

RWRE in d=1

Env. $\omega = (p_x^\omega)_{x\in\mathbb{Z}}$ is i.i.d. with values in $[\kappa, 1-\kappa]$ for $\kappa > 0$ small.

this environment is called uniformly elliptic

prob. to jump for $x \to x \pm 1$ is lower-bounded.

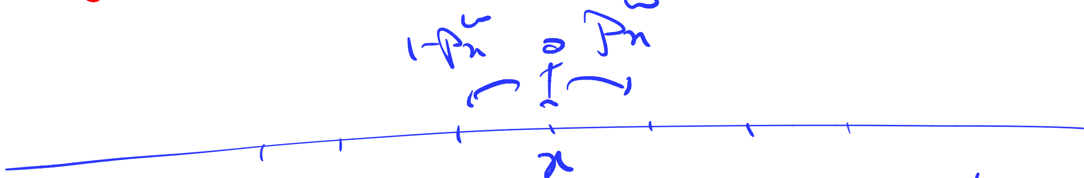
Given ω, $(X_n)_{n\geq 0}$ is a Markov Chain s.t.

$X_0 = 0$,

$$P^\omega(X_{n+1} = x \underset{-1}{+1} \mid X_n = x) = \underset{1-p_x^\omega}{p_x^\omega}, \quad \forall x \in \mathbb{Z}$$

P_0^ω : quenched law , $\mathbb{P}$: law ω.

$\mathbb{P}_0 = \mathbf{E}[P^\omega(\cdot)]$ annealed law. (X_n not a Markov chain under $\mathbb{P}$)



* Uniform ellipticity prevents local traps
  $\rightsquigarrow$ positive chance to escape quickly any finite region.

* It is still reversible   true only in $d=1$

Define, $\forall x \in \mathbb{Z}$, $\rho_x^\omega = \dfrac{1-p_x^\omega}{p_x^\omega}$.

$\rho_x^\omega = \frac{1-p_x^\omega}{p_x^\omega}$

Th (Solomon, '75)

A) 1) If $E[\log \rho_0^\omega] < 0$ then $\lim_{n\to+\infty} X_n = +\infty$ $\mathbb{P}_0$-a.s.

transience

2) If $E[\log \rho_0^\omega] > 0$ then $\lim_{n\to+\infty} X_n = -\infty$ $\mathbb{P}_0$-a.s.

recurrence 3) If $E[\log \rho_0^\omega] = 0$ then $\limsup X_n = +\infty$ & $\liminf X_n = -\infty$, $\mathbb{P}_0$-a.s.

B) $\frac{X_n}{n} \to v$ $\mathbb{P}_0$-a.s. where

1) If $E[\rho_0^\omega] < 1$ then $v = \frac{1-E[\rho]}{1+E[\rho]} > 0$;

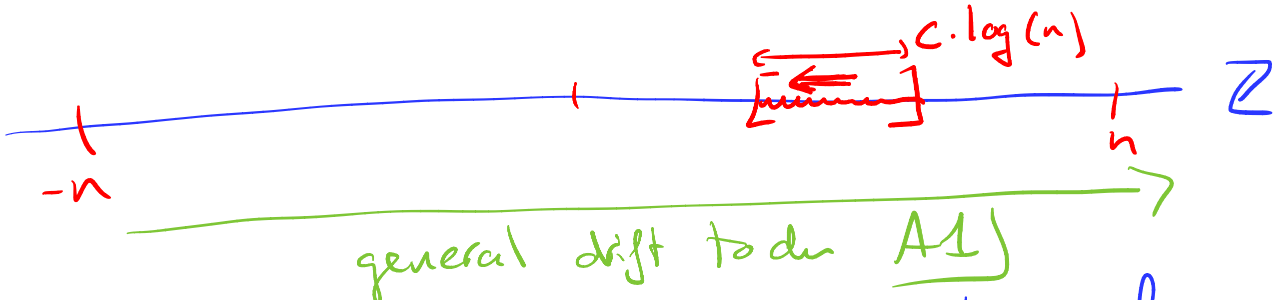
2) If $E[\rho_0^\omega] > 1$ then $v = \frac{E[\rho^{-1}]-1}{E[\rho^{-1}]+1} < 0$;

3) If $\frac{1}{E[\rho^{-1}]} \leq 1 \leq E[\rho]$ then $v = 0$.

Ex. of env. in A1) & B3), hence transient to $+\infty$ with $v=0$

$P_x^\omega = \frac{3}{4}$ w.p. $\frac{7}{12}$ & $P_x^\omega = \frac{1}{4}$ w.p. $\frac{5}{12}$.

Why are these walks trapped???



Prob. that an interval of size $c\log(n)$ have all vertices with $P_x^\omega = \frac{1}{4}$

$$\left(\frac{5}{12}\right)^{c\log(n)} = e^{-c'\log(n)} = \frac{1}{n^{c'}}$$

choose c so that $c' = 1$.

Hence, we are likely to see such an interval between 0 & n.

For the random walk to cross such an interval for the first time, it will a time

$$\exp(c'' \cdot c\log(n)) \gg n$$

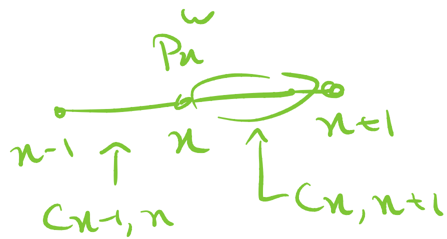
given by the law of the env.

Then it will take a time $\gg n$ for the RW to hit n.

We can rephrase this walk as a random walk in random conductances <u>but</u> not i.i.d.

$$\frac{c_{x,x+1}}{c_{x-1,x}+c_{x,x+1}}$$

<u>Idea:</u> $$\frac{1-p_x}{p_x} = \frac{c_{x,x+1}}{c_{x-1,x}} = \frac{r_{x-1,x}}{r_{x,x+1}}$$



Fix $\begin{cases} c_{0,1} = 1 \\ r_{0,1} = 1 \end{cases}$ then, $$\underset{x \geq 1}{r_{x-1,x}} = \prod_{k=1}^{x-1} \frac{1-p_k^\omega}{p_k^\omega} = e^{\sum_{k=1}^{x-1} \log(\rho_k^\omega)}$$

<u>$x<0$</u>: $$r_x = \prod_{k=x+1}^{0} \frac{p_k^\omega}{1-p_k^\omega}$$

$$R(0 \leftrightarrow x) = \sum_{i=1}^{x} r_i = \sum_{i=1}^{x} e^{\sum_{k=1}^{x-1} \log(\rho_k^\omega)}$$

Proof of A] (recurrence / transience).

For $x, y \in \mathbb{Z}$, $x \le y$, define

$$* \quad \Pi_{x,y} = \prod_{z=x+1}^{y} \rho_z^{\omega} \qquad (=1 \text{ if } y = x)$$

$$* \quad f_0(x,\omega) = \begin{cases} \overbrace{-\sum_{z=0}^{x-1} \overbrace{\Pi_{0,z}}^{= \frac{r_{z,z+1}}{r(0,1)}}}^{\frac{R(0 \leftrightarrow x)}{r(0,1)}} & \text{if } x \ge 0, \\ \underbrace{\sum_{z=x}^{-1} \left(\frac{1}{\Pi_{z,0}} \right)}_{\frac{R(x \leftrightarrow 0)}{r(-1,0)}} & \text{if } x < 0. \end{cases}$$

$$f_0(0,\omega) = 0, \quad f_0(1,\omega) = -1.$$

and $\quad p_x^\omega f_0(x+1,\omega) + (1-p_x^\omega) f_0(x-1,\omega) = f_0(x,\omega).\}$

$\forall z \in \mathbb{Z},\ z>0,$

$$\Pi_{0,z} = \prod_{y=1}^{z} \rho_y^\omega = \exp\left[\sum_{y=1}^{z} \log(\rho_y^\omega)\right]$$

LLN

$$\underset{z \sim +\infty}{\overset{P\text{-a.s.}}{=}} \exp\left[z \cdot \left(E[\log \rho_0^\omega] + o(1)\right)\right].$$

Similarly, for $z<0$

$$\Pi_{z,0} \underset{z\sim-\infty}{\overset{P\text{-a.s.}}{=}} \exp\left[z\left(E[\log(\rho)] + o(1)\right)\right].$$

Let use assume $\underline{A1}$: $E[\log(\rho_0^\omega)] < 0,$

hence $\begin{cases} \lim_{x \to +\infty} f_0(x,\omega) \text{ exists \& is in } (-\infty,-1] \\ \lim_{x \to -\infty} f_0(x,\omega) = +\infty. \end{cases}$

Moreover, by harmonicity of $f_0(\cdot,\omega)$, we have that $(f_0(X_n,\omega))_{n\geq 0}$ is a martingale w.r.t. $(\sigma(X_n))$ under P^ω, bounded below (under A1)

thus it converges <u>a.s.</u> to a <u>finite</u> limit by the Martingale Convergence theorem.

Here, the limit can <u>only</u> be $\underline{\lim_{x \to +\infty} f_0(x,\omega)}$.

This implies: for a.e. ω,

$$\lim X_n = +\infty \quad P_0^\omega\text{-a.s.}$$

This proves A1), A2) by symmetry

- Now assume A3) $E[\log(\rho_0)] = 0$.

Let us recall a Chung-Fuchs-type result:

<u>Lemma</u>: (Y_n) i.i.d., finite-valued r.v.'s,

<u>If</u> $\sum_{n\geq 1} \frac{1}{n} \mathbb{P}\left(\sum_{k=1}^{n} Y_k > 0\right) = \sum_{n\geq 0} \frac{1}{n} \mathbb{P}\left(\sum_{k=1}^{n} Y_k < 0\right) = +\infty$

<u>then</u> $\liminf Y_n = -\infty$ & $\limsup Y_n = +\infty$ a.s.

Here, for $Y_x = \log(\rho_0^\omega)$, this is easily proved from the CLT.

We obtain that

$$\lim_{x\to+\infty} f_0(x,\omega) = \lim_{x\to+\infty} - \underbrace{\sum_{j=0}^{x-1} e^{\sum_{k=1}^{j-1}\log(\xi_k^\omega)}}_{\text{infinitely many terms} \geq 1} = -\infty.$$

$$\lim_{x\to-\infty} f_0(x,\omega) = +\infty,$$ similarly.

$n\wedge T = \min\{n, T\}$

Use optional stopping theorem with the martingales $(f_0(X_{n\wedge T},\omega))_n$ and $(f_0(X_{n\wedge S},\omega))_n$

$$T = \inf\{k\geq 0 : X_k \geq A\} \quad \text{for } A>0.$$

$$S = \inf\{k\geq 0 : X_k \leq -A\}$$

and prove that $P_0^\omega(T<\infty) = P_0^\omega(S<\infty) = 1.$

$$\Longleftrightarrow P_0^\omega(S<\infty, T<\infty) = 1.$$

That will hold $\forall A>0.$

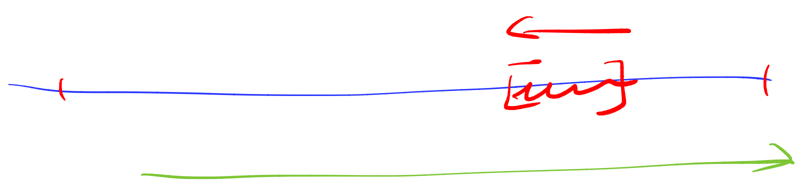
$\Longrightarrow (X_n)$ is recurrent.

OST: $E[M_{T_1}] = E[M_0]$

This proves part A].

Part B : next week.

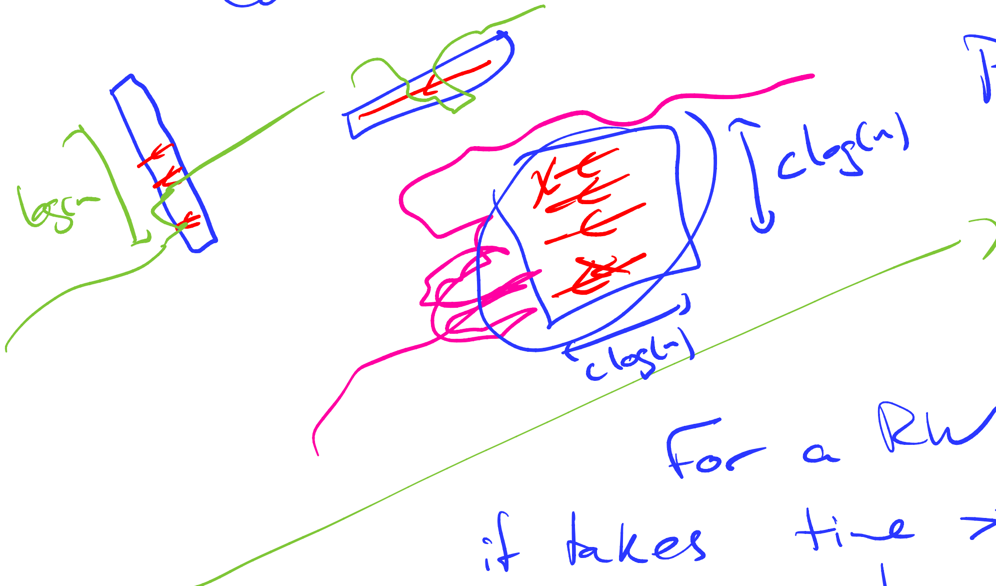
in $d=1$



Consider $\mathbb{Z}^d$ $(p_n^\omega)_n$ i.i.d. vectors

$p_n^\omega(n, x+e_i) \in [\kappa, 1-\kappa]$

$\{e_1, \dots, e_{2d}\}$



For a RW to be delayed

it takes time $\gg n$

You need an ~~at~~ typical region of volume

$\sim (\log(n))^d$

For the environment to build, it costs a prob:

$$\sim \exp\left(-c(\log(n))^d\right) \ll n^{-d}, \dots$$